# Nucleon partonic structure: concepts and measurements

## Part 6: TMDs

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HELMHOLTZ
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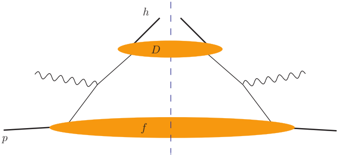
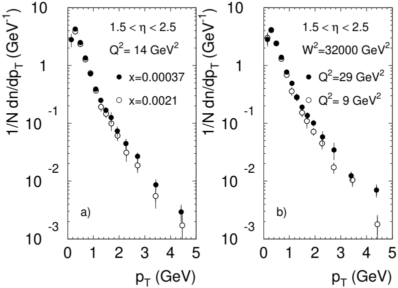
## Measured transverse momentum




- consider
  - Drell-Yan with measured small $q_T$ of $\gamma^*$
  - SIDIS with measured small $q_T$ of hadron
  - $e^+e^- \to h_1 h_2 + X$ with $h_1$, $h_2$ approx. opposite momenta and small relative $q_T$
- $k_T \sim m$ from collinear graphs matters in final state
  - can still neglect parton $k_T$ in hard scattering
  - but do not $\int d^2\boldsymbol{k}$ in parton densities and fragm. fcts.
    $\rightsquigarrow$ $k_T$ dependent/unintegrated PDFs
    also called TMDs (transverse-momentum distributions)
- theoretical framework: TMD factorisation
  - also called $k_T$ factorisation
    different from (but related to) $k_T$ factorisation at small $x$

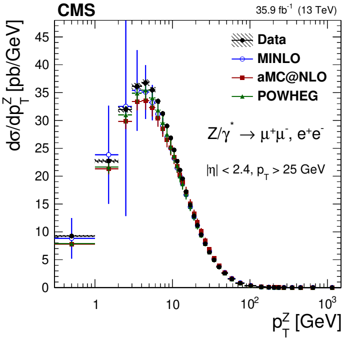
# Measured transverse momentum




$$e + p \to e + h^{\pm} + X$$

$h^{\pm}$ = charged hadron

H1, hep-ex/9610006




$$p + p \to \ell^+ \ell^- + X$$

CMS, arXiv:1909.04133

## $k_T$ dependent parton densities

$k_T$ integrated:

$$f_1(x) = \int \frac{dz^-}{4\pi}\, e^{iz^- p^+ x} \left\langle p,s \middle| \bar{q}(0)\gamma^+ W(0,\infty) W(\infty,z^-)\, q(z^-) \middle| p,s \right\rangle \Big|_{z^+=0,\, \boldsymbol{z}=\boldsymbol{0}}$$

$k_T$ dependent:

$$\int \frac{dz^-}{4\pi} \frac{d^2\boldsymbol{z}}{(2\pi)^2}\, e^{iz^- p^+ x}\, e^{-i\boldsymbol{k}\boldsymbol{z}} \left\langle p,s \middle| \bar{q}(0)\gamma^+ W(0,\infty) W(\infty,z^-,\boldsymbol{z})\, q(z^-,\boldsymbol{z}) \middle| p,s \right\rangle \Big|_{z^+=0}$$

- fields at different transv. positions
  implications on Wilson lines $\rightarrow$ later
- notation: write $k_T$ in text, omit $_T$ in boldface vector $\boldsymbol{k}$ etc.

## $k_T$ dependent parton densities

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$k_T$ dependent:

$$\int \frac{dz^-}{4\pi} \frac{d^2\boldsymbol{z}}{(2\pi)^2}\, e^{iz^- p^+ x}\, e^{-i\boldsymbol{kz}} \left\langle p,s \middle| \bar{q}(0)\gamma^+ W(0,\infty) W(\infty, z^-, \boldsymbol{z})\, q(z^-, \boldsymbol{z}) \middle| p,s \right\rangle \Big|_{z^+=0}$$

$$= f_1(x, \boldsymbol{k}^2) - \frac{\epsilon^{ij} \boldsymbol{k}^i \boldsymbol{s}^j}{m}\, f_{1T}^{\perp}(x, \boldsymbol{k}^2)$$

$$\begin{aligned} \epsilon^{12} &= -\epsilon^{21} = 1 \\ \epsilon^{11} &= \epsilon^{22} = 0 \end{aligned}$$

- fields at different transv. positions
  implications on Wilson lines → later
- correlations between spins and transv. momentum
  e.g. Sivers function $f_{1T}^{\perp}$

## A zoo of distributions

- collinear twist 2 densities:

  $f_1$ unpol. quark in unpol. proton
  $g_1$ correlate $s_L$ of quark with $S_L$ of proton
  $h_1$ correlate $s_T$ of quark with $S_T$ of proton

