Gravitational waves from compact objects

Lecture III

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Leonardo Gualtieri
University of Rome "La Sapienza"

COMPACT BINARIES

Let us consider a binary system composed of two bodies with masses m_1, m_2, in the early inspiral stage, i.e. we assume

$$l_0 \gg R\,.$$

Then, weak field and slow velocity approximations are satisfied. To simplify the computations, we also assume that *the orbits are circular*.

We choose a frame in which the origin is the center-of-mass, and the bodies move in the $x-y$ plane. This is the well-known *two-body problem* of Newtonian mechanics: by defining

$$M = m_1 + m_2 \qquad \text{total mass}$$
$$\mu = \frac{m_1 m_2}{M} \qquad \text{reduced mass,}$$

the distances of the bodies from the origin are $r_1 = \frac{m_2 l_0}{M}$ and $r_2 = \frac{m_1 l_0}{M}$ ($r_1 + r_2 = l_0$). They move with the **Keplerian frequency**

$$\omega_K = \sqrt{\frac{GM}{l_0^3}} \qquad \Rightarrow \qquad P = \frac{2\pi}{\omega_K} = 2\pi\sqrt{\frac{l_0^3}{GM}}\,.$$

The motion is then given by

$$x_1 = r_1 \cos\omega_K t \qquad\qquad x_2 = -r_2\cos\omega_K$$
$$y_1 = r_1 \sin\omega_K t \qquad\qquad y_2 = -r_2\sin\omega_K\,.$$

The matter density is

$$\rho(t,x,y,z) = [m_1\delta(x-x_1)\delta(y-y_1) + m_2\delta(x-x_2)\delta(y-y_2)]\delta(x)\,;$$

by replacing in the quadrupole moment $q_{ij} = \int_V \rho x^i x^j d^3x$,

$$q_{xx} = m_1 x_1^2 + m_2 x_2^2 = \mu l_0^2 \cos^2\omega_K t = \frac{1}{2}\mu l_0^2\cos(2\omega_K t) + const.$$
$$q_{yy} = m_1 y_1^2 + m_2 y_2^2 = \mu l_0^2 \sin^2\omega_K t = -\frac{1}{2}\mu l_0^2\cos(2\omega_K t) + const.$$
$$q_{xy} = m_1 x_1 y_1 + m_2 x_2 y_2 = \mu l_0^2\cos\omega_K t\sin\omega_K t = \frac{1}{2}\mu l_0^2\sin(2\omega_K t)$$
$$q_{zi} = 0\,.$$

Thus,

$$q_{ij} = \frac{\mu l_0^2}{2} A_{ij}(t) + const.$$

with

$$A_{ij}(t) = \begin{pmatrix} \cos(2\omega_K t) & \sin(2\omega_K t) & 0 \\ \sin(2\omega_K t) & \cos(2\omega_K t) & 0 \\ 0 & 0 & 0 \end{pmatrix} .$$

The quadrupole formula gives

$$\begin{aligned} h_{ij}^{TT}(t, \vec{x}) &= \frac{2G}{rc^4} \mathcal{P}_{ijkl}(\theta, \phi) \ddot{q}_{kl} \left(t - \frac{r}{c}\right) \\ &= -4\omega_K^2 \frac{2G}{rc^4} \frac{\mu l_0^2}{2} \mathcal{P}_{ijkl} A_{kl} \left(t - \frac{r}{c}\right) . \end{aligned}$$

By defining $A_{ij}^{TT} = \mathcal{P}_{ijkl} A_{kl}$ (leaving implicit the dependence on θ, ϕ)

$$h_{ij}^{TT}(t, r) = -h_0 A_{ij}^{TT} \left(t - \frac{r}{c}\right)$$

where

$$h_0 = \frac{\mathcal{A}}{r} \qquad \mathcal{A} = \frac{4\mu M}{l_0} \frac{G^2}{c^4}$$

is the amplitude of the GW; the GW frequency is twice the orbital frequency: $\nu_K = 2\nu_{GW} = \frac{1}{\pi}\sqrt{\frac{GM}{l_0^3}}$.

