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Modern Cosm.,

Universe homogeneous and isotropic at large scales.
Described with FRWL metric

$$ds^2 = g_{\mu\nu}\, dx^\mu dx^\nu = -dt^2 + a^2(t)\left(\delta_{ij}\, dx^i dx^j\right)$$

$$\left(= +a^2(t)\left[-d\eta^2 + \delta_{ij}\, dx^i dx^j\right]\right)$$

conformal time

From geodesic equation $\Rightarrow$ $p \propto 1/a$ (momentum)

Dynamics described by Einstein equations:

$$G^{\mu}{}_{\nu} = \frac{8\pi G}{3} T^{\mu}{}_{\nu} \longrightarrow \text{Friedmann equations}$$

$T^{\mu}{}_{\nu} = \mathrm{diag}[-\rho, +P, +P, +P]$ stress-energy tensor of perfect fluid

From the conservation of $T^{\mu}{}_{\nu}$, we get the evolution of energy:

$$\rho \propto \begin{cases} a^{-3}; & \text{MATT.} \\ a^{-4}; & \text{RAD} \\ \text{const.}; & \Lambda \end{cases}$$

Microscopic description is given by DISTRIBUTION FUNCTION

$f = f(p,t)$ Note, no dependence on $\vec{x}, \hat{p}$ due to homog. and isotr.

Relation between macro and micro:

$$\rho = g_s \int \frac{d^3p}{(2\pi)^3} E\, f(p,t) \; ; \quad P = g_s \int \frac{d^3p}{(2\pi)^3} \frac{p^2}{3E} f(p,t)$$

Due to rapid interactions in the early Universe ($\Gamma < H$), particles are in local thermal equilibrium and we can use equilibrium distributions:

$$f_{FD,BE} = \frac{1}{e^{E/T(t)} \pm 1} \Longrightarrow \rho \propto \begin{cases} T^4; & \text{UR} \\ m n; & \text{NR} \end{cases}$$

[NB: $\rho_{tot} \propto g_* T^4$]

From II law of thermod. $\Rightarrow$ entropy is conserved: $s \propto g_{*s} T^3$

Particles decoupling [rule of the thumb: $\Gamma \sim H$] when (non) relativistic, keep equilibrium distr. with

$$T \propto \begin{cases} 1/a & \text{(UR at dec.)} \\ 1/a^2 & \text{(NR at dec.)} \end{cases}$$

The full evolution is described with Boltzmann equations:

$$\frac{d}{dt}[f] = C[f]$$

Liouville operator accounts for metric

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \frac{dx^i}{dt}\frac{\partial}{\partial x^i} + \frac{dp}{dt}\frac{\partial}{\partial p} + \frac{d\hat{p}^i}{dt}\frac{\partial}{\partial \hat{p}^i}$$

In homog. and isotr.

$$\Downarrow$$

$$\frac{df}{dt} = \frac{\partial}{\partial t} - Hp\frac{\partial}{\partial p}$$

$$\frac{dp}{dt} = \frac{dp}{da}\frac{da}{dt} = -p\frac{\dot{a}}{a}$$

and we get again $p \propto 1/a$

Collisional operator accounts for all interactions $1+2 \rightleftarrows 3\ 4$

In equilibrium, $C[f] = 0$

The full evolution of the Universe is given by the coupled system of EE and BE

PERTURBED UNIVERSE

We know the Universe is not fully hom. and isotr. While this description holds for the background evolution, we need to revise it to describe the observations. According to current understanding, inhom. and anisotropies evolved from tiny perturbations about the smooth background. In the most simple model of the early Universe, they originated from quantum fluctuations of the dominant (scalar) field at that time (e.g., we can think of time-shift of $\rho(t+dt)$). These scalar fluctuations are converted in metric perturbations and frozen as they exit the horizon during inflation. The same inflation mechanism also generates tensor perturbations to the metric.

These perturbations are "drawn" from a Gaussian distribution and thus are a possible stochastic realization (i.e., we live in a possible stochastic realization of the Universe).

The evolution of these perturbations as they re-enter the horizon at later times is governed by "deterministic" equations.