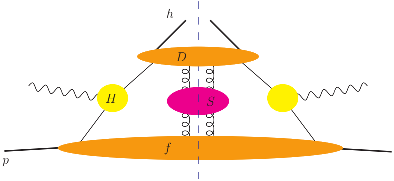
- $k_T$ dependent twist 2 densities:

  $f_1, g_1, h_1$ as above
  $f_{1T}^{\perp}$ correlate $k_T$ of quark with $S_T$ of proton (Sivers)
  $h_1^{\perp}$ correlate $k_T$ and $s_T$ of quark (Boer-Mulders)
  $g_{1T}, h_{1T}^{\perp}, h_{1L}^{\perp}$ three more densities

- analogous for fragmentation functions:
  - $f_1 \leftrightarrow D_1$ unpolarized
  - $h_1^{\perp} \leftrightarrow H_1^{\perp}$ Collins fragm. fct.

# TMD factorisation (SIDIS as example)

- take $Q$ large and $q_T$ small ($\sim m$ for power counting purposes)




- transverse-momentum dep't distribution and fragmentation fcts.
- only virtual corrections to hard subgraph
  no radiation of high-$p_T$ partons allowed

## TMD factorisation (SIDIS as example)

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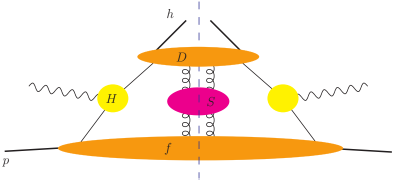
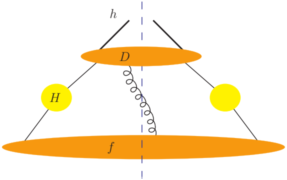



- soft gluon exchange does not cancel in sum over hadronic final state
  at leading-power accuracy gives soft factor in factorisation formula

$S =$ universal non-perturbative fct
$\rightarrow 1$ when integrate over $k_T$
cancellation between real



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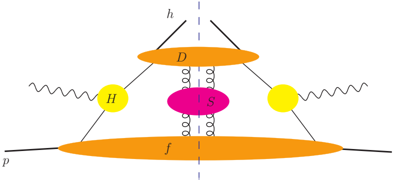


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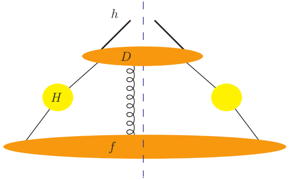
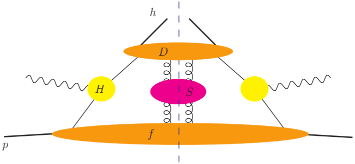


# SIDIS at low $q_T$

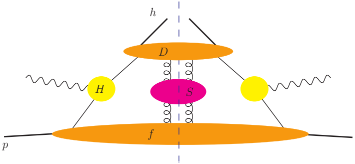



- factorisation formula

$$\frac{d\sigma_{\gamma^* p}}{dz\, d\boldsymbol{q}^2} = (\text{kin. fact.}) \times |H(\mu)|^2 \int d^2\boldsymbol{p}\, d^2\boldsymbol{k}\, d^2\boldsymbol{l}\; \delta^{(2)}(\boldsymbol{p} - \boldsymbol{k} + \boldsymbol{l} + \boldsymbol{q}) \times \sum_{i=q,\bar{q}} e_i^2\, f^i(x, \boldsymbol{p}, \mu)\, D^i(z, \boldsymbol{k}, \mu)\, S(\boldsymbol{l}, \mu)$$

- no $\int d^2\boldsymbol{k}$ in parton densities $\rightsquigarrow$ no DGLAP type evolution !!
- various azimuthal and spin asymmetries

# SIDIS at low $q_T$

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- simplifies if Fourier transform

  $f(\boldsymbol{p}) \to f(\boldsymbol{b})$, $D(\boldsymbol{k}) \to D(\boldsymbol{b})$, $S(\boldsymbol{l}) \to S(\boldsymbol{b})$:

$$\frac{d\sigma_{\gamma^* p}}{dz\, d\boldsymbol{q}^2} = (\text{kin.fact.}) \times |H(\mu)|^2 \int d^2\boldsymbol{b}\, e^{-i\boldsymbol{b}\boldsymbol{q}} \sum_{i=q,\bar{q}} e_i^2\, f^i(x, \boldsymbol{b}, \mu)\, D^i(z, \boldsymbol{b}, \mu)\, S(\boldsymbol{b}, \mu)$$

  note: $\boldsymbol{b}$ here not the same as $\boldsymbol{b}$ in GPDs (will later call $\boldsymbol{z}$)

- redefine $f$ and $D$ to each absorb factor $\sqrt{S}$

# SIDIS at low $q_T$

- TMD distributions also depend on rapidity parameter $\zeta$
- Collins-Soper equation and RGE for $f$ (same for $D$):

$$\frac{d}{d\ln\sqrt{\zeta}}\, f(x,b,\zeta,\mu) = K(b,\mu)\, f(x,b,\zeta,\mu)$$

$$\frac{d}{d\ln\mu}\, f(x,b,\zeta,\mu) = \gamma_F(\zeta,\mu)\, f(x,b,\zeta,\mu)$$ (no $x$ integral as in DGLAP eq.)

