Zaldarriaga & Seljak, PRD 55 1997 (astroph 9609170)
Baskaran, Grishchuk, Polnarev, PRD 74 2006 {
Chandrasekhar, Radiative Transfer,

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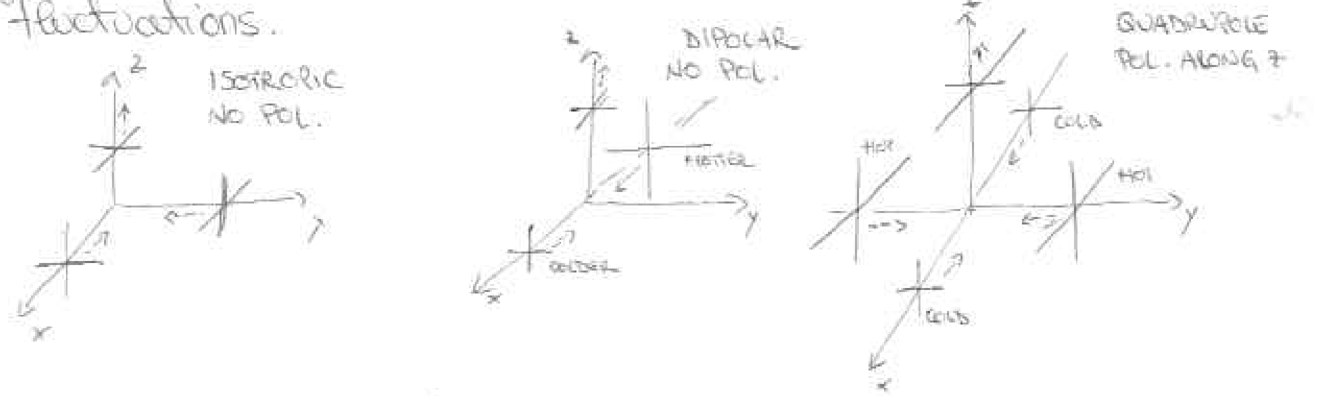
CMB

We observe fluctuations (anisotropies) of the CMB field both in temperature (T, total intensity) and (linear) polarization. T anisotropies are a direct consequence of metric perturbations. Pol. anisotropies are instead sourced not directly by metric, but by the quadrupolar T pattern thanks to Thomson scattering off free electrons. As for metric perturbations, we have both scalar and tensor CMB anisotropies.
Note that – as we will see later – if photons and electrons were always tightly coupled, we would 1) only have a monopole ($\ell=0$) and dipole ($\ell=1$) T fluctuations; 2) not have P fluctuations.
As the Universe expanded, photons start to decouple from electrons and "free-stream", generating (~transferring power to) higher moments $\ell>1$. A quadrupolar ($\ell=2$) T pattern is generated which makes it possible to generate linear polarization. Note that, due to the proximity of the γ-e decoupling and H recombination epochs, many free e^- start to recombinate by the time P polar. is sourced. This makes the magnitude of P fluctuations smaller (~1 order of magnitude) than T fluctuations.



Polarized radiation can be described with Stokes parameters:

$T = \langle |E_x|^2 \rangle + \langle |E_y|^2 \rangle$ → total intensity
$Q = \langle |E_x|^2 \rangle - \langle |E_y|^2 \rangle$ → intensity ($\propto$ n° of γ) along x-y
$U = 2\,\mathrm{Re}\langle E_x E_y^* \rangle$ → intensity along ~~[illegible]~~ 45°-direct.
$V = 2\,\mathrm{Im}\langle E_x E_y^* \rangle$ → circular pol.

Thomson scattering does not generate V, so we will assume $V=0$.

Stokes params are components of the polarization tensor:

$$P_{ab} = \frac{1}{2}\begin{pmatrix} T+Q & -(U-iV) \\ -(U+iV) & T-Q \end{pmatrix} \overset{V=0}{\Rightarrow} \begin{array}{l} T = \mathrm{Tr}[P_{ab}] \\ Q = (P_{11} - P_{22}) \\ U = -2P_{12} \end{array}$$

Note: we normalize P_{ab} such that $P_{ab} \propto \frac{\Delta I}{I}$, i.e. fractional fluctuations of the field [e.g., Θ in Dodelson]

T (and V) is a SCALAR quantity. Clearly, Q and U are not, since their definitions depend on the choice of a ref. system. Q and U transform as:

$$\begin{cases} Q' = Q\cos 2\varphi + U \sin 2\varphi \\ U' = -Q\sin 2\varphi + U\cos 2\varphi \end{cases}$$

From this expression, we can build 2 quantities with a defined SPIN.
Why important? $\rightarrow$ A function ${}_sf(\theta,\varphi)$ defined on the sphere has s-spin if, under rotation of a ref. system tangential to the sphere [] by angle α, transforms as: ${}_sf'(\theta,\varphi) = e^{-is\alpha}\,{}_sf(\theta,\varphi)$.

