SOURCE FUNCTIONS

Let's start from SCALAR Δ_T:

$$\Pi = (\Delta_{T_2} + \Delta_{P_0} + \Delta_{P_2})$$

$$\Delta_T' + ik\mu\Delta_T - \dot{\tau}\Delta_T = -\Phi' - ik\mu\Psi - \dot{\tau}\mu v_b - \dot{\tau}\left[\Delta_{T_0} - \frac{1}{2}P_2\Pi\right]$$

$$\Downarrow$$

$$\Delta_T(\eta_0,k,\mu) = \int_0^{\eta_0} d\eta\, e^{ik\mu(\eta-\eta_0)}\, e^{-\tau(\eta,\eta_0)}\left[-\dot{\tau}\left(\mu v_b + \Delta_{T_0} - \frac{1}{2}P_2\Pi\right) - \Phi' - ik\mu\Psi\right]$$

Note that, e.g. $\displaystyle\int_0^{\eta_0} d\eta\, \underbrace{e^{ik\mu(\eta-\eta_0)}}_{f'} \underbrace{e^{-\tau}(-ik\mu\Psi)}_{g} \overset{\text{integrate by parts, with vanishing boundaries}}{=} \int_0^{\eta_0} d\eta\, e^{ik\mu(\eta-\eta_0)}\underbrace{\frac{d}{d\eta}\left(e^{-\tau}\Psi\right)}_{\to(\dot{\tau}\Psi+\dot{\Psi})e^{-\tau}}$

so, each time we have $\mu \to \frac{1}{ik}\frac{d}{d\eta}$:

$$\downarrow$$

$$\Delta_T(\eta_0,k,\mu) = \int_0^{\eta_0} d\eta\, e^{ik\mu(\eta-\eta_0)}\, e^{-\tau}\left[\cancel{\dot{\tau}\left(\frac{v_b}{ik}+\Delta_{T_0}\right)} - \frac{1}{k}\frac{d}{d\eta}\left(-\dot{\tau}v_b\right) - \dot{\tau}\Delta_{T_0} - \dot{\tau}\frac{\Pi}{4} + \right.$$

$$\left. -\frac{3}{4k^2}\frac{d^2}{d\eta^2}\left(e^{-\tau}\dot{\tau}\Pi\right) - \Phi' + \frac{d}{d\eta}\left(e^{-\tau}\Psi\right)\right]$$

$$S_{T,S} = e^{-\tau}\left[-\Phi' - \tau'\left(\Delta_{T_0} + \frac{\Pi}{4}\right)\right] + \frac{d}{d\eta}\left(e^{-\tau}\left(\Psi - \frac{i\tau' v_b}{k}\right)\right) - \frac{3}{4k^2}\frac{d^2}{d\eta^2}\left(e^{-\tau}\tau'\Pi\right)$$

Define the VISIBILITY FUNCTION $g(\eta) = -\tau' e^{-\tau(\eta)}$ with $\underbrace{\int_0^{\eta_0} g(\eta)\,d\eta = 1}_{\text{Probability to scatter at } \eta}$

$g(\eta)$ is very peaked when last scatter is more likely, i.e., e^- are still not recombined in H and γ are decoupled from e^-.

$$\boxed{S_{T,S} = g(\eta)\left[\Psi + \Delta_{T_0}\right] + \frac{i}{k}\frac{d}{d\eta}\left[g(\eta)v_b\right] + e^{-\tau}\left[\Psi' - \Phi'\right] - \frac{3}{4k^2}\frac{d^2}{d\eta^2}\left(e^{-\tau}\tau'\Pi\right)}$$