Perturbation of species X → $\delta_X(t) \sim T[X]\ \delta_{\text{PRIMORDIAL}}$ → Encodes stochasticity

↑ Transfer function describes "determ." evolution

[Note, we assume ADIABATIC initial conditions, so we only need 1 field to evolve all]

Perturbations are small $\Rightarrow$ Easy to treat in Fourier space, where 1) each Fourier mode evolves independently (as long as we are in linear regime); 2) the system of PDE simplifies to ODE.

Perturbations are Gaussian $\Rightarrow$ All physics properties can be extracted from the variance of the distribution (NB: the average is 0 if we consider $\delta_X = \frac{X-\bar{X}}{\bar{X}}$):

$$\langle \delta_X(\vec{k})\, \delta_X^*(\vec{k}')\rangle = (2\pi)^3\, \underset{\substack{\uparrow \\ \text{Independence} \\ \text{of Fourier modes}}}{\delta_D^{(3)}(\vec{k}-\vec{k}')}\, \underset{\substack{\uparrow \\ \text{Rotational} \\ \text{invariance}}}{P_\delta(|\vec{k}|)}$$

$P_\delta(k)$ is the POWER SPECTRUM. Note, $\mathcal{P}_\delta(k) = k^3 P_\delta(k)$ is the variance in log. bin:

$$\sigma^2 \sim \int d^3k\, P_\delta = \int k^2 dk\, P_\delta = \int d\ln k\, \mathcal{P}_\delta$$

Inflation produces almost scale invariant spectrum $\mathcal{P}$:

$$\mathcal{P}_{SCAL.} = \underset{\substack{\uparrow \\ \text{Amplitude}}}{A_S}\, k^{n_S - 1} \rightarrow \text{Spectral index} \sim 1$$

[Note this is usually computed as the power spectrum of the gauge-invariant curvature perturbation $\mathcal{R}$, which is conserved on super-horizon scales,] at the time of horizon crossing $aH = k$]

As we will see, tensor perturbations have 2 DOF, $h_{+,\times}$, and the total PS is:

$$P_T = P_\times + P_+$$

where each DOF is given by $\cancel{\mathcal{P}_{\times,+}}$ $\langle h_{+,\times}(\vec{k})\, h^*_{+,\times}(\vec{k}')\rangle = (2\pi)^3 \delta_D^{(3)}(\vec{k}-\vec{k}')\, P_{+,\times}(k)$

and $\mathcal{P}_T(k) = \underset{\substack{\downarrow \\ \propto H_{INFLATION}}}{A_T}\, k^{n_T} \rightarrow$ tensor spectral index ~ 0 [Note, we usually use $r \equiv \frac{P_T}{P_R} = \frac{A_T}{A_S}$]

So, back to the δ_X, we can write: $\delta_X(\eta,\vec{k}) = \overbrace{[\delta_X(\eta,\vec{k})/\delta_{in}(\vec{k})]}^{\text{Transfer} \equiv \delta_X(\eta,k)}\, \delta_{in}(\vec{k})$

$$\langle \delta_X(\vec{k},\eta)\, \delta_X^*(\vec{k}',\eta)\rangle = (2\pi)^3 \delta_D^{(3)}(\vec{k}-\vec{k}')\, [\delta_X(k,\eta)]^2\, P(k)$$

So, inflation gives us $P_{S,T}(k)$. Let's see what we need to compute the transfer. Again, the evolution is given by the system of Einstein-Boltzmann equations. We need to derive their perturbed version.

First, we perturb the METRIC $g^{\mu\nu}$ which is now, in principle, a 4×4 full tensor. However, the 16 DOF can be reduced, because 1) $g^{\mu\nu}$ is symmetric 2) we can absorb 4 DOF with the gauge choice. We remain with 6 DOF, that can be identified based on their behavior under rotations into SCALAR (2), VECTOR (2) and TENSOR (2). We neglect vector, because they are not sourced in the early Universe.

The decomposition theorem states that, at linear order, different types of perturbations to the metric are independent from each other. Therefore, we can treat them separately when deriving the corresponding evolution equations.