- "cusp anomalous dimension"

$$\frac{dK(b,\mu)}{d\ln\mu} = \frac{d\gamma_F(\zeta,\mu)}{d\ln\sqrt{\zeta}} = -\gamma_K(\mu) = -C_F\,\frac{2\alpha_s(\mu)}{\pi} + \ldots$$

$C_F = \frac{N_c^2-1}{2N_c}$ for colour group $SU(N_c)$

## SIDIS at low $q_T$

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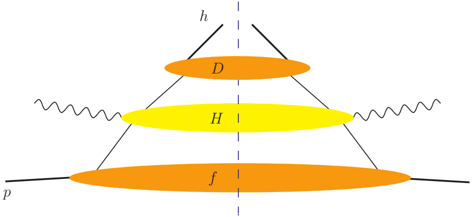
- solution:

$$\frac{f(x,b,\zeta,\mu)}{f(x,b,\zeta_0,\mu_0)} = \exp\left\{-\int_{\mu_0}^{\mu}\frac{d\mu'}{\mu'}\left[\gamma_K(\mu')\ln\frac{\sqrt{\zeta}}{\mu'} - \gamma_F(\mu'^2,\mu')\right] + K(b,\mu_0)\ln\frac{\sqrt{\zeta}}{\sqrt{\zeta_0}}\right\}$$

- $\exp\{\ldots\}$ = Sudakov factor
- in exponent have “double logarithms” $\ln^2(\mu/\mu_0)$ for $\zeta \sim \mu^2$
  in cross section set $\sqrt{\zeta} \sim \mu \sim Q$ and $\sqrt{\zeta}_0 \sim \mu_0 \sim q_T$
- $K(b,\mu)$ calculable in pert. theory only if $b$ is small
  need to interpolate between small and large $b$

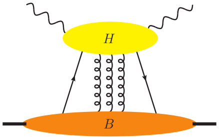
## Compare with collinear factorisation for large $q_T$




$$\frac{d\sigma_{\gamma^* p}}{dz\, d\boldsymbol{q}^2} = (\text{kin. fact.}) \times \int_x^1 \frac{d\hat{x}}{\hat{x}} \int_z^1 \frac{d\hat{z}}{\hat{z}}\, \delta\biggl( \frac{\boldsymbol{q}^2}{Q^2} - \frac{(1-\hat{x})(1-\hat{z})}{\hat{x}\hat{z}} \biggr) \\ \times \sum_{i,j=q,\bar{q},g} f_i\Bigl(\frac{x}{\hat{x}}, \mu^2\Bigr) D_j\Bigl(\frac{z}{\hat{z}}, \mu^2\Bigr)\, C_{ij}\Bigl(\hat{x}, \hat{z}, \ln\frac{\mu^2}{Q^2}\Bigr)$$

- $C_{ij}$ start at $\mathcal{O}(\alpha_s)$, must emit partons recoiling against $\boldsymbol{q}$
- convolution in momentum fractions

## Wilson lines in short-distance factorisation




- exchange of $> 2$ partons between $H$ and $B$ power suppressed
- except for $A^+$ gluon exchange
  $\rightsquigarrow$ resum to all orders

- $H^\mu(l)$ all components big
  $B^\mu(l) \propto l^\mu$ only plus-component big ($p^\mu$ and $l^\mu$ collinear momenta)
  $\rightsquigarrow \quad H_\mu B^\mu \approx H^- B^+$

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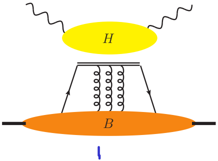


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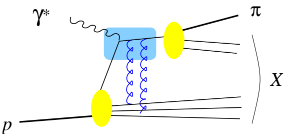
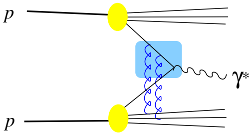
- Ward identities $\rightsquigarrow$ gluons removed from $H$
  in $B$ obtain Wilson line $W(a,b) = \mathrm{P} \exp\left[ig \int_a^b dz^- \, A^+(z)\right]_{z^+=0,\, \boldsymbol{z}=\boldsymbol{0}}$

$$q(x) \propto \int dz^- \, e^{ixp^+ z^-} \langle p| \, \bar{q}(0) \, W(0,z) \, \gamma^+ q(z) \, |p\rangle \Big|_{z^+=0,\, \boldsymbol{z}=\boldsymbol{0}}$$