If for instance the propagation direction is orthogonal to the orbital frequency, A_{ij} is already TT and thus $A_{ij}^{TT} = A_{ij}$. In this case,

$$\begin{aligned} h_{xx}^{TT} &= -h_{yy}^{TT} = -\frac{\mathcal{A}}{z} \cos[2\omega_K(t - z/c)] \equiv h_+ \\ h_{xy}^{TT} &= h_{yy}^{TT} = -\frac{\mathcal{A}}{z} \sin[2\omega_K(t - z/c)] \equiv h_+ ; \end{aligned}$$

in this case the two polarization have a dephasing of $\pi/2$, i.e. there is circular polarization. It can be shown that if instead $\hat{n}$ lies in the orbital plane, the GW has linear polarization.

ORBITAL PERIOD DECAY IN BINARY PULSARS

The first system in which these effects have been studied is PSR 1913+16, formed by two NSs, one of which is a pulsar, i.e. it sends a beam of radio waves towards us (which allows to measure the orbital parameters of the binaries with good accuracy); it was discovered in 1975 by Hulse and Taylor. From the analysis of the orbital motion, it was found that:

$$\begin{aligned}
&m_1 \sim m_2 \sim 1.4 M_\odot \quad (M_\odot \simeq 2 \cdot 10^{33}\mathrm{g}) \\
&l_0 \sim 2 \cdot 10^6\,\mathrm{km} \sim 2 - 3\, R_\odot \\
&P = 2.8\,10^4\, s \quad \nu_{GW} = 7.16 \cdot 10^{-5}\,\mathrm{Hz}\,.
\end{aligned}$$

Since $l_0 \gg R \sim 10$ km, the weak field and slow motion are very well satisfied in this system.

The amplitude of the emitted GW computed with the formulae we have derived is

$$h_0 = \frac{4\mu M G^2}{r l_0 c^4} \simeq 5 \cdot 10^{-23}\,.$$

However, these formulae are not correct in this case, because the orbit of PSR1913+16 is not circular: it has a large eccentricity, $e \simeq 0.617$. Repeating the computation including the eccentricity gives a larger amplitude, $h_0 \sim 10^{-22}$.

Present detectors are able to observe GWs with amplitudes of $10^{-23} - 10^{-22}$, but only if the frequency is in their sensitivity band, which, as I said, is about 100 Hz (very roughly, from few tens of Hz to around 1KHz). Since this signal has $\nu_{GW} \sim 10^{-4} Hz$, is it impossible to observe it with LIGO-Virgo.

However, it is possible to observe, using radiotelescopes, the decrease of the orbital period due to the energy loss by GW emission. Let us compute the gravitational luminosity:

$$L_{GW} = \frac{G}{5c^5} < \dddot{Q}_{ij} \dddot{Q}_{ij} > \ .$$

Since $Q_{ij} = \frac{\mu l_0^2}{2} A_{ij}$,

$$\dddot{Q}_{ij}(t) = \frac{\mu l_0^2}{2}(8\omega_K^3)\begin{pmatrix} \sin(2\omega_K t) & -\cos(2\omega_K t) & 0 \\ -\cos(2\omega_K t) & -\sin(2\omega_K t) & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

therefore

$$\dddot{Q}_{ij}\dddot{Q}_{ij} = 32\frac{\mu^2 G^3 M^3}{l_0^5}$$

and thus

$$L_{GW} = \frac{32}{5}\frac{G^4}{c^5}\mu^2 M^3 \left\langle \frac{1}{l_0^5} \right\rangle .$$

Far from the coalescence, as

$$l_0 \gg R \, ,$$

the **adiabatic approximation** is satisfied: *the orbital parameters do not significantly change in an orbital period.* Therefore, the energy loss by GWs in a period is much smaller than the orbital energy E_{orb}, and the system has time to adjust its parameters. Thus we have

$$\frac{dE_{orb}}{dt} + L_{GW} = 0 \, . \tag{1}$$

Within the adiabtic approximation, the average of l_0 over several wavelengths - which is equivalent to the average over several periods - is just l_0, which in that timescale is constant, and