Considering that $g(\eta)$ is sharply peaked at $\eta = \eta_{rec}$, and using

$\frac{dj_\ell(x)}{dx} = j_{\ell-1} - \frac{\ell+1}{x}j_\ell$ | Note, $j_\ell(k(\eta_0-\eta))$ tells us at which angular scale $\ell \sim k\eta_0$ a perturbation with k contributes

$$\Delta_{T_\ell}(k,\eta_0) = \int \frac{d\mu}{2} P_\ell \Delta_T(k,\eta_0,\mu) = \iint \frac{d\mu}{2} P_\ell e^{ik\mu(\eta-\eta_0)} d\eta \, S_{T,S}(k,\eta) = \int d\eta \, S_{T,S} \, j_\ell[k(\eta_0-\eta)]$$

$$\simeq \left[\Delta_{T_0}(k,\eta_R) + \Psi(k,\eta_R)\right] j_\ell[k(\eta_0-\eta_R)] + \Rightarrow \text{OBSERVED MONOPOLE TODAY (A)}$$

$$+ \frac{i v_b(k,\eta_R)}{k}\left[j_{\ell-1} - (\ell+1)\frac{j_\ell}{k}\right] + \Rightarrow \text{DIPOLE (Doppler) (B)}$$

$$+ \int_0^{\eta_0} d\eta \, e^{-\tau}\left[\Psi'(k,\eta) - \Phi'(k,\eta)\right] j_\ell + \Rightarrow \text{INTEGRATED SACHS-WOLFE (C)}$$

$$+ \frac{3}{4k^2}\Pi(k,\eta_R)\left[j_{\ell-2} - \frac{2\ell+1}{k} j_{\ell-1} + \frac{(\ell+1)^2}{k^2} j_\ell\right][k(\eta_0-\eta_R)] \Rightarrow \text{POLARIZATION CORRECTION (D)}$$

(A) We observe today $(\Delta_{T_0} + \Psi)(\eta_R)$ because γ had to travel out from potential wells at η_R

(B) Doppler shift generates a dipolar pattern. From the Boltz. eq. of baryons
$v_b \simeq -3i \Delta_{T_1} \rightarrow \text{DIPOLE} + \mathcal{O}(1/\tau')$
Note it is incoherent WRT monopole: (A) $\propto j_\ell$, (B) $\propto j'_\ell \Rightarrow \int dx \, j_\ell j'_\ell = 0$

(C) INTEGRATED SW only present when time-varying potentials, i.e. MR equality and Λ domination. Note, it adds coherently with (A) (when integrate by parts, (C) $\propto j_\ell$). Scales affected are those smaller than horizon at the time of time-varying potentials: $\ell \sim k(\eta_0 - \eta_{ISW})$, $\underbrace{k\eta_{ISW} > 1}_{\text{sub-hor. condition}} \Rightarrow \ell \lesssim \frac{\eta_0 - \eta_{ISW}}{\eta_{ISW}}$

If $\eta_{ISW} \ll \eta_0$ (Early ISW) $\Rightarrow \ell < \eta_0/\eta_{ISW}$
If $\eta_{ISW} \approx \eta_0$ (late ISW) $\Rightarrow$ only very small ℓ are affected

(D) Polarization and quadrupole feedback. Small but important contribution (spectra affected at 10% level), necessary for accurate predictions.

Let's got with scalar polarization

$$\Delta_P' + ik\mu\Delta_P - \dot{\tau}\Delta_P = \frac{\dot{\tau}}{2}(1-P_2)\Pi$$

$$\Downarrow$$

$$\mathring{\Delta}_P(\eta_0,k,\mu) = \int_0^{\eta_0} d\eta\, e^{ik\mu(\eta-\eta_0)} e^{-\tau(\eta,\eta_0)}\left[\frac{3}{4}(\mu^2-1)\Pi\dot{\tau}\right] =$$

$$= \frac{3}{4}(1-\mu^2)\int_0^{\eta_0} d\eta\, e^{ik\mu(\eta-\eta_0)} g(\eta)\,\Pi(k,\eta)$$