# Wilson lines in TMDs

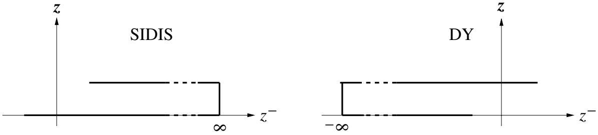



$$q(x, \boldsymbol{k}^2) \propto \int dz^- \, d^2\boldsymbol{z} \, e^{ixp^+z^-} e^{-i\boldsymbol{k}\boldsymbol{z}} \langle p| \, \bar{q}(0) \, W_P(0,z) \, \gamma^+ q(z) \, |p\rangle \Big|_{z^+=0}$$

- space-time structure of process $\rightsquigarrow$ path $P$ in Wilson line
- SIDIS: interactions after quark struck by photon
  DY: interactions before quark annihilates

# Wilson lines in TMDs

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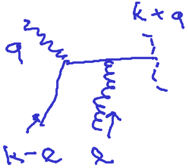
- space-time structure of process $\rightsquigarrow$ path $P$ in Wilson line
- SIDIS: interactions after quark struck by photon
  DY: interactions before quark annihilates
- obtain "staple like" paths
  - Feynman gauge: pieces at $z^- \to \pm\infty$ not important
  - light-cone gauge $A^+ = 0$: straight sections $\to 1$
    all effects from $z^- \to \pm\infty$

# Wilson lines at lowest order




$$\bar u(\hat k+q)\,(-ig\gamma^-)\,\frac{i}{(\hat k+q-\hat\ell)\gamma} =$$

$$k \to \hat k = (k^+, 0, \vec 0)$$

$$\ell \to \hat\ell$$

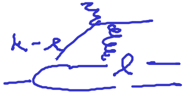
$$\ldots = \bar u(\hat k+q)\,\frac{-ig}{\ell^+}\,\hat\ell\cdot\gamma\,\frac{i}{(\hat k+q-\hat\ell)\gamma} = \bar u(\hat k+q)\; i\;\frac{ig}{\ell^+-i\epsilon}$$

$$\hat\ell\cdot\gamma \to (\hat k+q)-(\hat k+q-\hat\ell)$$

$$\frac{i}{(\hat k+q-\hat\ell)\gamma} = \frac{i(\hat k+q-\hat\ell)\gamma}{(\hat k+q-\hat\ell)^2+i\epsilon} = -\frac{(\hat k+q-\hat\ell)\gamma}{2q^-}\;\frac{i}{\ell^+-i\epsilon}$$

$$(\hat k+q-\hat\ell)^2 = 2q^-(-\ell^+)$$

## Wilson lines at lowest order




$$\int d\ell^+ \frac{i}{\ell^+ - i\epsilon}\, d^2\ell_T \int \frac{dz^-}{2\pi}\, \frac{dy^-}{2\pi}\, \frac{d^2y_T}{(2\pi)^2}\, e^{i(k^+-\ell^+)z^- + i\ell^+ y^-}$$

$$\times\, e^{-i(\vec{k}_T-\vec{\ell}_T)\vec{z}_T - i\vec{\ell}_T\vec{y}_T}\,(+igA^+(y^-,\vec{y}_T))\; q(z^-,\vec{z}_T)$$

$$(y^+ = z^+ = 0)$$

$$= \int \frac{dz^-}{2\pi}\, e^{ik^+z^- - i\vec{k}_T\vec{z}_T} \int dy^- \int \frac{d\ell^+}{2\pi}\, e^{i\ell^+(y^- - z^-)}\, \frac{i}{\ell^+ - i\epsilon}$$

$$\times\; igA^+(y^-,\vec{z}_T)\, q(z^-,\vec{z}_T)$$

$$= \text{—— " ——} \int_{z^-}^{\infty} dy^-\; igA^+(y^-,\vec{z}_T)\cdot q(z^-,\vec{z}_T)$$

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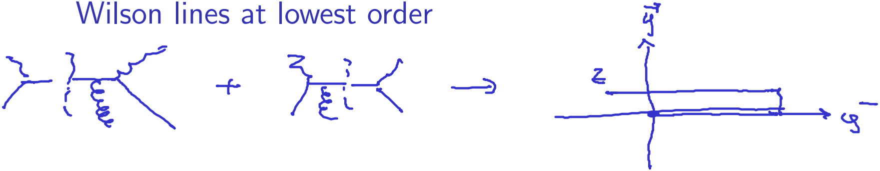
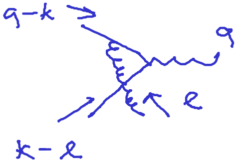
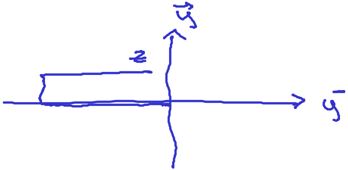