For SCALAR perturbations, we use the conformal Newtonian gauge, where the metric is:

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \Rightarrow h_{00} = -2\Psi \;;\; h_{0i} = 0 \;;\; h_{ij} = 2\Phi\delta_{ij}$$

For TENSOR perturbations, we use

$$h_{00} = -1 \;;\; h_{0i} = 0 \;;\; \underbrace{h_{ij}}_{\text{Transverse Traceless}} = \begin{pmatrix} h_+ & h_\times & 0 \\ h_\times & -h_+ & 0 \\ 0 & 0 & 0 \end{pmatrix} = \sum_s h_s \underbrace{\varepsilon_s^{ij}}_{\text{Polarization tensor}} \;;\; \varepsilon_s = \varepsilon_{+,\times} / \varepsilon_{R,L}$$

From the metric, we can already compute the perturbed version of the Liouville operator. We need to retain, in principle, all terms

SCALAR

Derive the 4-momentum $P^\mu = (P^0, P^i)$

- $P^0 \longrightarrow g_{\mu\nu}P^\mu P^\nu = -m^2 \Rightarrow -(1+2\Psi)P^{0\,2} + \overset{g_{ij}P^iP^j = p^2}{p^2} = -m^2 \Rightarrow \boxed{P^0 \simeq (1-\Psi)E}$
- $P^i \longrightarrow p^2 = g_{ij}P^iP^j = a^2(\delta_{ij} + 2\Phi\delta_{ij})\hat{p}^i\hat{p}^j \Rightarrow \boxed{P^i = (1-\Phi)\frac{p}{a}\hat{p}^i}$
- $\frac{dx^i}{dt} = \frac{dx^i}{d\lambda}\frac{d\lambda}{dt} = \frac{P^i}{P^0} = \frac{\hat{p}^i}{a}\frac{p}{E}(1-\Phi+\Psi)$
- $\frac{dp^i}{dt} = \frac{dp^i}{d\lambda}\frac{d\lambda}{dt} = \frac{1}{P^0}\frac{d}{d\lambda}\left(a(1+\Phi)P^i\right) = \frac{1}{P^0}\left[a(1+\Phi)\overset{\text{geodesic}}{\frac{dP^i}{d\lambda}} + P^i\left(P^0\frac{d}{dt} + P^k\frac{d}{dx^k}\right)(a(1+\Phi))\right.$

Using $\frac{dp}{dt} = \frac{d}{dt}\sqrt{\delta_{ij}p^i p^j} = \delta_{ij}\frac{p^i}{p}\frac{dp^j}{dt}$

$\Downarrow$

$$\Rightarrow \frac{dp}{dt} = \underbrace{-(H+\dot\Phi)p}_{\substack{\text{Hubble}\\ \text{friction}\\ \text{in perturbed}\\ \text{Universe}}} \underbrace{- \frac{E}{a}\hat p^i \frac{\partial \Psi}{\partial x^i}}_{\substack{\cancel{\text{Cosmological}}\\ \text{Gravitational redshift}}}$$

$\left(\hat p^i \frac{\partial\Psi}{\partial x^i} \lessgtr 0 \Rightarrow \frac{\text{into well}}{\text{out of well}} \Rightarrow \frac{\text{gain}}{\text{loose}} \text{ energy}\right)$

[Note that $g_{ij} = a^2\delta_{ij}(1+2\Phi) = \delta_{ij}\bar a^2$; $\bar a = a\sqrt{1+2\Phi} \simeq a(1+\Phi)$

So that $\frac{\dot{\bar a}}{\bar a} = H + \dot\Phi$]

→ Rivedere indici

Using $\frac{d\hat p^i}{dt} = \frac{d}{dt}\left(\frac{p^i}{p}\right) = \frac{E}{ap}\left(\delta^{ik} - \hat p^i \hat p^k\right)\left(\frac{p^2}{E^2}\Phi - \Psi\right)_{,k}$

$\Downarrow$

Change in direction is due to interveening gradients of potentials.
Note this is a first-order effect only (will be useful later)