Once we can define a definite spin, we can expand the angular dependence of ${}_sf$ over an appropriate set of spin-weighted spherical harmonics ${}_sY(\theta,\varphi)_{\ell m}$:

$$\int_0^{2\pi} d\varphi \int_{-1}^{1} \underbrace{\sin\theta\, d\theta}_{d\cos\theta \,\equiv\, d\mu}\; {}_sY^*_{\ell' m'}\, {}_sY_{\ell m} = \delta_{\ell\ell'}\delta_{mm'}$$

$$\sum_{\ell m} {}_sY^*_{\ell m}(\theta,\varphi)\, {}_sY_{\ell m}(\theta',\varphi') = \delta(\varphi-\varphi')\,\delta(\theta-\theta')$$

$\Rightarrow$ T is a spin-0 quantity: $\boxed{T = \sum_{\ell,m} a^T_{\ell m}\, Y_{\ell m}(\theta,\varphi)}$

$\Rightarrow$ $(Q\pm iU)$ are spin-2 quantities: $(Q\pm iU)' = e^{\mp 2\alpha i}(Q\pm iU)$

$\boxed{Q+iU = \sum_{\ell,m} {}_2a_{2,\ell m}\; {}_2Y_{\ell m}(\theta,\varphi)}$ $\boxed{Q-iU = \sum_{\ell,m} {}_2a_{2,\ell m}\; {}_{-2}Y_{\ell m}(\theta,\varphi)}$

Due to the dependence on the ref. frame, Q and U are not easy to handle. Fortunately, we can construct 2 scalar quantities by acting on $(Q\pm iU)$. These 2 scalars are 2 additional (to T, V) invariants of P_{ab}.

The 2 scalars can be constructed by exploiting a property of spin-s functions, i.e., the existence of spin-raising ($\eth$) and spin-lowering ($\bar{\eth}$) operators such that:

$$\underbrace{(\eth\, {}_sf)'}_{{}_sf'_+} = e^{-i(s+1)\alpha}\, \eth\, {}_sf \qquad \text{and} \qquad (\bar{\eth}\, {}_sf)' = e^{-i(s-1)\alpha} (\bar{\eth}\, {}_sf)$$

$$= e^{-i(s+1)\alpha}\, {}_sf_+ \qquad\qquad {}_sf'_- = e^{-i(s-1)\alpha}\, {}_sf_-$$

So, acting twice on $Q \pm iU$, we can build $s=0$ functions:

$$\bar{\eth}^2 (Q+iU) = \sum_{\ell m} \left[\frac{(\ell+2)!}{(\ell-2)!}\right]^{1/2} a_{2,\ell m}\, Y_{\ell m}(\theta,\varphi)$$

$$\eth^2 (Q-iU) = \sum_{\ell m} \left[\frac{(\ell+2)!}{(\ell-2)!}\right]^{1/2} a_{-2,\ell m}\, Y_{\ell m}(\theta,\varphi)$$

[Note, $\eth\, {}_sf = -\sin^s\theta \left[\frac{\partial}{\partial\theta} + i\frac{1}{\cos\theta}\frac{\partial}{\partial\varphi}\right]\sin^{-s}\theta\, {}_sf$, from which we can

$\bar{\eth}\, {}_sf = -\sin^{-s}\theta \left[\frac{\partial}{\partial\theta} - i\frac{1}{\cos\theta}\frac{\partial}{\partial\varphi}\right]\sin^{s}\theta\, {}_sf$ compute $\eth^2_{-2}f$ $\bar{\eth}^2_2 f$:

$$\bar{\eth}^2_{+2} f = \left(-\frac{\partial}{\partial\mu} + \frac{m}{1-\mu^2}\right)^2 \left((1-\mu^2)\, {}_2f(\mu,\varphi)\right)$$

$$\eth^2_{-2} f = \left(-\frac{\partial}{\partial\mu} - \frac{m}{1-\mu^2}\right)^2 \left[(1-\mu^2)\, {}_{-2}f(\mu,\varphi)\right]$$

[$\partial\mu = \partial\cos\theta = -\sin\theta\,\partial\theta$

$\partial/\partial\varphi$ takes down im from $e^{im\varphi}$] usual φ-depend.

Note: if ${}_2f \equiv {}_2f(\mu)$ only, then $\bar{\eth}^2 = \eth^2$]

When we apply those to ${}_sY_{\ell m}$, we convert ${}_{\pm 2}Y_{\ell m}$ to $Y_{\ell m}$ and take out the normalization factor with !.

It is more common to define new quantities which are combinations of $\bar{\eth}^2(Q \pm iU)$ and $\eth^2(Q-iU)$:

$$E(\theta,\varphi) = \sum_{\ell,m} \left[\frac{(\ell+2)!}{(\ell-2)!}\right]^{1/2} a_{E,\ell m}\, Y_{\ell m}(\theta,\varphi) \;; \quad a^E_{\ell m} = -\frac{(a_2 + a_{-2})}{2}$$

$$B(\theta,\varphi) = \sum_{\ell,m} \left[\frac{(\ell+2)!}{(\ell-2)!}\right]^{1/2} a_{B,\ell m}\, Y_{\ell m}(\theta,\varphi) \;; \quad a^B_{\ell m} = i\frac{(a_2 - a_{-2})}{2}$$

Note E (as T) is a pure scalar, while B (as V) is a pseudo-scalar. This ~~is~~ has important consequences: since in the standard cosmo model we do not expect parity-violating sources, we also ~~do not~~ expect vanishing TB and EB correlations.