Drell-Yan




$$\frac{i}{(q-\hat{k}+\hat{l})^2+i\epsilon} = \frac{i}{2q^-} \quad \frac{1}{l^+ + i\epsilon}$$



## Rapidity divergences revisited

- arise as $\int_0 d\ell^+/\ell^+$ from region $\ell^+ \to 0$ at nonzero $\ell^-$
  $\rightsquigarrow$ negative rapidity, should not be inside TMD
- need to regulate $\rightsquigarrow$ rapidity "cutoff" parameter $\zeta$

$$\frac{d}{d\ln\sqrt{\zeta}} = \frac{d}{d(\text{rapidity})}$$


- $\int d^2\boldsymbol{k} \rightsquigarrow$ divergences cancel between real and virtual graphs
  $\rightsquigarrow$ not present in usual PDFs (or GPDs)

Wilson line has physical consequences

- transverse proton polarization $\rightsquigarrow$ anisotropic $\boldsymbol{k}$ distribution

$$f_{q/p\uparrow}(x, \boldsymbol{k}) = f_1(x, \boldsymbol{k}^2) + \frac{(\boldsymbol{S} \times \boldsymbol{k}\,) \cdot \boldsymbol{p}}{m|\boldsymbol{p}\,|}\, f_{1T}{}^{\perp}(x, \boldsymbol{k}^2)$$

- induces anisotropic $p_T$ distribution in SIDIS (Sivers effect) observed experimentally
- time reversal changes sign of $(\boldsymbol{S} \times \boldsymbol{k}) \cdot \boldsymbol{p}$
  $\rightsquigarrow$ Sivers function $= 0$ ??

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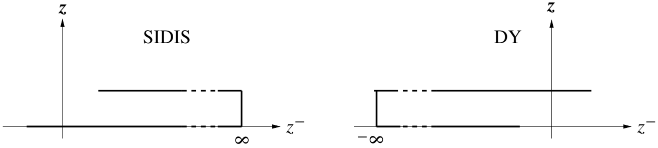
- induces anisotropic $p_T$ distribution in SIDIS (Sivers effect) observed experimentally
- time reversal changes sign of $(\boldsymbol{S}\times\boldsymbol{k})\cdot\boldsymbol{p}$
  $\rightsquigarrow$ Sivers function $= 0$ ??
- no: time reversal interchanges Wilson lines for
  SIDIS (future pointing) and DY (past pointing)

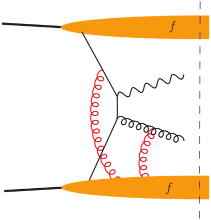
  $\rightsquigarrow$ $f_{1T}^{\perp,\mathrm{SIDIS}}(x,\boldsymbol{k}^2) = -f_{1T}^{\perp,\mathrm{DY}}(x,\boldsymbol{k}^2)$ J. Collins '02

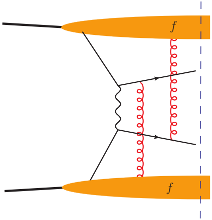


## More complicated processes

- examples: $pp \to \gamma + \text{jet} + X$, $pp \to \pi + \text{jet} + X$
- more partons in initial and final state
  $\rightsquigarrow$ more complicated Wilson lines
  $\rightsquigarrow$ more parton densities and fragm. functions

Bomhof, Mulders, Pijlman, Buffing '04-'15




- two-loop analysis $\rightsquigarrow$ breakdown of TMD factorisation

Mulders, Rogers '10

# Relation between high-$q_T$ and low-$q_T$ descriptions

- for $q_T \gg m$ calc. $k_T$ dependent densities from coll. ones:


$$f_1^i(x, \boldsymbol{k}^2; \zeta, \mu) = \frac{1}{\boldsymbol{k}^2} \sum_j \int_x^1 \frac{dx'}{x'} K^{ij}\Big(\frac{x}{x'}, \ln\frac{\boldsymbol{k}^2}{\zeta}\Big) f_1^j(x'; \mu)$$

$K$ closely related with DGLAP splitting functions $P$

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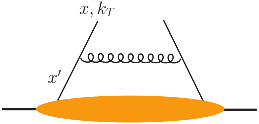
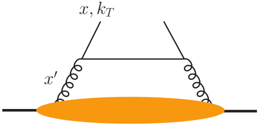



$$f_1^i(x, \boldsymbol{b}^2; \zeta, \mu) = f_1^i(x; \mu) + \sum_j \int_x^1 \frac{dx'}{x'} \widetilde{K}^{ij}\Big(\frac{x}{x'}, \ln\frac{\mu^2}{\zeta}, \ln(\mu^2 \boldsymbol{b}^2)\Big) f_1^j(x'; \mu)$$

