

Gravitational waves from compact objects

Lecture IV

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GW150914

As we have seen, using the quadrupole formula it is possible to measure (at least approximately) the chirp mass of the binary. For GW150914, this led (taking into account the cosmological expansion)

$$\mathcal{M} = 28.6_{-1.5}^{+1.6} M_{\odot}.$$

The measurement of chirp mass does not tell us the values of the two masses m_1 , m_2 , but it gives useful information. The function

$$\mathcal{M} = \mu^{3/2} M^{2/5} = \frac{m_1^{3/5} (M - m_1)^{3/5}}{M^{1/5}} = 28.6 M_{\odot}$$

can be drawn in the $m_1 - M$ plane, showing that

$$M \geq M^{min} = 65.7 M_{\odot}.$$

Moreover, in GW150914 the frequency of the signal increased from $\nu_{GW} \sim 35$ Hz to $\nu_{GW} \sim 150$ Hz. So, just before merger, the frequency was $\nu_{GW} = \bar{\nu}_{GW}$ where

$$\bar{\nu}_{GW} = 150 \text{ Hz}.$$

Since (note that these formulae give just orders of magnitude, since the weak field approximation is not satisfied)

$$\nu_{GW} \sim \frac{1}{\pi} \sqrt{\frac{GM}{l_0^3}},$$

then the orbital distance just before merger was $l_0 \sim \bar{l}_0$

$$\bar{l}_0 = \frac{(GM)^{1/3}}{(\bar{\nu}_{GW}\pi)^{2/3}}.$$

Note that (if we only know the chirp mass) we don't really know l_0 , since we don't know M , but we know a minimum possible value of \bar{l}_0 :

$$\bar{l}_0 \geq \bar{l}_0^{min} = \frac{(GM^{min})^{1/3}}{(\bar{\nu}_{GW}\pi)^{2/3}} = 340 \text{ km}.$$

On the other hand, the Schwarzschild radius corresponding to the total mass is

$$R_s = \frac{2GM}{c^2} \geq R_s^{min} = \frac{2GM^{min}}{c^2} = 194 \text{ km} .$$

Therefore, just before merger,

$$\frac{l_0}{R_s} \sim \frac{\bar{l}_0}{R_s} = \frac{(GM)^{1/3}}{(\bar{\nu}_{GW}\pi)^{2/3}} \frac{c^2}{2GM} \leq \frac{(GM^{min})^{1/3}}{(\bar{\nu}_{GW}\pi)^{2/3}} \frac{c^2}{2GM^{min}} = \frac{\bar{l}_0^{min}}{R_s^{min}} = 1.75 ,$$

i.e., just before merger, $l_0 \lesssim 1.75R_s$. Since the most compact stars are NSs, for which $R \sim 2.5R_s$, we can conclude that the two bodies are BHs.

Moreover, the velocities of the bodies just before merger are

$$v \sim \bar{v} = \bar{l}_0 \frac{\bar{\nu}_{GW}}{2} \geq \bar{l}_0^{min} \frac{\bar{\nu}_{GW}}{2} = 0.1c :$$

this analysis also tells us that these BHs move, near the merger, at relativistic velocities.

We stress that these conclusions rely only on the comparison of the observations with the quadrupole formula.

Using more advanced models of the waveform (see later), it has been possible to understand most of the feature of this system. Let me summarize these results.

GW150914 was emitted by **two BHs** with masses $m_1 = 35.6_{-3.0}^{+4.6} M_\odot$, $m_2 = 30.6_{-4.4}^{+3.0} M_\odot$. After the merger, the remnant was a BH with $M_{fin} = 63.1_{-3.0}^{+4.3} M_\odot$ and dimensionless angular momentum $\chi = \frac{cJ}{GM^2} = 0.69_{-0.04}^{+0.05}$.

