Gravitational waves from compact objects

Lecture I

Lectures for the 2022 GGI APCG School

Leonardo Gualtieri
University of Rome “La Sapienza”

In this course, I assume basic knowledge of general relativity (GR).
Most of the topics can be found here:

- Ferrari, Gualtieri, Pani, *General Relativity and its Applications*, CRC Press;
- Maggiore, *Gravitational Waves (Vol.I)*, Oxford Univ. Press.

Furthermore, the following are also useful:

- Shapiro, Teukolsky, *Black Holes, White Dwarfs and Neutron Stars*, Wiley;
- Carroll. *Spacetime and Geometry*, Addison Wesley;
- Misner, Thorne, Wheeler, *Gravitation*, Freeman;
- Wald, *General Relativity*, Univ. Chicago Press.

GRAVITATIONAL WAVES

I will briefly recall the main concepts of the theory of gravitational waves (GWs).

General Relativity (GR) predicts the existence of GWs: any change in a matter-energy distribution affects the gravitational field; to preserve causality, such modification has to propagate through a wave at finite velocity.

Since the gravitational interaction is described in GR by the spacetime metric, GWs are *metric waves*, changing the proper distance between events:

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu \quad (\mu, \nu = 0, \ldots, 3) \, .$$

Let us consider a spacetime which is a small perturbation of Minkowski spacetime:

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \qquad |h_{\mu\nu}| \ll 1 \, , \tag{1}$$

where $\eta_{\mu\nu} = \mathrm{diag}(-1, 1, 1, 1)$. Note that we are using Minkwoskian coordinates, $\{x^\mu\} = (x^0, x^i)$ where $x^0 = ct$ and $\{x^i\}$ $(i = 1, 2, 3)$ is an orthogonal Cartesian frame. In general, I will use Greek indices $\mu, \nu = 0, \ldots, 3$ as spacetime indices, and Latin indices $i, j = 1, 2, 3$ as space indices in a Minkowskian frame.

This is the **weak-field approximation**, satisfied by all astrophysical phenomena except those close to the surface of a black hole (BH) or of a neutron star (NS). In this approximation, $h^\mu{}_\nu = \eta^{\mu\alpha} h_{\alpha\nu} + O(h^2)$,

$$\Gamma^\alpha_{\mu\nu} = \frac{1}{2} \eta^{\alpha\beta} (h_{\beta\mu,\nu} + h_{\beta\nu,\mu} - h_{\mu\nu,\beta}) + O(h^2) = O(h)$$

and

$$h_{\mu\nu;\alpha} = h_{\mu\nu,\alpha} + O(h^2) \, .$$

Therefore, if we neglect $O(h^2)$ terms with respect to $O(1)$ terms, the operations of raising an lowering of indices and covariant derivatives of $h_{\mu\nu}$ are like in flat spacetime: formally, we can consider $h_{\mu\nu}$ as a *field on Minkowski spacetime*.

Similarly, since the stress-energy tensor on the background vanishes, $T_{\mu\nu} = O(h)$: the stress-energy tensor is the source of the perturbation, and can be treated as a tensor field in flat spacetime.

By replacing $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ into Einstein's equations

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \frac{8\pi G}{c^2}T_{\mu\nu}$$

and neglecting $O(h^2)$ terms, i.e. *linearizing Einstein's equations*, gives:

$$\Box_F \bar{h}_{\mu\nu} = -\frac{16\pi G}{c^2}T_{\mu\nu} \tag{2}$$

$$\bar{h}^{\mu}{}_{\alpha,\mu} = 0 \tag{3}$$

where $\Box_F = \eta^{\mu\nu}\frac{\partial^2}{\partial x^\mu \partial x^\nu} = -\frac{1}{c^2}\frac{\partial^2}{\partial t^2} + \nabla^2$ is the D'Alembertian operator in flat space, and

$$\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h \quad (h = h^{\mu}{}_{\mu}) \,.$$

Eq. (3), which is the linearization of $\Gamma^{\alpha}_{\mu\nu}g^{\mu\nu} = 0$, is the **harmonic gauge condition**, and fixes the gauge freedom. Indeed, in general, we can describe the same physical system by changing coordinates; in particular, for a coordinate transformation $O(h)$, i.e. $x^\alpha \to x'^\alpha = x^\alpha + \epsilon^\alpha(x)$ with $\epsilon^\alpha = O(h)$, Eq. (1) is preserved: $g^{\mu'\nu'} = \eta^{\mu'\nu'} + h^{\mu'\nu'}$ with $|h^{\mu'\nu'}| \ll 1$.