Why we define E and B in analogy to electric and magnetic fields?
We can decompose P_{ab} into 2 parts: a trace and a symmetric trace-less.
The trace gives T and is decomposed in $Y_{\ell m}$. The traceless part, as any generic symm. and traceless tensor in 2d, can be decomposed similarly to the vector decomposition into a gradient (curl-free) of a scalar field and a curl (divergenceless) of a vector field.
We identify the gradient with E and the curl with B.

Now, while we will evolve T, E, B (or, T, $(Q \pm iU) \to E, B$) with Boltzmann eqs., we are interested in their statistics encoded in the power spectrum (or 2-point correlation function; remember inflation spectra):

$$\langle X(\theta,\varphi)\, X^*(\theta',\varphi')\rangle = \sum_{\ell m}\sum_{\ell' m'} \langle a^X_{\ell m}\, a^{X*}_{\ell' m'}\rangle\, Y_{\ell m}(\theta,\varphi)\, Y^*_{\ell' m'}(\theta',\varphi') \;;\; X = T, E, B$$

Using 1) $a_{\ell m}$ are independent random variable, so their covariance is diagonal $\Rightarrow \delta_{\ell\ell'}$; 2) the fields are rotationally invariant $\Rightarrow \delta_{mm'}$, we have:

$$\boxed{\langle a^X_{\ell m}\, a^{X*}_{\ell' m'}\rangle = \underbrace{C_\ell}_{\text{Power spectrum}}\, \delta_{\ell\ell'}\, \delta_{mm'}}$$

[Note: $C_\ell = \frac{1}{2\ell+1} \sum_{m=-\ell}^{\ell} \langle a_{\ell m}\, a^*_{\ell m}\rangle$]

Substituting and using that $P_\ell(\mu) = \frac{4\pi}{2\ell+1} \sum_{m=-\ell}^{\ell} Y_{\ell m}(\theta,\varphi)\, Y^*_{\ell m}(\theta',\varphi')$, we have

$$\langle X(\theta,\varphi)\, X(\theta',\varphi')\rangle = \sum_\ell \frac{2\ell+1}{4\pi}\, C_\ell\, P_\ell(\mu)$$

we observe this

We use this to predict observables

We will now derive the set of Boltzmann eqs. for photons. We follow a classical approach (as opposed to the quantum approach, see e.g., Kosowsky, 1994) which adopts the equation of radiative transfer (see e.g. Chandrasekhar).

$$\underbrace{\frac{dI}{dt}}_{\text{Variation of the intensity along the direction of propagation}} = \underbrace{-\cancel{n_e\sigma_T} I}_{\text{Loss due to scattering off direction of propagation}} + \cancel{\tfrac{\tau}{4\pi}} \overset{\text{Source}}{S} = +\overset{\text{Differential Optical depth}}{\dot{\tau}} I - \underbrace{\frac{\dot{\tau}}{4\pi}\int d\Omega' \overset{\text{Scattering matrix}}{P(\Omega,\Omega')} I(\Omega')}_{\text{In our case, the source is the intensity scattered from other directions and channelled into the direction of propagation}}$$

$$\tau = \int^{t} n_e \sigma_T \, dt \;\rightarrow\; \dot{\tau} = -n_e\sigma_T$$

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$$I \longrightarrow \vec{n} = \begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix} = \begin{pmatrix} T+Q \\ T-U \\ -2U \end{pmatrix}$$

Components of P_{ab}

$$\vec{n} = \vec{n}(\eta, \vec{x}, \tilde{p}, \hat{p}) = \simeq \underbrace{\vec{n}_0}_{\text{Unperturbed}} + \underbrace{\vec{n}_1}_{\text{Perturbed}} = n_0(\eta, \tilde{p}) \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \vec{n}_1(\eta, \tilde{p}, \hat{p}, \vec{x})$$

In the n_0 case, we only have T

Before putting $\vec{n}$ in the Boltzmann eqs., let's see what $P(\Omega', \Omega)$ is. $P(\Omega', \Omega)$ is telling how $I(\Omega')$ is scattered and channelled ~~through~~ Ω. Note that we work in a spherical coordinate system, with $\hat{p}$ along the radial direction and θ, φ identifying a plane tangential to the sphere. However, we know how to relate the components of $\vec{n}$ before and after the scattering IN THE SCATTERING PLANE (SP):

$$\vec{n}_{SP} \propto \overset{\text{Phase matrix}}{R(\Theta)}\, \vec{n}'_{SP}$$

Θ → scattering angle (between zz', yy' in the fig.)