We need to derive the expression for Δ_E using $\eth^2$ (remember $\bar{\eth}^2 = \eth^2$ for scalar pol.): $\eth^2 f = \bar{\eth}^2 f = \left(\frac{\partial}{\partial\mu}\right)^2\left[(1-\mu^2)f\right]$

$$\Delta_E = \eth^2\Delta_P = -\frac{3}{4}\int d\eta\, \cancel{e^{ik\mu(\eta-\eta_0)}}\, g(\eta)\,\Pi(k,\eta)\,\underbrace{\partial_\mu^2\left[(1-\mu^2)^2 e^{i\mu x}\right]}_{-(1+\partial_x^2)^2(x^2 e^{i\mu x})} \qquad x = k(\eta-\eta_0)$$

$$(1+\partial_x^2)^2 \Rightarrow (1+\partial_x^4+2\partial_x^2)$$

Then, expand (actually, we first expand, then apply $\eth^2$):

$$\cancel{\Delta_{E,\ell} = \int\frac{d\mu}{2}P_\ell\Delta_E = \frac{3}{4}\int d\eta\, g(\eta)\Pi(k,\eta)\int\frac{d\mu}{2}(1-\mu^2)e^{ikx}P_\ell = \frac{3}{4}\int d\eta\, g\,\Pi(1-}$$

$$\Delta_{E,\ell} = \int\frac{d\mu}{2}P_\ell\,\Delta_E = \frac{3}{4}\int d\eta\, g(\eta)\,\Pi(k,\eta)\underbrace{\int\frac{d\mu}{2}(1+\partial_x^2)^2\left[x^2 e^{i\mu x}P_\ell\right]}_{x^2 j_\ell} =$$

[Note, we can take the derivative out and integrate first over μ]

$$= \frac{3}{4}\int d\eta\, g(\eta)\,\Pi(k,\eta)\left[1+\partial_x^2\right]^2\left(x^2 j_\ell(x)\right)$$

$$S_{E,S}(k,\eta) = \frac{3}{4}\left[1+\partial_x^2\right]^2(x^2 j_\ell)\Pi = \frac{3}{4}\frac{j_\ell(k(\eta-\eta_0))}{[k(\eta-\eta_0)]^2}\Pi(k,\eta)$$

Use Bessel property

$$j_\ell'' + \frac{2j_\ell'}{x} + \left[1-\frac{\ell(\ell+1)}{x^2}\right]j_\ell = 0$$

In the peaked limit of $g(\eta)$:

$$\Delta_{E,\ell}(k,\eta_0) \simeq \frac{3}{4}\Pi(k,\eta_R)\frac{j_\ell(k(\eta_0-\eta_R))}{[k(\eta_0-\eta_R)]^2}$$

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In the tightly-coupled limit, we have 1) $\Pi \sim \Delta_{T_2}$; 2) $\Delta_{T_2} \propto -\frac{k}{\tau'} \Delta_{T_1}$, so:

$$\Delta_{E_\ell}(k,\eta_0) \propto -\frac{k}{\tau'} \Delta_{T_1}(k,\eta_e) \frac{j_\ell(k(\eta_0-\eta_e))}{[k(\eta_0-\eta_e)]^2}$$

⇒ The pol. anisotropy is smaller than Δ_T by k/τ' ⇒ Δ_E is generated by Δ_{T_2}, which is suppressed in the early Universe due to scattering

⇒ $\Delta_{E_\ell} \propto \Delta_{T_1}$, so it shares the same features. We will see later that those are 1) acoustic oscillations ; 2) be out-of-phase wrt Δ_{T_0}

⇒ There is no ISW. The metric does not directly affect Δ_{E_ℓ}, which can be regarded as a pristine picture of the CMB field

So, in order to evolve the full set of Δ_{T,E_ℓ}, we only need to know a few ells, to be inserted in the source functions $S_{T,E}$. Let's see them.
We start from the full Boltz. eq. for $\Delta_{T,P}$, expand $\Delta_{T,P}$ in Leg. polyn. and project it to different P_ℓ, using the recursion $(\ell+1)P_{\ell+1} = (2\ell+1)P_\ell \mu - \ell P_{\ell-1}$:

$$\Delta'_{T_0} = -k\Delta_{T_1} - \Phi'$$

$$\Delta'_{T_1} = \frac{k}{3}\left(\Delta_{T_0} - 2\Delta_{T_2} + \Psi\right) + \tau'\left(-\frac{v_b i}{3} + \Delta_{T_1}\right)$$

$$\Delta'_{T_2} = \frac{k}{5}\left(2\Delta_{T_1} - 3\Delta_{T_3}\right) + \tau'\left(\Delta_{T_2} - \frac{\Pi}{10}\right)$$