Some remarks:

- $m_1 + m_2 > M_{fin}$; the difference $\Delta M \simeq 3.1 M_\odot$ has been emitted in GWs. This is an enormous amount of energy: $E_{GW} \simeq 3.1 M_\odot c^2 \simeq 5.5 \cdot 10^{54}$ erg, about ten times the emission from the most energetic supernovae.
- Near the merger, the orbital velocity was

$$v \sim 0.5 c.$$

This is the only known case of an astronomical object at relativistic velocities.

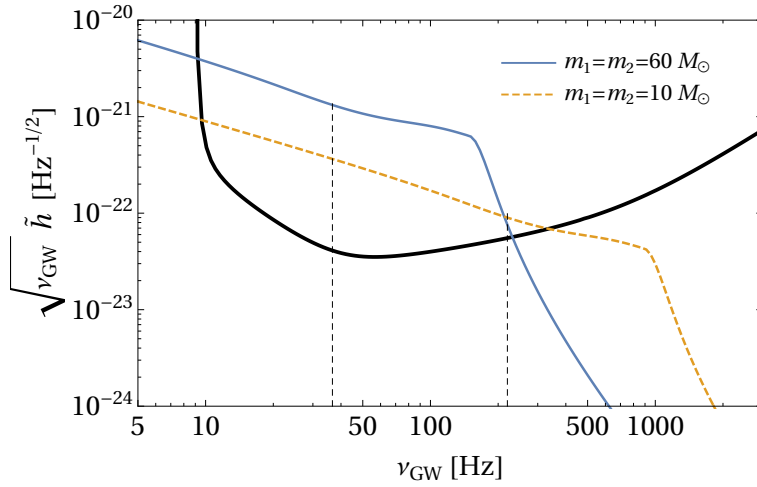
- $\chi = 0.69$ is quite a large spin: the maximum allowed value for a stationary (Kerr) BH is $|\chi| = 1$. It arises, due to angular momentum conservation, from the orbital angular momentum of the binary. In BH-BH coalescences, this is a typical value for the spin of the remnant. The measured values of the spins of the two bodies of the binary, instead, is compatible with zero.
- The eccentricity of the observed orbits is $e \simeq 0$. This is due to the fact that during the inspiral the orbit *circularize*, i.e. its eccentricity decreases, thus in late inspiral the orbits are circular.

In subsequent years, LIGO and the Italian detector Virgo observed tens of such signals. In the third and last observation run, this occurred on a weekly basis. Joint analysis with Virgo allowed, in some cases, localization of the source in the sky, with a precision of a few degrees.

Observed masses range from $\gtrsim 5 M_{\odot}$ (but some candidates may be lighter) to $\sim 100 M_{\odot}$, and the spins of the final BHs are typically $\chi \sim 0.7$, but in some cases they can be in a larger range, from ~ 0.14 to ~ 0.83 .

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Let's consider the noise curve of the interferometer. The coalescence signal looks like the curves in the figure, with an amplitude which change smoothly as the frequency increases, has a small bump at merger, and then rapidly drops down. Roughly speaking, the maximum frequency is that of the last stable orbit, after which there is the coalescence; typically, the maximum frequency scales (like all frequencies involved in BH systems) as the inverse of the mass.

For larger masses, on one hand $h_0 \sim \mathcal{M}^{5/3}$ is larger, on the other hand $\nu_{\text{GW}}^{\text{max}} \sim 1/M$ is smaller. Therefore, a large mass leads to a louder signal, but if the mass is too large the signal remains less in the sensitivity band of the detector, and is more difficult to be observed.

The first NS-NS signal, GW170817, has been detected in 2017. The two bodies have $m_1 \sim m_2 \sim 1.4 M_\odot$: much smaller masses than those of GW150914 (and in general of BH-BH binaries). On the other hand, there are much more NS-NS than BH-BH binaries, and thus it has been possible to observe a much close system, at just $r \sim 40 Mpc$. These two effects compensate, and \mathcal{M}/r is comparable.