$$R(\Theta) = \begin{pmatrix} \cos^2\Theta & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \cos\Theta \end{pmatrix} \Leftarrow$$

$n^f_{\parallel} \propto n'_{\parallel}$ → Along the SP
$n^f_{\perp} \propto \cos^2\Theta\, n'_{\perp}$ → $\perp$ to SP
$n^f_3 \propto \cos\Theta\, n_3$ → U comp. ($E_\parallel \cdot E^*_\perp$)

However, the ref. system does not necessarily coincides with the SP.
So, we 1) first need to rotate $\vec{n}'$ from the primed ref. system to the SP;
2) then apply $n_{SP} \propto R(\Theta)\, n'_{SP}$; 3) then rotate back to the ref. system:

$$\vec{n} \propto \underbrace{L(\pi - i_2)}_{\text{Angle between SP and meridian plane}}\ \underbrace{R(\Theta)}_{\text{scattering angle}}\ \underbrace{L(-i_1)}_{\text{Angle between scattering plane and primed meridian plane}}\ \vec{n}' = P(\Omega, \Omega')\, \vec{n}'$$

With some algebra, the explicit form of $P(\Omega, \Omega')$ can be derived (see e.g., Chandrasekhar Sec. 17.2). For our purposes, it is sufficient to say:

$$P(\mu, \varphi, \mu', \varphi') = Q\Big[P^{(0)}(\mu, \mu') + \sqrt{(1-\mu^2)(1+\mu'^2)}\ \underbrace{P^{(1)}(\mu, \varphi, \mu', \varphi')}_{\sim e^{\pm i(\varphi - \varphi')}} + \underbrace{P^{(2)}(\mu, \varphi, \mu', \varphi')}_{\sim e^{\pm i2(\varphi-\varphi')}} \Big]$$

Important: the scattering (and so P) does not mix $\mu\varphi$ dependence:

$$① \quad \frac{1}{4\pi}\int d\varphi'\, P^{(0)} e^{im'\varphi'} = \frac{1}{2} P^{(0)} \delta_{m'0}$$

$$② \quad \frac{1}{4\pi}\int d\varphi'\, P^{(1)} e^{im'\varphi'} = \frac{3}{8}\Big[\underbrace{\Pi^{(1)}(\mu, \mu')\, e^{i\varphi}}_{P^{(1)}} \delta_{m',+1} + \Pi^{(-1)}(\mu,\mu')\, e^{-i\varphi} \delta_{-1,m'} \Big], \qquad \Pi^{(-1)} = \Pi^{(1)*}$$

$$③ \quad \frac{1}{4\pi}\int d\varphi'\, P^{(2)} e^{im'\varphi'} = \frac{3}{8}\Big[\Pi^{(2)}(\mu,\mu')\, \delta_{2m'}\, e^{2i\varphi} + \Pi^{(-2)}\, e^{-2i\varphi} \delta_{-2m'} \Big]$$

$\Rightarrow$ Single out the φ-dep. of $P^{(m)}$

In our case, we only need ① and ②:

$$Q = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} \qquad P^{(0)} = \frac{3}{4}\begin{pmatrix} 2(1-\mu^2)(1-\mu'^2) + \mu^2\mu'^2 & \mu^2 & 0 \\ \mu'^2 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\Pi^{(2)} = \Pi^{(-2)*} = \frac{1}{2}\begin{pmatrix} \mu^2\mu'^2 & -\mu^2 & -i\mu^2\mu' \\ -\mu'^2 & 1 & +i\mu \\ \mu\mu'^2 & -i\mu & \mu\mu' \end{pmatrix}$$

Now we have everything to get started with Boltzmann

SCALAR

In this case, let's amend the transfer equation remembering that the standard form applies to a rest frame. However, in the perturbed Universe, we have seen that e^- acquire a small bulk velocity $\vec{v}_b$. So, along $\hat{p}$, γ are subject to a Doppler shift due to moving e^- (what matters is the $\vec{v}_b$ component along $\hat{p}$, so $\vec{v}_b \cdot \hat{p}$. Also, in cosmology, $\vec{v}$ are IRROTATIONAL, so that $\vec{v}_b \cdot \hat{p} = v_b \cos\underset{\hat{k}\cdot\hat{p}}{\theta} = v_b \mu$ and $\vec{v}_b = \hat{k} v_b$)

$$\Delta \vec{n}_{Doppler} = \frac{\partial n}{\partial \tilde{p}} \Delta \tilde{p}_{Doppler} = \frac{\partial n}{\partial \tilde{p}} \tilde{p}\, \hat{p} \cdot \vec{v}_b = \frac{\partial n}{\partial \tilde{p}} \tilde{p}\, v_b \mu$$

$\Downarrow$

$$\frac{d\vec{n}}{dt} = + \dot{\tau} \vec{n} + \dot{\tau} \Delta \vec{n}_{Dopp.} \mp \frac{\dot{\tau}}{4\pi} \int d\Omega'\, P(\Omega, \Omega') \vec{n}(\Omega')$$

