Gravitational waves from compact objects

Lecture II

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GW GENERATION

As we have seen, the solution of linearized Einstein's equations with source is

$$\bar{h}_{\mu\nu}(t,\vec{x}) = \frac{4G}{c^4}\int_V \frac{T_{\mu\nu}\left(t-\frac{|\vec{x}-\vec{x}'|}{c},\vec{x}'\right)}{|\vec{x}-\vec{x}'|}d^3x'\,. \tag{1}$$

If, besides the assumption of weak field ($g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ with $|h_{\mu\nu}| \ll 1$), we assume that the source is far away from the observer,

$$|\vec{x}| = r \gg \epsilon$$

(ϵ linear dimension of the source), and we assume the **slow motion approximation**

$$\epsilon \ll \lambda \quad \text{which is equivalent to} \quad v \ll c\,. \tag{2}$$

where λ is the GW wavelength and v is the typical velocity on the source, Eq. (1) reduces to the **quadrupole formula**:

$$\begin{aligned} \bar{h}^{\mu 0} &= 0 \\ \bar{h}^{ij} &= \frac{4G}{c^4 r}\ddot{q}^{ij}\left(t-\frac{r}{c}\right)\,. \end{aligned} \tag{3}$$

where the overdot denotes derivative with respect of time, and we have defined the **quadrupole tensor**

$$q^{ij}(t) = \frac{1}{c^2}\int_V T^{00}(t,\vec{x})x^i x^j d^3x = \int_V \rho(t,\vec{x})x^i x^j d^3x\,, \tag{4}$$

and we remind that on the source, due to the weak field and slow motion approximation, Newtonian mechanics holds, and thus the matter density is $\rho = T^{00}/c^2$.

Eq. (3) is not in the TT-gauge. If the observer detect a wave in the direction $\hat{n} = \frac{\vec{x}}{r}$, which in spherical coordinates has components $n^i = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$, we have that in the TT-gauge $n_i h^{TT\,ij} = 0$ and $h^{TT\,\mu}_{\mu} = 0$. These conditions can be imposed by applying to the metric perturbation the **TT-projector**

$$\mathcal{P}_{ijkl} \equiv P_{ik}P_{jl} - \frac{1}{2}P_{ij}P_{kl} \quad \text{where} \quad P_{ij} = \delta_{ij} - n_i n_j \, .$$

Since $\delta^{ij}\mathcal{P}_{ijkl} = n^i\mathcal{P}_{ijkl} = \cdots = 0$ and $\mathcal{P}\mathcal{P} = \mathcal{P}$, it projects rank two tensor in the TT subspace of the tensor space. Therefore,

$$h^{TT}_{ij}(t, r, \theta, \phi) = \mathcal{P}_{ijkl}(\theta, \phi)h_{ij}(t, r) = \frac{2G}{c^4 r}\mathcal{P}_{ijkl}(\theta, \phi)\frac{d^2}{dt^2}Q_{kl}\left(t - \frac{r}{c}\right) \, .$$

Note that we have defined the *reduced quadrupole moment* $Q_{ij} = q_{ij} - \frac{1}{3}\delta_{ij}q^k{}_k$, which is traceless, for later use (in the equation above the trace of q is irrelevant since it is set to zero by the projector).

This equation tells us that the GWs far away from a weak field, slow motion source depend on the time derivatives of its *quadrupole moment*. This implies that, for instance, a stationary source like an axially symmetric, rotating star, does not emit GWs, since $\rho = const$. Spherically symmetric source do not emit GWs even if they are dynamical (thanks to the Birkhoff theorem). In general, in order to emit GWs, we need *asymmetry*; and, due to the time derivatives, we need rapidly evolving sources to have strong GWs.

THE ENERGY CARRIED BY GWS

In GR, it is not possible to define a tensor quantity describing the local density of energy and momentum because at any event P it is possible to choose a LIF, in which the metric is locally Minkwoskian and the gravitational field in P vanishes. However, it is possible to define a quantity which leads to a definition of the energy carried by GWs and, under certain conditions, does not depend on the coordinate sytem.

We remind that the stress-energy tensor satisfies a conservation law in flat spacetime $T^{\mu\nu}{}_{,\nu} = 0$:

$$\frac{1}{c}\frac{\partial}{\partial t}\int_V T^{\mu 0} d^3x = -\int_V \frac{\partial T^{\mu i}}{\partial x^i} d^3x = -\int_{\partial V} T^{\mu i} n^i dS$$

so $\int_V T^{\mu 0} d^3x$, the energy and momentum in V, are conserved quantities, their change being equal to the flux of the corresponding currents outside the voundary ∂V. In curved spacetime, instead, the stress-energy tensor satisfies $T^{\mu\nu}{}_{;\nu} = 0$, which is not a conservation law. This is related to the fact that $T^{\mu\nu}$ describes the energy and momentum of non-gravitational fields, which are not conserved, since they do not take into account those associated to the gravitational field.