$$\Delta'_{T_\ell} = \frac{k}{2\ell+1}\left(\ell\Delta_{T_{\ell-1}} - (\ell+1)\Delta_{T_{\ell+1}}\right) - \tau'\Delta_{T_\ell} \quad ; \ \ell > 2$$

$$\Delta'_{P_\ell} = \frac{k}{2\ell+1}\left(\ell\Delta_{P_{\ell-1}} - (\ell+1)\Delta_{P_{\ell+1}}\right) + \tau'\left[\Delta_{P_\ell} - \frac{\Pi}{2}\left(\delta_{\ell 0} + \frac{\delta_{\ell 2}}{5}\right)\right] \ ; \ \cancel{\ell > 2} \ \cancel{\text{Note}}$$

Let's start from Δ'_{T_ℓ}. In the tightly coupled regime ($\tau \gg 1$), we have:

$\Delta'_{T_\ell} \sim \frac{\Delta_{T_\ell}}{\eta} \ll \frac{\tau}{\eta}\Delta_{T_\ell}$: ~~Neglecting~~ $\Delta_{T_\ell} \sim \frac{1}{\tau'}\frac{k}{2\ell+1}\left(\ell\Delta_{T_{\ell-1}} - (\ell+1)\Delta_{\ell+1}\right)$ so that

$\Delta_{T_\ell} \ll \Delta_{T_{\ell-1}}$ ⇒ higher order are suppressed by scattering. Same consideration applies to Δ_{T_2} (remember that Π is sourced when γ-e ~ decoupling) and to Δ_{P_ℓ}.

So, in tight-coupling, we have:

$\Delta'_{T_0} \simeq -k\Delta_{T_1} - \Phi'$

$\Delta'_{T_1} = \frac{k}{3}(\Delta_{T_0} + \Psi) + \tau'\left(\Delta_{T_1} - \frac{i v_b}{3}\right)$

We add to this the eq. for v_b:

$v_b = -3i\Delta_{T_1} + \frac{R}{\tau'}\left(v_b' + \frac{a'}{a} v_b + ik\Psi\right)$

$R = 3/4\,(\rho_b/\rho_\gamma) \rightarrow \begin{cases} 0\text{; early times} \\ \text{const; late times} \end{cases}$

We will study this system in 3 regimes: large, intermediate and small angular scales ($k \ll 1$; $k \sim 1$; $k \gg 1$).

=) LARGE SCALES

We can neglect all terms with k, so that:

$\Delta'_{T_0} \simeq -\Phi' \Rightarrow \Delta_{T_0} \simeq -\Phi + \text{const.}$

$\Delta'_{T_1} \simeq \frac{\Delta_{T_1}}{\eta} \ll \tau'\left(\Delta_{T_1} - \frac{i v_b}{3}\right) \Rightarrow \Delta_{T_1} \simeq \frac{i v_b}{3}$ | This justifies the discussion in Δ_{E_ℓ}

The const. in $\Delta_{T_0} \simeq -\Phi + C$. can be obtained from initial conditions set by inflation. If adiabatic, solving Einst. equations at early times gives $\Delta_{T_0} \simeq -\Phi + \mathcal{R}$ | At $\eta = 0 \Rightarrow \Delta_{T_0} = +\Phi/2 \Rightarrow$ const. must be such that this holds const $= \mathcal{R} = -3/2\Psi = 3/2\Phi$

Using the evolution equation for Φ in the large-scale regime (again, Einst.) we have $\Phi \simeq \frac{3}{5}\mathcal{R} \Rightarrow \Delta_{T_0}(k, \eta_R) = \frac{2}{5}\mathcal{R}(k)$

Remember that the observed anisotropy is $\Delta_{T_0} + \Psi$ (SW):

$$(\Delta_{T_0} + \Psi)(k, \eta_R) \simeq (\Delta_{T_0} - \Phi)(k, \eta_R) \simeq -\frac{1}{5}\mathcal{R}(k)$$

This expression allows to relate $\Delta_{T_0} + \Psi$ to the dark matter fluctuations, by deriving the exps. of $\delta_c(k, \eta_R)$ from its Boltz. + adiabatic initial conditions: $\delta_c(k, \eta_R) = \frac{6}{5}\mathcal{R}(k)$:

$$\boxed{(\Delta_{T_0} + \Psi)(k, \eta_R) = -\frac{1}{6}\delta_c(k, \eta_R)}$$

=) Note, if $\delta_c > 0$ the CMB anis. is $< 0 \Rightarrow \gamma$ loose energy when climbing out overdense regions at REC.