$\Downarrow$ Moving to η, Fourier and $\tilde{p}$ (and $\mu = \cos\theta$)

(A)

$$\frac{\partial \vec{n}_1}{\partial \eta} + ik\mu \vec{n}_1 \mp a\dot{\tau} \vec{n}_1 = \left[\Phi' + ik\mu\Psi + a\tau \cancel{k} v_b \mu\right] \underbrace{\tilde{p} \frac{\partial \vec{n}_0}{\partial \tilde{p}}}_{= \vec{n}_0 f_0 \text{ with } f_0 = \frac{\partial \ln \vec{n}_0}{\partial \ln \tilde{p}}} \mp \frac{a\dot{\tau}}{4\pi} \int d\Omega'\, P(\Omega, \Omega') \vec{n}_1(\Omega')$$

We can further simplify:

-> The only explicit dependence on $\tilde{p}$ comes from $n_0 f_0$, so we can factor it out

-> The φ-dep. can be expanded in harmonics:

$$\vec{n}_1(\eta, \mu, \tilde{p}, \varphi) = n_0 f_0 \sum_m \vec{n}_1(\eta, \mu)\, e^{im\varphi} \qquad \left[\vec{n}_0 = n_0 \hat{u} = n_0 \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right]$$

Equivalent to say we have ∞ eqs., one for each m

When we put this in (A), the integral single out terms with $|m| \leq 2$ (remember how P is decomposed and integrated).

In addition, the term in [ ] brackets does not depend on φ, so it is expanded with $m = 0$. Putting things together, all ~~eqs.~~ solutions with $|m| > 2$ satisfy homogeneous equations: if they are 0 at the beginning, they will always be zero (or, if not, they will quickly decay in absence of source terms). So, we have:

$$\vec{n}_1(\eta, \mu, \varphi, \tilde{p}) = n_0 f_0 \vec{n}_1(\eta, \mu)$$

and

$$f_{BE} \sim f_0 - p \frac{\partial f}{\partial p} \Theta, \quad \vec{n} = \vec{n}_0 + n_0 f_0 \vec{n}_1 \Rightarrow \vec{n}_1 \sim -\vec{\Theta}$$

$$\left[\frac{\partial}{\partial \eta} + ik\mu \mp a\dot{\tau}\right] \vec{n}_1(\eta, \mu) = \left[\Phi' + ik\mu\Psi + a\tau \cancel{k} v_b \mu\right] \hat{u} \mp \frac{a\dot{\tau}}{4\pi} \int d\Omega'\, R P^{(0)}(\mu, \mu') \vec{n}_1(\eta, \mu')$$

$$[\cdots]\,\vec{n}_1 = [\cdots]\,\hat{u} \mp \frac{3}{8} a\dot{\tau} \int d\mu' \begin{bmatrix} (2+3\mu^2\mu'^2-2\mu^2-2\mu'^2)\,n_1+\mu^2 n_2 \\ \mu'^2 n_1 + n_2 \\ 0 \end{bmatrix} \Rightarrow \int a P^{(0)}\, \vec{n}_1\, d\varphi'$$

$\Rightarrow$ I IMPORTANT RESULT $\Rightarrow$ n_3 component satisfies a homogeneous equation.
Remember that $n_3 \sim 2U \Rightarrow$ SCALAR PERT. DO NOT GENERATE U MODES!
(Note: of course, when we fix a ref. system, we need to convert Q-U modes computed in the spherical system to the new one $\rightarrow$ U' modes can appear as rotations of Q modes)

$\Rightarrow$ II RESULT $\Rightarrow$ Only 2 comp. of $\vec{n}_1$ (or, better, linear combinations of them) are independent. We choose: (remember $\vec{n}_1 \sim -\Delta\hat{I}/I$)

$$\alpha = n_1+n_2 = -2\underset{\substack{\downarrow \\ \text{This is } \Delta_T \text{ in e.g. Zaldarriaga}}}{T} \;;\quad \beta = n_1-n_2 = -2\underset{\substack{\downarrow \\ \text{This is } \Delta_P}}{Q} \quad\Rightarrow\quad n_1 = \frac{\alpha+\beta}{2}\;;\; n_2 = \frac{\alpha-\beta}{2}$$

Let's write the eq. for α:

$$[\cdots]\,\alpha = 2[\cdots] \mp \frac{3}{8} a\dot{\tau} \underbrace{\int \frac{d\mu'}{2}\Big[(2+3\mu^2\mu'^2-2\mu^2-\mu'^2)(\alpha+\beta)+(1+\mu^2)(\alpha-\beta)\Big]}_{\text{Let's simplify this using the properties of Legendre pol.}}$$

$P_0 = 1$
$P_1 = \mu$
$P_2 = \frac{1}{2}(3\mu^2-1)$ $\Rightarrow$ $\mu^2 = \frac{1}{3}(2P_2+P_0)$