The harmonic choice of the gauge has the advantage of simplifying the GW equations to the form (2). Note that once it is fixed, there is some residual gauge freedom: if, in the harmonic gauge, we change coordinates $x^\alpha \to x'^\alpha = x^\alpha + \epsilon^\alpha(x)$ with $\Box_F \epsilon^\alpha = 0$, the new coordinates are still in the harmonic gauge.

Let e us now consider linearized Einstein's equations in vacuum:

$$\Box_F \bar{h}_{\mu\mu} = \left(-\frac{1}{c^2}\frac{\partial^2}{\partial t^2} + \nabla^2\right)\bar{h}_{\mu\nu} = 0\,.$$

It describes *waves* moving with velocity c: the GWs. This is a linear equation, satisfying the superposition principle, then the solution space is a vector space.

A basis of the solution space is given by *monochromatic plane waves*:

$$\bar{h}_{\mu\nu} = \mathcal{R}\left[A_{\mu\nu}e^{ik_\alpha x^\alpha}\right] \qquad (e^{ik_\alpha x^\alpha} = e^{-i\omega t + \vec{k}\cdot\vec{x}})$$

with $A_{\mu\nu}$ polarization tensor, $k^\alpha = \left(\frac{\omega}{c}, \vec{k}\right)$ wave 4-vector. The wave equation implies that k^α is a null vector, while the gauge condition implies that the wave is transverse: $k^\alpha A_{\alpha\beta} = 0$. The space part of k^α, $\vec{k}$, is the wave vector, and

$$k_\alpha k^\alpha = -\frac{\omega^2}{c^2} + |\vec{k}|^2 = 0 \quad \Rightarrow \quad |\vec{k}| = \frac{2\pi}{\lambda} = \frac{\omega}{c}$$

therefore the wavelength of the GW is $\lambda = \frac{2\pi c}{\omega}$.

Far away from the source, the wavefront is not plane, but locally it looks like a plane wave; if the source-observer line is aligned with the x-axis, the wave looks locally, near the observer, as a plane wave propagating along x: $\bar{h}_{\mu\nu} = \bar{h}_{\mu\nu}(x,t) = \bar{h}_{\mu\nu}\left(t - \frac{x}{c}\right)$.

By imposing the harmonic gauge condition $\bar{h}^\mu{}_{\nu,\mu} = 0$ and exploiting the residual gauge freedom, the GW can be put in the **transverse traceless (TT-) gauge**:

$$\bar{h}^{0\mu} = \bar{h}^{x\mu} = \bar{h}^\alpha{}_\alpha = 0$$

(note that the latter implies that $\bar{h}_{\mu\nu} = h_{\mu\nu}$). Thus, in the TT-gauge the only non-vanishing components of the metric perturbation are:

$$\begin{aligned} h_{yy} &= -h_{zz} \equiv h_+ \\ h_{yz} &= h_{zy} \equiv h_\times \ : \end{aligned}$$

a GW has two degrees of freedom (two independent polarizations)

$$h^{TT}_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & h_+ & h_\times \\ 0 & 0 & h_\times & -h_+ \end{pmatrix}\,.$$

GW DETECTION

Which is the effect of GWs on a distribution of matter? Can we detect GWs?

Consider one or more free particles at rest in flat spacetime. Suddenly, a GW passes. Each particle follows a geodesic $x^\mu(\tau)$, thus $u^\mu = dx^\mu/d\tau$ satisfies the geodesic equation

$$\frac{du^\alpha}{d\tau} + \Gamma^\alpha_{\mu\nu} u^\mu u^\nu \,.$$

If we consider a single particle, we can always choose a locally inertial frame (LIF) centered on it, where $\Gamma^\alpha_{\mu\nu} = 0$: the particle does not see any gravitational field, and remains at rest.