=) Note, $\Delta T/T \sim 1/6\, \Delta\rho_c/\rho_c$. Any cosmological model, to be viable, MUST account for this prediction

⇒ INTERMEDIATE SCALES

We can combine the tight-coupling eqs. for Δ_T, v_b to get a single, second order eq. for Δ_{T_0}:

$$\boxed{(\Delta_{T_0}+\Phi)''+\frac{a'}{a}\frac{R}{1+R}(\Delta_{T_0}+\Phi)'+k^2c_s^2\Delta_{T_0}=-\frac{k^2}{3}\Psi}$$ Forced harmonic oscillator

The monopole undergoes acoustic oscillations with $c_s=\frac{1}{\sqrt{3}}\frac{1}{\sqrt{1+R(\eta)}}$

The solution for $\Delta_{T_0}+\Phi$ can be built from the homogeneous solutions $S_1=\sin(kr_s(\eta))$; $S_2=\cos(kr_s(\eta))$, where $\underbrace{r_s}_{\text{Sound horizon}}=\int_0^{\eta_s}d\eta\, c_s(\eta)$

$$(\Delta_{T_0}+\Phi)(k,\eta)=C_1^{(k)}S_1(\eta)+C_2(k)S_2(\eta)+\frac{k}{\sqrt{3}}\int_0^{\eta}d\tilde{\eta}\,[\Phi-\Psi]\sin(k(r_s(\eta)-r_s(\tilde{\eta})))$$

C_1, C_2 set from initial conditions:

$\Delta_{T_0}'+\Phi'(\eta=0)\to 0 \Rightarrow C_1=0$

$C_2=(\Delta_{T_0}+\Phi)(k,0)$

Note: initial conditions from inflation only excite cosine modes! This is key to have coherent oscillations when k-modes re-enter the horizon.
Alternative models where perturb. are generated inside the horizon must account for this (they generally predict both sin and cos $\neq 0$).

Note also, $k_{peak}\simeq n\pi/r_s$ ($\to \ell_{peak}\sim k_p\eta_e$). However, spectra peak at slightly different ℓ_p because of the zeros of the Bessel.

From $\Delta_{T_0}'=-k\Delta_{T_1}-\Phi' \Rightarrow \Delta_{T_1}\simeq -1/k\,\Delta_{T_0}' \Rightarrow$ Dipole is OUT-OF-PHASE WRT monopole.

⇒ SMALL SCALES

At scales $k \gg \tau'$, i.e., $k \gg k^{MFP}$, γ are very frequently scattered (i.e., the length of the perturbation is much smaller than the photon mean free path). At these scales, oscillations are exponentially damped

Let's now compute the spectra:

=> TEMPERATURE

$$C_\ell^{TT} = \frac{1}{2\ell+1} \sum_m \langle a^T_{\ell m} a^{*T}_{\ell m} \rangle = \frac{1}{2\ell+1} \sum_m \left| \int d\Omega \, Y_{\ell m}(\theta,\varphi) \, T(\theta,\varphi) \right|^2 =$$

$$= \frac{1}{2\ell+1} \sum_m \left| \int d\Omega \, Y_{\ell m} \left[\int dk^3 \, \delta^{in}_{sc}(k) \, \Delta_T(\eta_0, k, \mu) \right] \right|^2 =$$

$$= \int (4\pi)^2 \, d\ln k \, P_S(k) \left[\int_0^{\eta_0} d\eta \, S_T(k,\eta) \, j_\ell(k,\eta) \right]^2 =$$

$$= 4\pi^2 \int d\ln k \, P_S(k) \, \Delta^2_{T\ell}(\eta_0, k)$$

-> Large scales: we directly see imprints of initial conditions (remember the monopole ~ $R(k)$). If the P_S is scale invariant, we can prove that

$\ell(\ell+1) C_\ell \propto A_S$

At these scales, we also see the late ISW.