$$L_{GW} = \frac{32}{5}\frac{G^4}{c^5}\mu^2 M^3 l_0^5 \, .$$

The orbital energy can be computed within Newtonian mechanics. The kinetic energy is $E_K = \frac{1}{2}m_1(\omega_K r_1)^2 + \frac{1}{2}m_2(\omega_K r_2)^2 = \frac{G\mu M}{2l_0}$ while the potential energy is $U = -\frac{G\mu M}{l_0}$, therefore

$$E_{orb} = -\frac{G\mu M}{2l_0} \quad \Rightarrow \frac{d}{dt}E_{orb} = -\frac{E_{orb}}{l_0}\frac{dl_0}{dt}$$

and thus replacing in Eq. (1)

$$\frac{1}{l_0}\frac{dl_0}{dt} = \frac{L_{GW}}{E_{orb}} = -\frac{64}{5}\frac{G^3}{c^5}\frac{\mu M^2}{l_0^4} \, . \tag{2}$$

The integration of Eq. (2) in time gives $\frac{64}{5}\frac{G^3}{c^5}\mu M^2 dt = -l_0^3 dl_0$ and thus, if $l_0^{in} = l_0(t=0)$,

$$l_0^4 = (l_0^{in})^4 - \frac{256}{5}\frac{G^3}{c^5}\mu M^2 t = (l_0^{in})^4 \left(1 - \frac{t}{t_C}\right)$$

where $t_C = \frac{5}{256}\frac{c^5}{G^3}\frac{(l_0^{in})^4}{\mu M^2}$ is called *coalescence time* and gives an estimate of the time needed for coalescence. Note that this is just an estimate: this formula is only aggurate as $\frac{t}{t_C} \ll 1$, and it is not valid at all as $t \to t_C$.

Thus,

$$l_0(t) = l_0^{in}\left(1 - \frac{t}{t_C}\right)^{1/4} :$$

the orbital separation decreases with time. Moreover, since $\omega_K = \sqrt{\frac{GM}{l_0^3}}$, the orbital period is

$$P(t) = \frac{2\pi}{\omega_K(t)} = 2\pi\sqrt{\frac{l_0^3(t)}{GM}} = P^{in}\left(1 - \frac{t}{t_C}\right)^{3/8} :$$

the orbital period decreases as well.

Since $P^2 \propto l_0^3$, Eq. (2) gives

$$\frac{1}{P}\frac{dP}{dt} = \frac{3}{2}\frac{1}{l_0}\frac{dl_0}{dt} = -\frac{96}{5}\frac{G^3}{c^5}\frac{\mu M^2}{l_0^4}$$

and replacing $\frac{1}{l_0^4} = \frac{1}{G^{4/3}M^{4/3}}\left(\frac{2\pi}{P}\right)^{8/3}$,

$$\frac{1}{P}\frac{dP}{dt} = -\frac{96}{5}\frac{G^{5/3}}{c^5}\mu M^{2/3}\left(\frac{2\pi}{P}\right)^{8/3} .$$

Since PSR 1913+16 has $e \simeq 0.617$, this computation should be performed taking into account the eccentricity. The computation gives the same expression as in the circular case, with a correction factor depending on e:

$$\frac{1}{P}\frac{dP}{dt} = -\frac{96}{5}\frac{G^{5/3}}{c^5}\mu M^{2/3}\left(\frac{2\pi}{P}\right)^{8/3}\frac{1}{(1-e^2)^{7/2}}\left(1 + \frac{73}{24}e^4 + \frac{37}{96}e^4\right) .$$

For PSR1913+16, this expression gives

$$\frac{dP}{dt} \simeq -2.4 \cdot 10^{-12} .$$

Note that the relative change in P during P is dP/dt, which is very small, so the adiabatic approximation is satisfied with very good accuracy in this system.

By including all other effects present (e.g., the Doppler effect due to the motion with respect to Earth, etc.), the observed value coincides with the predition of GR:

$$\frac{\dot{P}_{obs}}{\dot{P}_{GR}} = 1.0012 \pm 0.0021\,.$$

This result has been the first (indirect) evidence of the existence of GWs; for this reason, Hulse and Taylor got the Nobel prize in 1993.

Today we have observations from several NS-NS binaries; some of them are double pulsars, allowing for much more accurate measurements. These observations confirm the predictions of GR with even better accuracy.