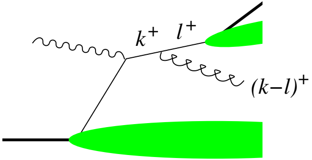
$\widetilde{K}$ closely related with DGLAP splitting functions $P$

## Comparison between high-$q_T$ and low-$q_T$ descriptions

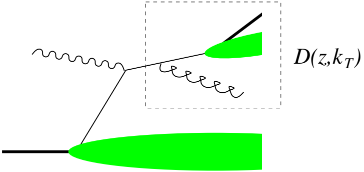
- collinear fact. requires $q_T \gg m$
  $k_T$ fact. requires $q_T \ll Q$
  $\rightsquigarrow$ in region $m \ll q_T \ll Q$ both approaches are valid
- compare $q_T \gg m$ limit of $k_T$ fact. result
  with $q_T \ll Q$ limit of coll. fact. result
  $\rightsquigarrow$ full agreement for unpol. cross section
  Collins, Soper, Sterman '85; Bacchetta et al. '08
- detailed comparison also for various spin asymmetries
  e.g. Sivers asy. in SIDIS or Drell-Yan at low $q_T$ (Sivers fct.) and high $q_T$ (Qiu-Sterman fct.)
  Ji, Qiu, Vogelsang, Yuan '06; Koike, Vogelsang, Yuan '07

## Correspondence at level of graphs

high-$q_T$ calculation

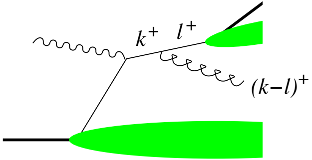



low-$q_T$ calculation with $q_T \gg m$

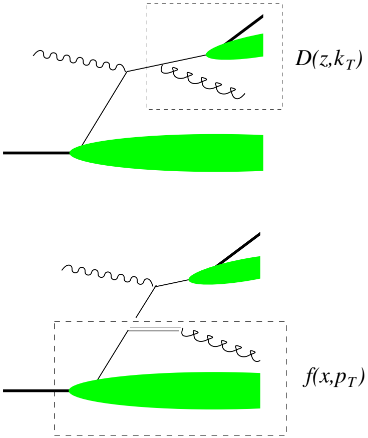


## Correspondence at level of graphs

high-$q_T$ calculation




low-$q_T$ calculation with $q_T \gg m$



## Transverse momentum vs. position

- variables related by 2d Fourier transforms, e.g.
  - quark fields $\tilde{q}(\boldsymbol{k}, z^-) = \int d^2\boldsymbol{z}\, e^{i\boldsymbol{z}\boldsymbol{k}} q(\boldsymbol{z}, z^-)$
  - proton states $|p^+, \boldsymbol{b}\rangle = \int d^2\boldsymbol{p}\, e^{-i\boldsymbol{b}\boldsymbol{p}} |p^+, \boldsymbol{p}\rangle$
- in bilinear operators

$$\overline{\tilde{q}}(\boldsymbol{k})\, \tilde{q}(\boldsymbol{l}) = \int d^2\boldsymbol{y}\, d^2\boldsymbol{z}\, e^{-i(\boldsymbol{y}\boldsymbol{k} - \boldsymbol{z}\boldsymbol{l})}\, \overline{q}(\boldsymbol{y})\, q(\boldsymbol{z})$$

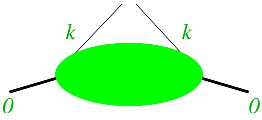
$$\boldsymbol{y}\boldsymbol{k} - \boldsymbol{z}\boldsymbol{l} = \tfrac{1}{2}(\boldsymbol{y} + \boldsymbol{z})(\boldsymbol{k} - \boldsymbol{l}) + \tfrac{1}{2}(\boldsymbol{y} - \boldsymbol{z})(\boldsymbol{k} + \boldsymbol{l})$$

'average' transv. momentum $\leftrightarrow$ position difference
transv. momentum transfer $\leftrightarrow$ 'average' position

- 'average' transv. mom. and position not Fourier conjugate

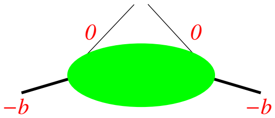
## Mind the difference

TMDs




$\int d^2\boldsymbol{z}\, e^{-i\boldsymbol{z}\boldsymbol{k}}\, \langle \mathbf{0}|\bar{q}(-\frac{1}{2}\boldsymbol{z}) \dots q(\frac{1}{2}\boldsymbol{z})|\mathbf{0}\rangle$

impact parameter distributions




$\int d^2\boldsymbol{\Delta}\, e^{-i\boldsymbol{b}\boldsymbol{\Delta}}\, \langle -\frac{1}{2}\boldsymbol{\Delta}|\bar{q}(\mathbf{0}) \dots q(\mathbf{0})|\frac{1}{2}\boldsymbol{\Delta}\rangle$