However, other features of the signal are different: since ν_{GW}^{max} is much larger than for BH-BH binaries, the signal remained in the bandwidth ~ 100 s. Besides that, while the early inspiral of a NS-NS binary is like as that of a BH-BH binary, the late inspiral is slightly different: tidal interaction leads to observable finite-size effects (see later).

Moreover, in GW170817 (also thanks to Virgo) there was accurate localization in the sky; then, telescopes pointed in that direction, finding electromagnetic counterparts of the signal, in optical, infrared, radio, gamma bands (an approach called *multimessenger astronomy*). Note that BH-BH binaries are instead not believed to be associated to significant electromagnetic emission.

After GW170817, at least another NS-NS signal (may be more) has been detected, together with two BH-NS signals, all of them without detectable electromagnetic counterparts.

ON THE QUADRUPOLE FORMULA

Let us consider the quadrupole formula

$$h_{ij}^{TT}(t, \vec{x}) = \frac{2G}{rc^4} \mathcal{P}_{ijkl}(\theta, \phi) \frac{d^2}{dt^2} Q_{kl} \left(t - \frac{r}{c} \right).$$

It tells that the GWs depend (mainly) on the changes of the (reduced) **quadrupole moment**

$$Q_{ij}(t) = \int_V \rho(t, \vec{x}) \left(x^i x^j - \frac{1}{3} \delta^{ij} r^2 \right) d^3x.$$

To better understand this fact, let us consider, in electrodynamics, a system of charges $\{q_r\}_{r=1\dots}$ at positions $\{\vec{c}_r\}$. The *electric dipole moment* of the system is

$$\vec{d}_{EM} = \sum_r q_r \vec{x}_r.$$

This system emits (mainly) dipolar radiation, whose field is $\sim \dot{\vec{d}}_{EM}$ and whose energy flux is $\sim \ddot{\vec{d}}_{EM}$.

Let us now consider a system of masses $\{m_r\}_{r=1\dots}$ at positions $\{\vec{c}_r\}$. The *gravitational dipole moment* of the system is

$$\vec{d}_G = \sum_r m_r \vec{x}_r.$$

Since

$$\dot{\vec{d}}_G = \sum_r m_r \vec{v}_r = \vec{P}$$

momentum of the system, which for an isolated system is constant (in Newtonian physics, but our conclusions also apply to GR), then $\ddot{\vec{d}}_G = \vec{0}$: **no gravitational dipole emission is present.**

The GW radiation, instead, comes from the derivatives of the quadrupole moment $q_{ij} = \sum_r m_r x^i x^j$, and (like in the electromagnetic case) there are subleading contribution from the higher-order multipole moments.

Summarize, in the *multipole expansion*, the electromagnetic emission starts from the dipole, the gravitational emission starts from the quadrupole.

Let us be more precise: where does the multipole expansion come from?

Let us assume weak field and distant source ($r \gg \epsilon$ and also $r \gg \lambda_{GW}$), but *not* slow motion. The typical velocities of the source are $v \sim \nu\epsilon < c$; we ask v/c to be small enough that an expansion in v/c is possible and well defined, but large enough that the higher-order terms in this expansion are not negligible.

Under these assumption,

$$h_{ij}^{TT} = \frac{4G}{c^4} \mathcal{P}_{ijkl} \int_V d^3x' \frac{T_{kl} \left(t - \frac{|\vec{x} - \vec{x}'|}{c}, \vec{x}' \right)}{|\vec{x} - \vec{x}'|}$$

and since

$$|\vec{x}'| < \epsilon \ll r = |\vec{x}|,$$

$$h_{ij}^{TT} = \frac{4G}{c^4 r} \mathcal{P}_{ijkl} \int_V d^3x' T_{kl} \left(t - \frac{|\vec{x} - \vec{x}'|}{c}, \vec{x}' \right). \quad (1)$$