If we want to measure the gravitational field we have to consider the relative motion of different particles. Let us consider, then, two nearby particles A, B. Let us choose the TT-gauge:

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = (\eta_{\mu\nu} + h^{TT}_{\mu\nu}) dx^\mu dx^\nu \,.$$

At $\tau \leq 0$, $h^{TT}_{\mu\nu} = 0$, $u^\mu_{A,B} = (1,0,0,0)$. At $\tau = 0$,

$$\frac{du^\alpha_{A,B}}{d\tau} = -\Gamma^\alpha_{00} = -\frac{1}{2}\eta^{\alpha\beta}[2h^{TT}_{0\beta,0} - h^{TT}_{00,\beta}] = 0$$

being $h^{TT}_{\mu 0} = 0$, therefore the particles, in this coordinate frame, remain at rest and their space separation

$$\delta x^i = x^i_B - x^i_A$$

is constant. However, this is just an artifact of the coordinate (gauge) choice: the *proper distance* between A and B changes. If, for instance, the GW propagates along x and the particle lies in the y-axis,

$$\Delta l = \int ds = \int_{y_A}^{y_B} \sqrt{g_{yy}} dy \simeq (y_B - y_A + \left[1 + \frac{1}{2} h^{TT}_{yy}\left(t - \frac{x}{c}\right)\right] \,.$$

The appropriate way to describe the motion of two free nearby particles is in term of a *tensor equation*: the **geodesic deviation equations**:

$$\frac{D^2\delta x^\alpha}{d\tau^2} \equiv u^\gamma(u^\beta \delta x^\alpha_{;\beta})_{;\gamma} = R^\alpha{}_{\beta\mu\nu}u^\beta u^\mu \delta x^\nu$$

where δx^α is the separation 4-vector between the nearby geodesics, and u^α is their tangent vector.

Actual measurements by a local observer are performed on a LIF $\{\xi^\alpha\}$ centered on a particle, e.g. A:

$$\xi^i_A = 0 \qquad \xi^i_B = \xi^i \qquad ds^2 = \eta_{\alpha\beta}d\xi^\alpha d\xi^\beta + O(\xi^2)\,.$$

Solving the geodesic deviation equation in this frame, with $\xi^i(t=0) \equiv \xi^i_0$, gives

$$\xi^j(t) = \xi^j_0 + \frac{1}{2}\delta^{ji}h^{TT}_{ik}\xi^k_0\,.$$

For a wave propagating along the x axis,

$$h^{TT}_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & h_+ & h_\times \\ 0 & 0 & h_\times & -h_+ \end{pmatrix}\,.$$

If it is monochromatic, with period $P = 2\pi/\omega$, a circular ring of particles on the $y-z$ plane deforms - for each polarization - into an ellipse after $P/4$; returns circular after $P/2$, becomes an ellipse orthogonal to the previous one after $3P/4$; and returns circular after P. In the polarization $'+'$, the axes of the ellipses are aligned with the y, z axes, while in the polarization $'\times'$ they are inclined of 45^o with respect to them. In the general case, there is a superposition of these polarizations, and then a superposition of these deformations.

GWs can be detected using a Michelson interferometer: a beam of light is separated in two by a beam splitter; the two beams move in orthogonal arms, are reflected by mirrors, join again and are revealed by a detector.

This instrument was used in the XIX in the famous experiment leading to the formulation of special relativity. Modern interferometers are much more sophisticated, with laser light, going back and forth in a Fabry-Perot cavity hundred of times before reaching the detector, which is not a simple screen but is a photodetector. Moreover, more advanced systems isolate the detector from the external noise. But the basic concept is the same

It is worth stressing that the numbers of wavelengths in the arm does not change when the GW passes, since both the arm and the wavelength change. Still, a dephasing is present because there is a *time delay* between the two paths.

Let the arms be in the $y-z$ plane, and let a wave with $'+'$ polarization (for simplicity) propagate in the x direction. Then,

$$ds^2 = -c^2dt^2 + dx^2 + (1+h_+)dy^2 + (1-h_+)dz^2 \,.$$

Let l_0 be the length of the arm, and ω the GW frequency. Moreover, let us assume for simplicity

$$\lambda_{GW} = \frac{2\pi c}{\omega} \gg l_0 \,. \tag{4}$$

Therefore, h_+ is roughly constant through the arm.

By imposing $ds^2 = 0$ for a light ray, we find that the ray along y and the ray along z take, to go back and forth through the arm, a time (respectively)

$$t_{(y)} = \frac{2l_0}{c}\left(1 + \frac{h_+}{c}\right) \qquad t_{(z)} = \frac{2l_0}{c}\left(1 - \frac{h_+}{c}\right)$$

therefore the rays have a time delay

$$\Delta t = t_{(y)} - t_{(z)} = \frac{2l_0}{c}h_+$$

and a shift $c\Delta t = 2l_0h_+$. The same occurs for the $'\times'$ polarization.