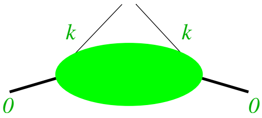
(longitudinal variables not shown for simplicity)

Fourier conjugates:

| | | |
|---|---|---|
| average transv. momentum | $\leftrightarrow$ | difference of transv. positions |
| $q(x, \boldsymbol{k})$ | | Wilson lines, Sudakov resummation, ... |
| difference of transv. momenta | $\leftrightarrow$ | average transv. position |
| $H(x, \boldsymbol{\Delta})_{\xi=0}$ | | $q(x, \boldsymbol{b})$ |

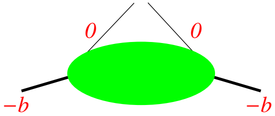
# Mind the difference

TMDs




$$\int d^2 z \, e^{-i z k} \, \langle \mathbf{0} | \bar{q}(-\tfrac{1}{2} z) \ldots q(\tfrac{1}{2} z) | \mathbf{0} \rangle$$

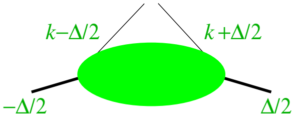
impact parameter distributions




$$\int d^2 \boldsymbol{\Delta} \, e^{-i b \boldsymbol{\Delta}} \, \langle -\tfrac{\mathbf{1}}{\mathbf{2}} \boldsymbol{\Delta} | \bar{q}(\mathbf{0}) \ldots q(\mathbf{0}) | \tfrac{\mathbf{1}}{\mathbf{2}} \boldsymbol{\Delta} \rangle$$

more general:

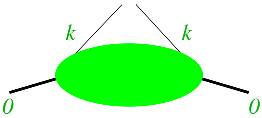
GTMDs




$$\int d^2 z \, e^{-i z k} \, \langle -\tfrac{\mathbf{1}}{\mathbf{2}} \boldsymbol{\Delta} | \bar{q}(-\tfrac{1}{2} z) \ldots q(\tfrac{1}{2} z) | \tfrac{\mathbf{1}}{\mathbf{2}} \boldsymbol{\Delta} \rangle$$

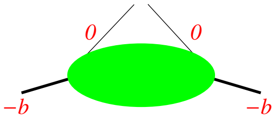
# Mind the difference

TMDs




$$\int d^2\boldsymbol{z}\, e^{-i\boldsymbol{z}\boldsymbol{k}}\, \langle \boldsymbol{0}|\bar{q}(-\tfrac{1}{2}\boldsymbol{z}) \dots q(\tfrac{1}{2}\boldsymbol{z})|\boldsymbol{0}\rangle$$

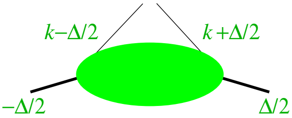
impact parameter distributions




$$\int d^2\boldsymbol{\Delta}\, e^{-i\boldsymbol{b}\boldsymbol{\Delta}}\, \langle -\tfrac{1}{2}\boldsymbol{\Delta}|\bar{q}(\boldsymbol{0}) \dots q(\boldsymbol{0})|\tfrac{1}{2}\boldsymbol{\Delta}\rangle$$

more general:

GTMDs




$$\int d^2\boldsymbol{z}\, e^{-i\boldsymbol{z}\boldsymbol{k}}\, \langle -\tfrac{1}{2}\boldsymbol{\Delta}|\bar{q}(-\tfrac{1}{2}\boldsymbol{z}) \dots q(\tfrac{1}{2}\boldsymbol{z})|\tfrac{1}{2}\boldsymbol{\Delta}\rangle$$

Fourier transf. from $\boldsymbol{\Delta}$ to $\boldsymbol{b}$
$\rightsquigarrow$ Wigner functions