We define the **Landau-Lifshitz stress-energy pseudo-tensor**

$$t^{\mu\nu} = \frac{c^4}{8\pi G}\left(\Gamma^{\delta}_{\alpha\beta}\Gamma^{\sigma}_{\delta\sigma} g^{\mu\alpha} g^{\nu\beta} + \cdots 15 \text{ terms with the same structure}\right) . \quad (5)$$

In Minkowski space, or in a LIF, the Christoffel symbols vanish, and thus $t^{\mu\nu} = 0$. Remarkably, this quantity satisfies, together with the stress-energy tensor, a conservation law:

$$\frac{\partial}{\partial x^\nu}[(-g)(t^{\mu\nu} + T^{\mu\nu})] = 0 . \quad (6)$$

Eq. (6) is just a reformulation of Einstein's equations: it can be found by replacing the definition of $t^{\mu\nu}$ in terms of Christoffel's symbols, and replacing $T^{\mu\nu}$ with Einstein's equations.

Therefore, *in an asymptotically flat spacetime*, if we consider a large volume V such that ∂V is in the region where $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ with $|h_{\mu\nu}| \ll 1$, if we define

$$P^\mu = \int_V (-g)(t^{\mu 0} + T^{\mu 0}) d^3x \tag{7}$$

Eq. (6) gives

$$\frac{\partial}{\partial x^0} P^\mu = -\int_{\partial V} (-g)(T^{\mu i} + t^{\mu i}) n^i dS$$

and if $T^{\mu i}$ and $t^{\mu i}$ are negligible on ∂V, P^μ are constant quantities.

These quantities can be interpreted as the *global four-momentum* in V; in them, $T^{\mu 0}$ gives the contribution of non-gravitational fields, while $t^{\mu 0}$ gives the contribution of the gravitational field. As $V \to \infty$, they are the energy and momentum of the entire spacetime.

Remarkably, while $t^{\mu\nu}$ is not a tensor (it transforms as a tensor only for a subset of the general coordinate transformations, the linear transformations), and thus can not give the local energy and momentum densities of the gravitational field, in can be shown that any coordinate transformation reduces to a Lorentz transformation on ∂V, and that the integrated quantity P^μ transform as a Lorentz 4-vector for such transformation (plus higher-order terms in h). Therefore, the global energy and momentum of spacetime, P^μ, which include the contribution of the gravitational field, are well defined.

In addition, $t^{\mu\nu}$ allows to define quantities which are "local" on an appropriate scale. Let us consider a perturbed spacetime

$$g_{\mu\nu} = g^{(0)}_{\mu\nu} + h_{\mu\nu}$$

and let us call λ the characteristic length of the perturbation (in the case of the GW, λ can be the GW wavelength); let us call L the characteristic length og the background spacetime. Let assume that

$$\lambda \ll L\,.$$

We define the **Brill-Hartle average** $< \cdots >$ as the average over several λ. Then, it can be shown that the Brill-Hartle average of the LL pseudotensor, $< t^{\mu\nu} >$, transforms as a tensor for coordinate transformations $O(h)$. In this case, $< t^{\mu\nu} >$ describes the energy and momentum density (in a scale much larger than λ) of the perturbation.

Let us now consider a GW generated by a source and observed in P far away from the source. I P, it appears like a plane wave. If we choose a frame $(Oxyz)$ centered on the source such that the $x-axis$ is aligned with the propagation of that wave, given the cooresponding spacetime coordinate $\{x^\alpha\} = (ct, x, y, z)$, then the wave in P, in the TT gauge, has the form

$$g_{\mu\nu} = \eta_{\mu\nu} + h^{TT}_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 + h_+(t - x/c) & h_\times(t - x/c) \\ 0 & 0 & h_\times(t - x/c) & 1 - h_+(t - x/c) \end{pmatrix} .$$

The **energy flux** carried by the wave (which moves in direction x), i.e. the energy crossing per unit time a surface orthogonal to x, per unit surface, is

$$\frac{dE_{GW}}{dtdS} = c < t^{0x} > .$$

If we replace Eq. (5) we have an expression bilinear in the Christoffel symbols, which are linear in the first derivatives of h_+ and $h_\times$. Note also that being the metric perturbation function of $t - x/c$,

$$\frac{\partial}{\partial x^0} h_+ = \frac{1}{c}\dot{h}_+ , \qquad \frac{\partial}{\partial x^1} h_+ = -\frac{1}{c}\dot{h}_+$$

and the same applies to $h_\times$. Therefore, $t^{\mu\nu}$ i bilinear in $\dot{h}_+$ and $\dot{h}_\times$. The explicit computation gives

$$t^{0x} = \frac{c^2}{16\pi G}[(\dot{h}_+)^2 + (\dot{h}_\times)^2]$$

and therefore

$$\frac{dE_{GW}}{dtdS} = \frac{c^3}{16\Pi G} < [(\dot{h}_+)^2 + (\dot{h}_\times)^2] > .$$