In actual interferometers, Eq. (4) is not satisfied. The computation without this simplification gives

$$c\Delta t \simeq 2l_0h_+\frac{\sin(\omega l_0/c)}{\omega l_0/c}$$

which for a given $\omega = 2\pi c/\lambda$ is maximum for $l_0 = \lambda/4$. For instance, in LIGO-Virgo, the arms are 4 km long, but since the light is reflected hundreds of times in the Fabry-Perot cavity, the effective length is $l_0 = 750$ km. Thus, the detectos is best suited for GWs at frequency $\nu \sim 100$ Hz. In the case of the space detector LISA, $l_0 = 2.5 \cdot 10^6$ km, best suited for GWs at frequency $\nu =\sim 10^{-2}$ Hz.

GW GENERATION

Let us consider linearized Einstein's equations with source, Eqs. (**??**). They can be solved in terms of retarded functions:

$$\bar{h}_{\mu\nu}(t,\vec{x}) = \frac{4G}{c^4}\int_V \frac{T_{\mu\nu}\left(t-\frac{|\vec{x}-\vec{x}'|}{c},\vec{x}'\right)}{|\vec{x}-\vec{x}'|}d^3x'\,, \tag{5}$$

where V is the volume of the source, and we assume $T_{\mu\nu}=0$ on its boundary ∂V. It can be shown that this solution automatically satisfied the harmonic gauge condition (3).

Eq. (5) is derived on the assumption of weak field (1). We shall make two further assumptions.

- We shall assume that the observer is far away from the source. Then, if ϵ is the linear dimension of the source and $r=|\vec{x}|$ is the source-observer distance,

$$\epsilon \ll r\,.$$

- We shall assume that the source is much smaller than the wavelength of the emitted GWs:

$$\epsilon \ll \lambda = \frac{c}{\nu} \quad \Rightarrow \quad \epsilon\nu \ll c\,. \tag{6}$$

  Since ν^{-1} is the timescale of source changes, $\epsilon\nu \sim v$ typical velocity on the source, therefore the condition (6) is equivalent to $v \ll c$. For this reason, it is called **slow motion approximation**.

  Under these assumptions, with some simple manipulation (I'll come back later on this), Eq. (5) becomes:

$$\bar{h}_{\mu\nu}(t,\vec{x}) = \frac{4G}{c^4}\frac{1}{r}\int_V T_{\mu\nu}\left(t-\frac{r}{c},\vec{x}'\right)d^3x'\,.$$

Since $T_{\mu\nu} = O(h)$, as I said it can be treated as a field on Minkowski spacetime. Then, it satisfies the flat-space conservation law

$$T^{\mu\nu}{}_{,\nu} = 0\,.$$

By integrating over V,

$$\frac{1}{c}\frac{\partial}{\partial t}\int_V T^{\mu 0} d^3x = -\int_V \frac{\partial T^{\mu k}}{\partial x^k} d^3x = -\int_{\partial V} T^{\mu k} n^k dS = 0$$

because $T^{\mu\nu} = 0$ on ∂V. Here n^k is the unit vector normal to ∂V and dS is the surface elment on ∂V.

Therefore, $\int_V T^{\mu 0} d^3x = const.$ and

$$\bar{h}^{\mu 0}(t, \vec{x}) = \frac{4G}{c^4}\frac{1}{r}\int_V T^{\mu 0}\left(t - \frac{r}{c}, \vec{x}'\right) d^3x' = const.$$

If we are interested only in the oscillating solution (remember that these are solutions of a linear equation and form a vector space), we can consider the solutions in which this constant is zero:

$$\bar{h}^{\mu 0} = 0\,.$$

With a similar derivation, using $T^{\mu\nu}{}_{,\nu} = 0$ it is possible to demonstrate the *tensor virial theorem*:

$$\int_V T^{ij} d^3x = \frac{1}{2c^2}\frac{\partial^2}{\partial t^2}\int_V T^{00} x^i x^j d^3x\,.$$

Since we are in weak field, slow motion approximation, Newtonian physics applies to the source, and $\frac{1}{c^2}T^{00} = \rho$ matter density. Then, if we define the **quadrupole tensor**

$$q^{ij}(t) = \frac{1}{c^2}\int_V T^{00}(t, \vec{x}) x^i x^j d^3x = \int_V \rho(t, \vec{x}) x^i x^j d^3x\,, \tag{7}$$

we have the **quadrupole formula**:

$$\begin{aligned} \bar{h}^{\mu 0} &= 0 \\ \bar{h}^{ij} &= \frac{4G}{c^4 r}\frac{d^2}{dt^2} q^{ij}\left(t - \frac{r}{c}\right)\,. \end{aligned} \tag{8}$$