GWS FROM INSPIRALLING COMPACT BINARIES

Let is now consider the GW signal emitted in the inspiral. It is

$$h_{ij}^{TT}(t,r,\theta,\phi) = -h_0 \mathcal{P}_{ijkl}(\theta,\phi) A_{kl}\left(t-\frac{r}{c}\right)$$

where

$$h_0 = \frac{4\mu M G^2}{l_0 c^4 r} \qquad A_{kl}\left(t-\frac{r}{c}\right) = \begin{pmatrix} \cos[2\omega_K\left(t-\frac{r}{c}\right)] & \sin[2\omega_K\left(t-\frac{r}{c}\right)] & 0 \\ \sin[2\omega_K\left(t-\frac{r}{c}\right)] & \cos[2\omega_K\left(t-\frac{r}{c}\right)] & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

In the timescale of the orbital period P, this is a monochromatic wave with frequency

$$\nu_{GW} = 2\frac{\omega_K}{2\pi} = \frac{1}{\pi}\sqrt{\frac{GM}{l_0^3}}\,.$$

In a timescale $\gg P$, l_0 is a decreasing function of time:

$$l_0 = l_0^{in}\left(1-\frac{t}{t_C}\right)^{1/4},$$

therefore the frequency and the amplitude change with time as well:

$$\nu_{GW}(t) = \nu_{GW}^{in}\left(1-\frac{t}{t_C}\right)^{-3/8} \qquad \text{and} \qquad h_0(t) = \frac{4\mu M G^2}{r l_0^{in} c^4}\left(1-\frac{t}{t_C}\right)^{-1/4}.$$

This signal is called **chirp** because, as in a bird's chirp, the frequency and the amplitude both increas with time.

It is useful to express the amplitude as a function of the frequency (both quantities measurable), removing the dependence on l_0:

$$h_0(\nu_{GW}) = \frac{4\pi^{2/3}G^{5/3}}{c^4 r}\mu M^{2/3}\nu_{GW}^{2/3}\,.$$

This expression depends on m_1 and m_2 through a particular combination: the **chirp mass** $\mathcal{M}$:

$$\mathcal{M} \equiv \mu^{3/5}M^{2/5}\,.$$

Thus, $\mathcal{M}^{5/3} = \mu M^{2/3}$ and

$$h_0(\nu_{GW}) = \frac{4\pi^{2/3}G^{5/3}}{c^4}\frac{\mathcal{M}}{r}(\mathcal{M}\nu_{GW}^{2/3})\,.$$

Another useful relation which can be easily found by manipulating these relations is

$$\nu_{GW}(t) = \frac{5^{3/8}}{8\pi} \left(\frac{c^3}{G\mathcal{M}} \right)^{5/8} \frac{1}{(t_C - t)^{3/8}} \,. \tag{3}$$

In a timescale $\gg P$, the wave is not monochromatic anymore, and the phase of the oscillation is the integral

$$\phi(t) = \int_0^t 2\pi\nu_{GW}(t)dt + \phi_{in} = -2 \left[\frac{c^3(t_C - t)}{5G\mathcal{M}} \right]^{5/8} + \phi_{in} \,.$$

The gravitational wave is then

$$h_{ij}^{TT} = -h_0 \mathcal{P}_{ijkl} A_{kl} \left(t - \frac{r}{c} \right)$$

where

$$A_{kl} = \begin{pmatrix} \cos\phi(t) & \sin\phi(t) & 0 \\ \sin\phi(t) & \cos\phi(t) & 0 \\ 0 & 0 & 0 \end{pmatrix} \,.$$

GW150914

On September 14th 2015, the interferometric detector LIGO, formed by two interferometers in Livingstone (Louisiana) and Hanford (Washington) detected for the first time a GW signal, which was emitted by the coalescence of two compact bodies. The signal was given the name GW150914. It is formed of three stages: the inspiral, the merger and the ringdown. Let us focus, for the moment, on the **inspiral**.

When the GW was loud enough to be detected, the weak field, slow motion and adiabatic approximation were already violated to some extent. The quadrupole formula gives ony a first approximation of the signal in this stage; it allows to extract (approximately) the chirp mass of the binary, but - since the waveform depends on the masses only through $\mathcal{M}$ it is impossible to extract the individual masses m_1, m_2 using the quadrupole formula: they can only be found using the PN expansion.