For a generic choice of the frame, this expression can be written as:

$$\frac{dE_{GW}}{dtdS} = \frac{c^3}{32\pi G} < \sum_{jk} (\dot{h}^{TT}_{ij})^2) >$$

and by replacing the quadrupole formula,

$$\frac{dE_{GW}}{dtdS} = \frac{G}{8\pi c^5 r^2} < \sum_{jk} (\dddot{Q}_{ij}^{TT})^2) > \ .$$

Finally, let us compute the **GW luminosity**, i.e. the energy emitted per time unit by the source in GWs (in all directions):

$$L_{GW} = \frac{dE_{GW}}{dt} = \int \frac{dE_{GW}}{dtdS} r^2 d\Omega = \frac{G}{2c^5}\frac{1}{4\pi}\int d\Omega < \sum_{ij} (\mathcal{P}_{ijkl}(\theta,\phi) Q_{kl}(t-r/c))^2 > \ .$$

By replacing $\mathcal{P}_{ijkl} = P_{ik}P_{jl} - \frac{1}{2}P_{ij}P_{kl}$ and $P_{ij} = \delta_{ij} - n^i n^j$ with $n^i = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta$ and performing the integrals, one finds that

$$L_{GW}(t,r) = \frac{G}{5c^2} < \dddot{Q}_{ij}(t-r/c)\dddot{Q}_{ij}(t-r/c) > \ . \tag{8}$$

COMPACT BINARIES

Let us consider a **binary system** composed of two **compact objects**, with orbital separation l_0.

A compact object is a body whose **compactness** $C = \frac{GM}{c^2 R}$ (M, R are its mass and radius, and C is a dimensionless quantity) is not negligible. In the case of a BH, as "radius" we take the horizon radius. For a Schwarzschild BH, $r_h = 2GM/c^2$, therefore $C = 0.5$; for a rotating BK C is even larger. To our knowledge, the only compact objects in the Universe are BHs and NS (the latter having typically $C \sim 0.2$. In any case, C is always smaller than 1.

Note that in weak field approximation, near a body of mass M $g_{00} =\simeq -1 + \frac{2GM}{c^2 r}$, thus $h_{00} = \frac{2GM}{c^2 r}$. So near the surface of a compact object, $r \sim R$ and h_{00} is not negligible: the weak field approximation breaks down. However if we assume

$$l_0 \gg R$$

then each body feels a gravitational field with $h_{00} \sim \frac{2GM}{c^2 l_0} \ll \frac{2GM}{c^2 R} < 1$: *the weak field approximation is satisfied as long as* $l_0 \gg R$.

As we shall see, the orbital velocity is of the order $v \sim \sqrt{GM l_0}$, therefore

$$\frac{v}{c} \sim \sqrt{\frac{R}{l_0}}\sqrt{\frac{GM}{c^2 R}} < \sqrt{\frac{R}{l_0}} \ll 1 \quad \Leftrightarrow \quad l_0 \gg R\,.$$

Thus, is $l_0 \gg R$ both weak field and slow motion conditions are satisfied. In this case, the system is well described by Newtonian mechanics (at least, in the timescale of the orbital motion - say, the orbital period P), and the quadrupole formula is accurate.

In a timescale $\gg P$, the effect of GW emission piles up, leading to a decrease of the orbital energy. As we shall see, this determines a *decrease of both the orbital separation* l_0 *and of the orbital period* P. This phenomenom is called **inspiral**.

At a certain point, l_0 becomes comparable to R. In this stage of *late inspiral*, the quadrupole formula only gives a first approximation of the emitted GWs. More advaned, semi-analytical techniques such as the so-called **post-Newtonian (PN) expansion**, are needed to accurately model the waveform.

Then, the two body coalesce into a single object; in this stage, called **merger**, the quadrupole formula does not even give an approximate description of the phenomenom, and even PN approaches do not work: in order to model this stage and the resulting waveform we need to solve numerically Einstein's equations *without any approximation*. This is done with **numerical relativity**, in which fully non-linear Einstein's equations are solve with parallel computing.

Finally, there is the **ringdown**, in which the final object - typically a BH - oscillates in its proper oscillation frequency, emitting GWs and rapidly becomeing a stationary BH, described by the Kerr metric. This stage is decribed using perturbative approaches.