We are left with:

$$\boxed{\frac{d}{dt} = \frac{\partial}{\partial t} + \frac{\hat p^i}{a}\frac{p}{E}(1-\Phi+\Psi)\frac{\partial}{\partial x^i} - \left[(H+\dot\Phi)p - \frac{E}{a}\hat p^i \frac{\partial\Psi}{\partial x^i}\right]\frac{\partial}{\partial p} + \frac{d\hat p^i}{dt}\frac{\partial}{\partial \hat p^i}}$$

TENSOR

Derive 4-momentum

- $g_{\mu\nu}P^\mu P^\nu = -m^2 = -P^{0\,2} + p^2 \Rightarrow \boxed{P^0 = E}$
- Let's define $P^i = C\hat p^i \Rightarrow g_{ij}P^iP^j = p^2 = a^2(\delta_{ij}+h_{ij})C^2\hat p^i \hat p^j =$

$$= a^2C^2(1+h_{ij}\hat p^i\hat p^j) \Rightarrow C \simeq \frac{p}{a}\left(1 + \frac{1}{2}h_{ij}\hat p^i \hat p^j\right.$$

$$\boxed{P^i = \frac{p}{a}\left(1+\frac{1}{2}h_{jk}\hat p^j \hat p^k\right)\hat p^i}$$

$\Rightarrow \frac{dx^i}{dt} = \frac{P^i}{P^0} = \frac{\hat p^i}{a}\frac{p}{E}\left(1+\frac{1}{2}h_{jk}\hat p^j\hat p^k\right)$

With same procedure, we get $\frac{d\hat p^i}{dt}$:

$\frac{dp}{dt} = -p\left(H + \frac{1}{2}\dot h_{jk}\hat p^j \hat p^k\right) \Rightarrow$ Only Hubble friction

$\frac{d\hat p^i}{dt} = 0 \quad \Rightarrow$ No change in $\hat p^i$ due to tensor perturbations

Putting things together:

$$\boxed{\frac{d}{dt} = \frac{\partial}{\partial t} + \frac{\hat{p}^i}{a}\frac{p}{E}\left(1+\frac{1}{2}h_{kl}\hat{p}^k\hat{p}^l\right)\frac{\partial}{\partial x^i} - p\left[H + \frac{1}{2}\dot{h}_{kl}\hat{p}^k\hat{p}^l\right]\frac{\partial}{\partial p}}$$

In a perturbed Universe, the distribution function depends on $t, \vec{x}, p, \hat{p}^i$:

$$f \equiv f(t, \vec{x}, p, \hat{p}^i) \xrightarrow{\text{linearize}} f \simeq f_0 + f_1 \;;\; f_0 = f_{BE,FD}$$

The components of $T^{\mu\nu}$ are also perturbed, and off-diagonal terms appear:

$$T^0{}_0 = g_s \int \frac{d^3p}{(2\pi)^3} E\,(f_0+f_1) = \rho + \delta\rho$$

$$T^0{}_i = g_s \int \frac{d^3p}{(2\pi)^3} p_i\,(f_0+f_1) =$$

$$T^i{}_l = g_s \int \frac{d^3p}{(2\pi)^3} \frac{p^2 \hat{p}^i \hat{p}_l}{E}(f_0+f_1) = (P+\delta P)\,\delta^i_l + \Sigma^i{}_l$$

Anisotropic shear (off-diag)
(Note: if $E \sim m \gg p$, $T^i{}_l \sim 0$)
NR particles have vanishing T^i_l

From metric + stress-energy tensor, we can write the perturbed Einstein equations (not derived here)

SCALAR: 1) Poisson + II) $\Phi = -\Psi$ unless anisotropic stress $\neq 0$ ⇒ CONSTRAINT EQUATIONS (no ∂_t^2)
TENSOR: $h'' + 2\frac{a'}{a}h' + k^2 h = 0$