$\int \frac{d\mu}{2} X P_n = \int \frac{d\mu}{2} P_n \sum_m (2m+1)(-i)^m X_m P_m$
$\int \frac{d\mu}{2} P_n P_m = \frac{1}{2n+1}\delta_{mn}$ $\Rightarrow$ $\int \frac{d\mu}{2} X P_0 = X_0$, $\int \frac{d\mu}{2} X P_2 = -X_2$

$\Downarrow$ Vedi foglio

$$[\cdots]\,\alpha = 2[\cdots] \mp a\dot{\tau}\Big[\alpha_0 - \frac{1}{2}P_2(\alpha_2+\beta_0+\beta_2)\Big]$$

$\downarrow$ Move to $\underset{\substack{\Delta_T \\ \theta}}{T}$ and $\underset{\Delta_P}{Q}$ $\qquad \Delta_T = \Delta_T(\eta,\mu)k)$

$$\boxed{\Delta_T' + ik\mu\Delta_T \mp a\dot{\tau}\Delta_T = -\Phi' - ik\mu\Psi \mp a\dot{\tau}\mu v_b \mp a\dot{\tau}\Big[\Delta_{T_0} - \frac{P_2}{2}(\Delta_{T_2}+\Delta_{P_0}+\Delta_{P_2})\Big]}$$

$\downarrow$ let's include it in the def. of $\dot{\tau} = a n_e \sigma_T$

Repeat it for β:

$$[\dots]\beta = \cancel{0} \mp \frac{3}{8} a\dot{\tau} \int \frac{d\mu'}{2} \left[(2+3\mu^2\mu'^2 - 2\mu^2 - 3\mu'^2)(\alpha+\beta) + (\mu^2-1)(\alpha-\beta) \right]$$

[Note, we take out (P_2-1) NOT $(1-P_2)$]

$$[\dots]\beta = \pm \frac{a\dot{\tau}}{2} \left[1 - P_2 \right)(\alpha_2 + \beta_0 + \beta_2)$$

$\Downarrow$ Move to Δ_P $\qquad \Delta_P \equiv \Delta_P(\eta, \mu, k)$

$$\boxed{\Delta_P' + ik\mu\Delta_P \mp a\dot{\tau}\Delta_P = \pm \frac{a\dot{\tau}}{2}(1-P_2)(\Delta_{T_2} + \Delta_{P_0} + \Delta_{P_2})}$$

=> ① Scalar polariz. IS NOT SOURCED directly from metric perturb.

=> ② " " IS SOURCED from the quadrupole of T

Once the solutions Δ_T and Δ_P are found, we can write the pert. observed in a given direction on the sky:

$$T(\hat{n}) = \int d^3k\, \xi_S(k)\, \Delta_T(\eta_0, k, \mu)$$

$$(Q \pm iU)(\hat{n}) = \int d^3k\, \xi_S(k)\, e^{\mp 2i\varphi_{kn}}\, \Delta_P(\eta_0, k, \mu)$$

$\hookrightarrow$ Needed to rotate from $k \cdot \hat{n}$ to a fixed given frame

Now, since in the $k // \hat{z}$ frame $U=0$ and $Q(\Delta_P)$ does not depend on φ, we have $\bar{\eth}^2(Q+iU) = \eth^2(Q-iU)$ and ${}_2a_{\ell m} = {}_{-2}a_{\ell m}$ (we have exploited the fact that $\bar{\eth}^2$ and $\eth^2$ are invariant quantities: if their properties are true in a given system, they are always true).

=> ③ Scalar pert. only generate E modes

TENSOR

Let's do the same for tensor pert. Note: we do not consider Doppler here

$$\frac{d\vec{n}}{dt} = +\dot{\tau}\vec{n} \mp \frac{\dot{\tau}}{4\pi} \int d\Omega'\, P(\Omega, \Omega')\, \vec{n}(\Omega')$$

$\downarrow$ Use $\eta, \vec{p}, \mu$, Fourier

$$\frac{\partial \vec{n}_1}{\partial \eta} + ik\mu\vec{n}_1 + a\dot{\tau}\vec{n}_1 = \frac{1}{2} f_0 n_0 \sum_s \frac{\partial h_s}{\partial \eta} e^{\pm 2i\varphi}(1-\mu^2) \mp \frac{a\dot{\tau}}{4\pi} \int d\Omega'\, P(\Omega, \Omega')\, \vec{n}'(\Omega')$$

We can write 2 identical equations for the 2 comp. $s = +, \times$. We pick one and remember that: $\vec{n}_1 = \sum_s (1-\mu^2)\vec{n}_s e^{\pm 2i\varphi}$

We also consider only 1 harmonic, say $m=2$ in $e^{\mp 2i\varphi}$. We will see that $m=-2$ will give similar result:

$$\frac{\partial \vec{n}_{1+}}{\partial \eta} + ik\mu \vec{n}_{1+} \mp a\dot{\tau}\vec{n}_{1+} = \frac{1}{2} f_0 n_0 (1-\mu^2) e^{2i\varphi} \frac{\partial h_+}{\partial \eta}\hat{a} \mp \frac{a\dot{\tau}}{4\pi}\int d\Omega' \, P \, \vec{n}_{1+}$$