Note that the Taylor expansion of $|\vec{x} - \vec{x}'|$ around $\vec{x}' = 0$ is (since $\hat{n} = \vec{x}/r$)

$$|\vec{x} - \vec{x}'| = r \left(1 - \frac{\hat{n} \cdot \vec{x}'}{r} + O(\epsilon^2/r^2) \right)$$

and the second term is much smaller than the first, but it may be relevant if T_{kl} is rapidly oscillating. Indeed, the Fourier transform of T_{ij} is an integral of terms like

$$e^{i\omega \left(t - \frac{r}{c} + \frac{\hat{n} \cdot \vec{x}'}{c} \right)}$$

where $\omega(t - r/c) \gg 1$ being $r \gg \lambda_{GW}$, while

$$\omega \frac{\hat{n} \cdot \vec{x}'}{c} \sim \omega \frac{\epsilon}{c} \sim 2\pi \frac{v}{c}$$

which may be of the order of 2π and thus affect the complex exponential.

By expanding Eq. (1) we get

$$h_{ij}^{TT} = \frac{4G}{c^4 r} \mathcal{P}_{ijkl} \int_V d^3x' \left[T_{kl} \left(t - \frac{r}{c} \right) + \dot{T}_{kl} \left(t - \frac{r}{c} \right) \frac{\vec{x}' \cdot \hat{n}}{c} + \ddot{T}_{kl} \left(t - \frac{r}{c} \right) \left(\frac{\vec{x}' \cdot \hat{n}}{c} \right)^2 + \dots \right]$$

where the first term is

$$\int_x V d^3 x' T^{ij} = \frac{1}{2c^2} \int_V d^3 x' x'^i x'^j \ddot{T}^{00} = \frac{1}{2} \ddot{q}^{ij},$$

the second is

$$\frac{1}{c} \int_V d^3 x' \dot{T}^{ij} x'^k n^k,$$

the third is

$$\frac{1}{2c^2} \int_V d^3 x' \dot{T}^{ij} x'^k n^k x'^l n^l,$$

and so on. If the motion of the source has typical frequency ν , and thus typical velocities $v \sim \epsilon\nu$,

$$\dot{T}^{ij} \frac{\vec{x}' \cdot \mathbf{n}}{c} \sim \frac{v}{c} T^{ij}, \quad \ddot{T}^{ij} \left(\frac{\vec{x}' \cdot \mathbf{n}}{c} \right)^2 \sim \frac{v^2}{c^2} T^{ij} :$$

this is an expansion in the *velocities* of the source. If $v \ll c$, only the first term is present, and we recover the quadrupole formula.

With derivations similar to that of the tensor virial theorem, it is possible to express the integrals of this expansion in terms of the **mass multipole moments**:

$$\begin{aligned} \text{quadrupole: } Q^{ij}(t) &= \frac{1}{c^2} \int_V T^{00}(t, \vec{x}) x^{<i} x^{j>} d^3 x \\ \text{octupole: } Q^{ijk}(t) &= \frac{1}{c^2} \int_V T^{00}(t, \vec{x}) x^{<i} x^j x^{k>} d^3 x \end{aligned}$$

...

and of the **current multipole moments**:

$$\begin{aligned} \text{quadrupole: } S^{ij}(t) &= \frac{1}{c} \int_V T^{0<i}(t, \vec{x}) x^{j>} d^3 x \\ \text{octupole: } S^{ijk}(t) &= \frac{1}{c} \int_V T^{0<i}(t, \vec{x}) x^j x^{k>} d^3 x \end{aligned}$$

...

The angle parentheses denote the symmetric trace-free (STF) part of an Euclidean tensor:

$$x^{<i} x^{j>} = x^i x^j = \frac{1}{3} \delta^{ij} r^2$$

and similarly

$$x^{<i x^i x^k>} = x^i x^j x^k - \frac{3r^2}{15}(\delta^{ij} x^k + \delta^{jk} x^i + \delta^{ki} x^j),$$

and so on.