From metric + distr. funct., we can write the perturbed Boltzmann eq.
Let's move to Fourier ($\frac{\partial}{\partial x^i} \to ik\hat{k}^i$) and to conformal time ($t \to a\eta$) and to comoving $\tilde{p}$ ($\tilde{p} = ap \to \frac{d\tilde{p}}{dt} = a(pH + \frac{dp}{dt})$):

$$\frac{\partial f_1}{\partial \eta} + i\vec{k}\cdot\hat{p}\,\frac{\tilde{p}}{E}\left(1 \cancel{-\Phi+\Psi}\right)f_1 - \left[\frac{\partial\Phi}{\partial\eta} + i\vec{k}\cdot\hat{p}\,\Psi\right]\tilde{p}\,\frac{\partial f_0}{\partial\tilde{p}} = C[f_0+f_1] \qquad \text{SCALAR}$$

$$\frac{\partial f_1}{\partial \eta} + i\vec{k}\cdot\hat{p}\,\frac{\tilde{p}}{E}f_1 - \frac{1}{2}\underbrace{\frac{\partial h_{kl}}{\partial\eta}\hat{p}^k\hat{p}^l}_{\sum_s \frac{\partial h_s}{\partial\eta}\varepsilon_{s,kl}\hat{p}^k\hat{p}^l}\,\tilde{p}\frac{\partial f_0}{\partial\tilde{p}} = C[f_0+f_1] \qquad \text{TENSOR}$$

$$\sum_s \frac{\partial h_s}{\partial \eta} \varepsilon_{s,ij} \hat{p}^i \hat{p}^j \Rightarrow$$ Choose a spherical coord. system, where $\varepsilon_s = \varepsilon_{+,\times}$, $k = (0,0,1)$

$$\hat{p}^i = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)$$

$$\varepsilon_+ = \mathrm{diag}[1,-1,0] \; ; \; \varepsilon_\times = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow$$ Only terms that survive are:

$$\varepsilon_{+11}(\hat{p}^1)^2 + \varepsilon_{+22}(\hat{p}^2)^2$$
$$\varepsilon_{\times 12}\hat{p}^1\hat{p}^2 + \varepsilon_{\times 21}\hat{p}^2\hat{p}^1$$

$$\frac{\partial h_+}{\partial \eta}(\sin^2\theta\cos^2\varphi - \sin^2\theta\sin^2\varphi) + \frac{\partial h_\times}{\partial \eta}(2\sin^2\theta\cos\varphi\sin\varphi) =$$

$$= \sin^2\theta\left(\frac{\partial h_+}{\partial \eta}\underbrace{\cos 2\varphi}_{\frac{e^{2i\varphi}+e^{-2i\varphi}}{2}} + \frac{\partial h_\times}{\partial \eta}\underbrace{\sin 2\varphi}_{\frac{e^{2i\varphi}-e^{-2i\varphi}}{2i}}\right) = (1-\mu^2)\sum_s \frac{\partial h_s}{\partial \eta} e^{\pm 2i\varphi}$$

let's define $\cos\theta = \hat{k}\cdot\hat{p} = \mu$

Instead of evolving the full Boltzmann, it is useful to expand the perturbations in harmonics (after integrating opportunely over momentum p)*:

$$\cancel{X \neq \int d^3p} \qquad \underbrace{X}_{\text{Integrated distr. function over momentum}} = \sum_\ell (-i)^\ell (2\ell+1) \overbrace{X_\ell}^{\ell\text{-moment}} \underbrace{P_\ell(\mu)}_{\text{Legendre polynomial used to expand the angular dependence of } X}$$

With this normalization, $X = e^{i(\hat{k}\cdot\hat{x})}$ expands with coefficients $X_\ell = j_\ell(kx)$ → BESSEL

We obtain a system of ODE. For NR species, this is simply given by 2 equations: one for the density, one for velocity (remember stress was vanishing). For REL. species (and massive neutrinos) we have an infinite hierarchy of equations:

$\ell = 0 \Rightarrow$ density δ
$\ell = 1 \Rightarrow$ velocity θ
$\ell = 2 \Rightarrow$ anisotropic shear
$\vdots$

We will see that, for photons, we only have to evolve up to $\ell = 4$. $\ell > 4$ moments can be obtained from the solutions of $\ell \leq 4$.

* For collisionless species, easier to derive $\dot{\delta}, \dot{\theta}$ from $T^{\mu\nu}{}_{;\nu} = 0$.