The chirp mass can be extracted from Eq. (3),

$$\nu_{GW}(t) = \frac{5^{3/8}}{8\pi}\left(\frac{c^3}{G\mathcal{M}}\right)^{5/8}\frac{1}{(t_C - t)^{3/8}},$$

to be compared with the measured frequancy. In order to remove the dependence on t_C, we note that the measured frequency is a function of time, thus we have both ν_{GW} and $\dot{\nu}_{GW}$, to be compared with the time derivative of Eq. (3). Thus, by combining $\nu_{GW}(t)$ and $\dot{\nu}_{GW}(t)$, we obtain

$$\mathcal{M} = \frac{c^3}{G}\left[\frac{5}{96}\frac{1}{\pi^{8/3}}\dot{\nu}_{GW}\nu_{GW}^{-11/3}\right]^{3/5}.$$

This expression does not take into account the cosmological expansion, which affects GW propagation. Indeed, the observed GW frequency is different from the frequency at emission:

$$\nu_{GW}^{obs} = \frac{\nu_{GW}}{1+z}$$

with z cosomological redshift. Moreover,

$$\frac{d}{dt^{obs}} = \frac{1}{1+z}\frac{d}{dt} \quad \Rightarrow \quad \frac{d\nu^{obs}}{dt^{obs}} = \frac{\dot{\nu}_{GW}}{(1+z)}^{2}$$

therefore

$$\frac{d}{dt^{obs}}\nu_{GW}^{obs}\nu_{GW}^{obs\ -11/3}=\dot{\nu}_{GW}\nu_{GW}^{-11/3}(1+z)^{5/3}\,.$$

Since we measure $\frac{d}{dt^{obs}}\nu_{GW}^{obs}$ and ν_{GW}^{obs}, we can measure

$$\mathcal{M}^{obs}=\frac{c^3}{G}\left[\frac{5}{96}\frac{1}{\pi^{8/3}}\frac{d}{dt^{obs}}\nu_{GW}^{obs}\nu_{GW}^{obs\ -11/3}\right]^{3/5}=\mathcal{M}(1+z)\,,$$

i.e. $\mathcal{M}^{obs}\equiv(1+z)\mathcal{M}$, and to measure $\mathcal{M}$ we need to know z.

To measure z, we have to consider the amplitude h_0. It can be shown that the amplitude of a GW propagating in the curved background of an expanding universe has the same form of the amplitude of a wave propagating in Minkowski spacetime, with the distance r replaced with the proper distance at time t, $D(t)$:

$$h_0(t)=\frac{4\pi^{2/3}}{c^4}\frac{\mathcal{M}}{D(t)}(\mathcal{M}\nu_{GW})^{2/3}\,.$$

Since $\mathcal{M}=\mathcal{M}^{obs}/(1+z)$, $\nu_{GW}=\nu_{GW}^{obs}(1+z)$, if we define the **luminosity distance**

$$D_L=(1+z)D$$

we can express h_0 in terms of measurable quantities:

$$h_0(t^{obs})=\frac{4\pi^{2/3}}{c^4}\frac{\mathcal{M}^{obs}}{D_L(t^{obs}}(\mathcal{M}^{obs}\nu_{GW}^{obs})^{2/3}\,.$$

Assuming that the cosmological model and parameters $(H_0,\dots)$ are known, the function $D_L(z)$ is known. Then:

- from the measure of $\nu_{GW}^{obs}(t^{obs})$ we can extract $\mathcal{M}^{obs}$;
- from the measure of $h_0(t^{obs})$ we can extract D_L; for GW150914 this gives

  $$D_L=430^{+10}_{-170}\,\mathrm{Mpc}\sim 1\mathrm{ly}\,;$$

  note that this is a cosmological distance, so these corrections are relevant;

- assuming the cosmological model, this value of D_L corresponds to $z = 0.09 \pm 0.03$;
- finally we obtain

$$\mathcal{M} = \frac{\mathcal{M}}{1+z} = 28.6^{+1.6}_{-1.5}\, M_\odot\,.$$