Now, factor out $\tilde{p}$ dependence and expand φ dependence:

$$\vec{n}_{1+} = n_0 f_0 \sum_m \vec{n}_{1+}(\eta, \mu) e^{im\varphi}$$

Now, the integral singles out $|m| \leq 2$ terms. Since the metric has $m=2$, only $m=2$ eq. is not homogeneous:

$$\vec{n}_{1+} = n_0 f_0 \vec{n}_{1+}(\eta, \mu) e^{2i\varphi}$$

$$\frac{\partial \vec{n}_{1+}}{\partial \eta} + ik\mu \vec{n}_{1+} \mp a\dot{\tau}\vec{n}_{1+} = \frac{1}{2}(1-\mu^2)\frac{\partial h_+}{\partial \eta}\hat{a} \mp \frac{a\dot{\tau}}{4\pi}\int d\mu \, d\varphi' \, Q \, P^{(2)} \vec{n}_{1+}$$

$$[\dots]\vec{n}_{1+} = [\dots]\hat{a} \mp \frac{3}{8} a\dot{\tau} \int \frac{d\mu'}{2} \begin{bmatrix} \mu'^2\mu^2 n_1 - \mu^2 n_2 - i\mu^2\mu' n_3 \\ -\mu'^2 n_1 + n_2 + i\mu' n_3 \\ i2(\mu\mu'^2 n_1 - \mu n_2 + \mu\mu' n_3) \end{bmatrix}$$

It seems we have 3 indep. eqs.. However, only 2 combinations of n_1, n_2, n_3 satisfy non-trivial equations.

$$2i\mu(n_1 - n_2) \mp (1+\mu^2) n_3 = 0 \quad \text{satisfies homog. eq.} \quad \left[\begin{matrix} - \text{ is for } e^{2i\varphi} \\ + \quad '' \quad e^{-2i\varphi} \end{matrix}\right]$$

We can use it to write $\boxed{n_3 = n_3(n_1, n_2) = \mp 2i\mu(n_1 - n_2)}$

Similar to the scalar case, we choose:

$$\begin{aligned} n_1 + n_2 &= (1-\mu^2)\alpha = -2\cancel{(1-\mu^2)}\Delta_T \\ n_1 - n_2 &= (1+\mu^2)\beta = -2\cancel{(1+\mu^2)}\Delta_P \end{aligned} \quad \Rightarrow \quad \begin{aligned} n_1 &= \frac{1}{2}\left[(1-\mu^2)\alpha + (1+\mu^2)\beta\right] \\ n_2 &= \frac{1}{2}\left[(1-\mu^2)\alpha - (1+\mu^2)\beta\right] \end{aligned}$$

Let's write for α:

$$[\dots](1-\mu^2)\alpha = (1-\mu^2)\frac{\partial h_+}{\partial \eta} \mp \frac{3}{8} a\dot{\tau} \int \frac{d\mu'}{2}(1-\mu^2)\left[-\mu'^2 n_1 + n_2 + i\mu' n_3\right]$$

$$\frac{3}{8} a\dot{\tau}(1-\mu^2)\int \frac{d\mu'}{2}\left[(1-\mu'^2)^2\frac{\alpha}{2} - (1+\mu'^2)^2\frac{\beta}{2} - 2\mu'^2\beta\right]$$

Remember Legendre properties, to which we add:

$$\int \frac{d\mu}{2} X P_4 = X_4 \quad ; \quad \mu^4 = \frac{8}{35}P_4 + \frac{4}{7}P_2 + \frac{1}{5}P_0$$

We get (vedi foglio):

$$[\dots]\,\alpha = h'_+ \mp a\dot{\tau}\left[\frac{1}{10}\alpha_0 + \frac{1}{7}\alpha_2 + \frac{3}{70}\alpha_4 - \frac{3}{5}\beta_0 + \frac{6}{7}\beta_2 - \frac{3}{70}\beta_4\right]$$

$\downarrow$ Move to $\Delta_T^{T\prime}$

$$\boxed{\Delta_T^{T\prime} + ik\mu\,\Delta_T^T \mp a\dot{\tau}\,\Delta_T^T = -\frac{h'}{2} \mp a\dot{\tau}\left[\frac{\Delta_{T_0}^T}{10} + \frac{1}{7}\Delta_{T_2}^T + \frac{3}{70}\Delta_{T_4}^T - \frac{3}{5}\Delta_{P_0}^T + \frac{6}{7}\Delta_{P_2}^T - \frac{3}{70}\Delta_{P_4}^T\right]}$$

For β:

$$[\dots]\,\beta\,(1+\mu^2) = \frac{3}{8}a\tau\int\frac{d\mu'}{2}(1+\mu^2)\left[\mu'^2 n_1 - n_2 - i\mu' n_3\right]$$

$$\left[-(1-\mu'^2)^2\frac{\alpha}{2} + (1+\mu'^2)^2\frac{\beta}{2} + 2\mu'\beta\right]$$