parton momentum and position
within limits of uncertainty rel'n

## Relations

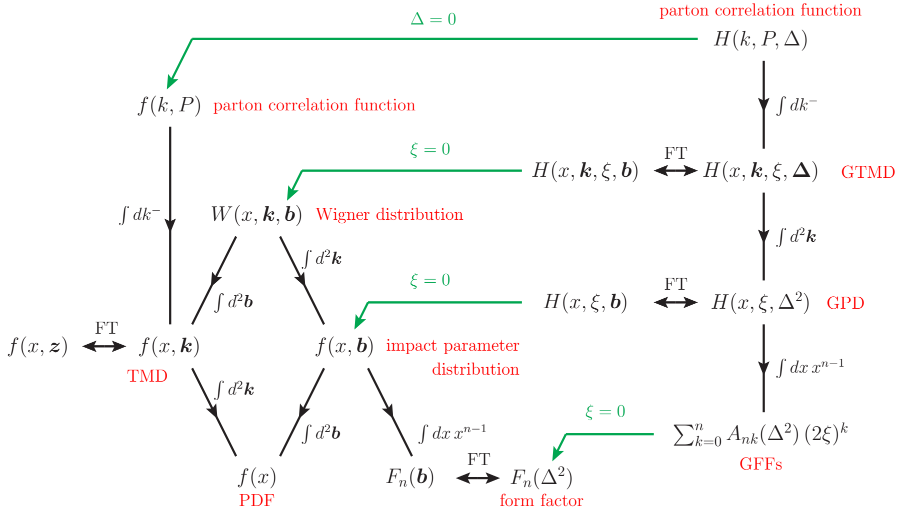
$$
\begin{tikzcd}
H(x,\boldsymbol{z},\boldsymbol{\Delta}) \arrow[r, leftrightarrow, "\mathrm{FT}\ \boldsymbol{z}\leftrightarrow\boldsymbol{k}"'] & H(x,\boldsymbol{k},\boldsymbol{\Delta}) \arrow[r, leftrightarrow, "\mathrm{FT}\ \boldsymbol{\Delta}\leftrightarrow\boldsymbol{b}"] \arrow[d, "\boldsymbol{\Delta}=\mathbf{0}"'] \arrow[dr, "\int d^2\boldsymbol{k}"] & W(x,\boldsymbol{k},\boldsymbol{b}) \arrow[dl, "\int d^2\boldsymbol{b}"] \arrow[dr, "\int d^2\boldsymbol{k}"] & \\
q(x,\boldsymbol{z}) \arrow[r, leftrightarrow, "\mathrm{FT}\ \boldsymbol{z}\leftrightarrow\boldsymbol{k}"'] & q(x,\boldsymbol{k}) \arrow[dr, "\int d^2\boldsymbol{k}"] & H(x,\boldsymbol{\Delta}) \arrow[r, leftrightarrow, "\mathrm{FT}\ \boldsymbol{\Delta}\leftrightarrow\boldsymbol{b}"'] \arrow[d, "\boldsymbol{\Delta}=\mathbf{0}"'] \arrow[dr, "\int dx\, x^{n-1}"] & q(x,\boldsymbol{b}) \arrow[dl, "\int d^2\boldsymbol{b}"] \\
 & & q(x) & F_n(\boldsymbol{\Delta})
\end{tikzcd}
$$

GPDs taken at zero skewness $\xi = 0$

$\int d^2\boldsymbol{k}$ needs UV regularization

$\boldsymbol{k}$ dep't functions have $\zeta$ dep'ce

# More relations




$\int d^2\boldsymbol{k}$ needs UV regularization

$\boldsymbol{k}$ dep't functions have $\zeta$ dep'ce

- naive:

$$\int d^2\boldsymbol{k}\, q(x,\boldsymbol{k}) = q(x)$$

cannot be true because $q(x,\boldsymbol{k}) \sim 1/\boldsymbol{k}^2$ at large $\boldsymbol{k}$

- correct:

$$\int_{\boldsymbol{k}^2<\mu^2} d^2\boldsymbol{k}\, q(x,\boldsymbol{k};\zeta,\mu) = q(x;\mu) + \text{calculable terms of } \mathcal{O}(\alpha_s)$$

- Fourier trf. from $\boldsymbol{k}$ to $\boldsymbol{z}$:

instead of $\int_{\boldsymbol{k}^2<\mu^2} d^2\boldsymbol{k}$ take $\int d^2\boldsymbol{k}\, e^{i\boldsymbol{k}\boldsymbol{z}}$ with $|\boldsymbol{z}| = 1/\mu$

oscillations suppress region $|\boldsymbol{k}| \gg 1/|\boldsymbol{z}|$

$$q(x,\boldsymbol{z};\zeta,\mu) = q(x;\mu) + \text{calculable terms of } \mathcal{O}(\alpha_s)$$

see earlier slide

## Summary of part 6

- TMD factorisation for measured $p_T \ll$ hard scale
- important differences with collinear factorisation, different evolution
- subtle dynamical effects due to gluons $\rightsquigarrow$ Wilson lines
- valid for restricted class of processes
  for some cases smooth theoretical transition to high $p_T$ regime
- theoreticallly controlled access to transverse parton momentum
- Wigner functions: unifying framework for describing
  transverse momentum and position

## Notes

# Notes

# Notes