The mass moments are associated to the mass-energy distribution (in the Newtonian limit $\frac{1}{c^2}T^{00} = \rho$ matter density); the current moments are associated to the mass-energy motion (in the Newtonian limit $\frac{1}{c}T^{0i} = \rho v^i$ matter flux).

The expansion in v/c of h_{ij}^{TT} is an *expansion in the multipole moments of the source*: it can be shown that this expansion (up to $O(v/c)$) is

$$h^{TTij}(t, \vec{x}) = \frac{2G}{c^4 r} \mathcal{P}^{ijkl}(\theta, \phi) \left[\ddot{Q}^{kl} - 2n^r(\theta, \phi) \dot{S}^{klr} + n^r(\theta, \phi) \ddot{Q}^{klr} + \dots \right] \left(t - \frac{r}{c} \right). \quad (2)$$

Which is the angular dependence of the GWs, e.g. of Eq. (2) is complex. This can be understood if the metric perturbation is expanded in **tensor spherical harmonics**.

Let's first consider a scalar field on flat space solution of the wave equation with a source

$$\square_F \Phi(t, \vec{x}) = T.$$

Far away from the source, the solution has the form

$$\Phi(t, \vec{x}) = \frac{1}{r} \psi \left(t - \frac{r}{c}, \theta, \phi \right).$$

At t, r constant, this is a function defined on the two-sphere, and can be expanded in *scalar spherical harmonics*

$$\Phi(t, \vec{x}) = \frac{1}{r} \sum_{l=0}^{\infty} \sum_{m=-l}^l \psi_{lm} \left(t - \frac{r}{c} \right) Y^{lm}(\theta, \phi).$$

Similarly, a vector field defined on the two-sphere can be expanded in the (complete) basis of *vector spherical harmonics*:

$$\{V_i^{(+)\,lm}(\theta, \phi), V_i^{(-)\,lm}(\theta, \phi)\}$$

where \pm stands for the behaviour under parity transformations (even or odd). Finally, any rank two symmetric traceless Euclidean tensor on the two-sphere can be expanded in the (complete) basis of *tensor spherical harmonics*:

$$\{T_{ij}^{(+)\,lm}(\theta, \phi), T_{ij}^{(-)\,lm}(\theta, \phi)\}.$$

In these bases, the tensors of a given l and different values of m form an irreducible representation of the rotation group. The representation with $l = 0$ has one component and is called the monopole; that with $l = 1$ has three components is called the dipole, that with $l = 2$ has five components and is called the quadrupole, and so on.

It can be shown that any gravitational wave far away from the source (requiring neither slow motion nor weak field *on the source*, although of course far way from the source the weak-field approximation is satisfied)

can be written as

$$h_{ij}^{TT}(t, \vec{x}) = \frac{G}{c^4 r} \sum_{l=2}^{\infty} \sum_{m=-l}^l \left[A_{lm}^{(+)}(t) T_{ij}^{(+)}(\theta, \phi) + A_{lm}^{(-)}(t) T_{ij}^{(-)}(\theta, \phi) \right]. \quad (3)$$

Remarkably, this expansion starts from the quadrupole $l = 2$: no monopole or dipole radiation is possible. Moreover, it can be shown that the coefficients $A_{lm}^{(\pm)}(t)$ are related to the multipole moments of the source as follows:

$$A_{lm}^{(+)}(t) = C_{i_1, \dots, i_l}^{lm} \frac{d^l}{dt^l} Q^{i_1, \dots, i_l}(t - r/c)$$

$$A_{lm}^{(-)}(t) = C_{i_1, \dots, i_l}^{lm} \frac{d^l}{dt^l} S^{i_1, \dots, i_l}(t - r/c).$$

where C_{i_1, \dots, i_l}^{lm} are constants. Thus, the wave is mainly quadrupolar (unless $v \rightarrow c$), because the quadrupole moment of the source generate the quadrupolar component of the wave.