$\downarrow$ Apply Legendre and move to $\Delta_P^{T\prime}$

$$\boxed{\Delta_P^{T\prime} + ik\mu\,\Delta_P^T \mp a\dot{\tau}\,\Delta_P^T = \pm a\dot{\tau}\left[\frac{\Delta_{T_0}^T}{10} + \frac{1}{7}\Delta_{T_2}^T + \frac{3}{70}\Delta_{T_4}^T - \frac{3}{5}\Delta_{P_0}^T + \frac{6}{7}\Delta_{P_2}^T - \frac{3}{70}\Delta_{P_4}^T\right]}$$

$\Rightarrow$ Δ_P^T is also sourced by $\Delta_{T_0}^T$

$\Rightarrow$ Δ_P^T is not directly sourced by metric.

Remember: these eqs. are for $s=+$. We want the full expression.
Remember that $\Delta \sim \delta_{in}\,\Delta_{transf.}$. For tensor, we have $\delta_{in} \to h_+, h_\times$ and $\Delta_{transf} \sim \Delta(\eta,\mu)\,e^{\pm 2i\varphi}$:

$$\Delta_T^T(\eta_0,\vec{k},\mu) = \left[(1-\mu^2)e^{2i\varphi}h_+(k) + (1-\mu^2)e^{-2i\varphi}h_\times(k)\right]\Delta_T^T(\eta,k,\mu)$$

$$\Delta_Q^T(\eta_0,\vec{k},\mu) = \left[(1+\mu^2)e^{2i\varphi}h_+(k) + (1+\mu^2)e^{-2i\varphi}h_\times(k)\right]\Delta_P^T(\eta,k,\mu)$$

$$\Delta_U^T(\eta_0,\vec{k},\mu) = i\left[2\mu e^{2i\varphi}h_+(k) - 2\mu e^{-2i\varphi}h_\times(k)\right]\Delta_P^T(\eta,k,\mu)$$

$\Rightarrow$ We also have non-vanishing U $\Rightarrow$ Tensor perturbations generate B modes!

For all the BE we derived, we can expand Δ_x in harmonics to extract the μ dependence. To do so, we expand Δ_x ~~on~~ in Leg. polyn., use the recursive relation $(\ell+1)P_{\ell+1}(\mu) = (2\ell+1)\mu P_\ell - \ell P_{\ell-1}$ and project the equations each time over a different P_ℓ, $\ell = 0, \dots, \infty$.
We get an infinite hierarchy of equations for Δ_{x_ℓ}, which solutions correspond to the quantities needed to compute C_ℓ:

$$C_\ell^{xy} \sim \int d\ln k \, P_{S,T}(k) \, \Delta_\ell^{x,ST} \Delta_\ell^{y,ST}$$

It seems we need to evolve the full hierarchy to get C_ℓ. In fact, we could simplify the thing by noting the following: each eq. can be written as

$$\underbrace{\Delta' + (ik\mu - \dot{\tau})\Delta}_{e^{-ik\mu\eta + \tau}\frac{d}{d\eta}(\Delta e^{ik\mu\eta - \tau})} = \hat{S} \quad , \quad \hat{S} \text{ encoding all RHS terms}$$

$\rightarrow$ Note: $\tau(\eta) = \int_\eta^{\eta_0} n_e \sigma_T a \, d\eta = -\dot{\tau}(\eta)$

$$\Downarrow$$

$$\Delta(\eta_0)\, e^{ik\mu\eta_0 - \tau_0} = \Delta(\eta_{in})\, e^{ik\mu\eta_{in} - \tau_{in}} + \int_{\eta_{in}}^{\eta_0} d\eta \, \hat{S}(\eta,\mu)\, e^{+ik\mu(\eta - \eta_0) - \tau(\eta)}$$

$e^{-\tau_0} = 1$

$e^{-\tau_{in}} \to 0 \Rightarrow$ Scattering erase pert.

So, the current value of Δ is simply given by the free-streaming of the Source function, which only contains a handful of relevant multipoles.
This can be further simplified by integrating by parts $\int d\eta \, \underbrace{e^{ik\mu\eta}}_{f'} \underbrace{[e^{-\tau(\eta)} S(\eta,\mu)]}_{g}$

$\rightarrow$ The boundary is zero (vanish as $\eta \to 0$ and are irrelevant as $\eta \to \eta_0$)
$\rightarrow$ The integral leads to substitutions $\mu \to \frac{1}{ik}\frac{d}{d\eta}$

The only remaining μ-term is $e^{ik\mu\eta}$, which can be expanded in Leg. (as the LHS)

$$\int \frac{d\mu}{2} e^{ik\mu(\eta-\eta_0)} P_\ell = \frac{1}{(-i)^\ell} \underbrace{j_\ell[k(\eta - \eta_0)]}_{\text{BESSEL}}$$

$$\Downarrow$$

$$\Delta_\ell(k, \eta_0) = \int_0^{\eta_0} d\eta \, S(k,\eta)\, j_\ell[k(\eta_0 - \eta)]$$