-> Intermediate scales: acoustic peaks. First peak = one full compression. We have to consider monopole + dipole + ISW.

1) Monopole and dipole are out-of-phase => this makes through less pronounced

2) " " are incoherent => when the dipole is added, it contributes less than expected (i.e., if $\Delta_{T_1} \sim 30\% \, \Delta_{T_0}$, $C_\ell^{DIP} \sim 10\% \, C_\ell^{TEM}$)

3) Early ISW affect $\ell \lesssim \eta_0/\eta_{[illegible]} \sim 100\text{-}200$. It is in phase with monopole, so it gives a big boost to C_ℓ where contributes (5% E-ISW Δ_T -> 10% effect in C_ℓ)

-> Small scales: exponential damping ($e^{-\ell/\ell_D}$) of oscillations on scales $\ell \sim k\eta \gg \ell_{DAMP} \sim k_D \eta$

⇒ POLARIZATION

$$C_\ell^{EE} = \frac{1}{2\ell+1} \langle |a_\ell^{EE}|^* \rangle = \frac{1}{2\ell+1} \overset{(\ell-2)!/(\ell+2)!}{\sum_m} \left| \int d\Omega\, Y_{\ell m}\, E(\hat{n}) \right|^2 =$$

$$= \frac{1}{2\ell+1} \frac{(\ell-2)!}{(\ell+2)!} \int d^3k\, P_S(k) \sum_m \left| \int \frac{3}{4} d\Omega\, Y_{\ell m} \int d\eta\, g(\eta)\, \Pi\, \frac{j_\ell(k\eta)}{[k(\eta_0-\eta)]^2} \right|^2 =$$

$$= \frac{(\ell+2)!}{(\ell-2)!} (4\pi)^2 \int d\ln k\, P_S(k) \left[\frac{3}{4} \int_0^{\eta_0} d\eta\, S_{E,S}(k,\eta) \right]^2 =$$

$$= \frac{\cancel{(\ell+2)!}}{\cancel{(\ell-2)!}} (4\pi)^2 \int d\ln k\, P_S(k)\, \tilde{\Delta}^2_{E_\ell}(k,\eta_0) \quad ; \quad \tilde{\Delta}_{E_\ell}(k,\eta_0) = \sqrt{\frac{(\ell+2)!}{(\ell-2)!}}\, \Delta_{E_\ell}(k,\eta_0)$$

We have acoustic oscillations (feature of the dipole) more pronounced and out-of-phase with C_ℓ^{TT} (C_ℓ^{TT} driven by monopole).
At large scales, we have a peak due to REIONIZATION: same physics of rec., but happening at later times ⇒ $g(\eta)$ gets a new peak at late η, which projects at small ℓ ⇒ due to new scattering, amplitude is suppressed at $\ell > k\eta_{rec} \sim 100$ exponentially.

$$C_\ell^{TE} = \frac{1}{2\ell+1} \sum_m \langle a^T_{\ell m} a^{E*}_{\ell m} \rangle = \int d\ln k\, P_S(k)\, \Delta_\ell^T(k,\eta_0)\, \Delta_\ell^E(k,\eta_0)$$

→ Crucial feature of C_ℓ^{TE} at $\ell < 200$. Remember that coherent oscillations are possible because inflation excites only cos modes. However, in $C_\ell^{TT,EE}$ we observe oscillations at $\ell \gtrsim 200$, i.e. modes within a H at recomb. One may argue that other effects generate these features (other than inflation). The anti-correlation in C_ℓ^{TE} at $\ell < 200$ tells us that 1) it is in agreement with T and E being in agreement with cos initial conditions set OUTSIDE the horizon at recombination.

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TENSOR

We compute the source functions and transfer functions for tensor CMB. Let's start with $\Delta_T^{(T)} \to$ omit T:

$$\tilde{\Delta}_T' + ik\mu\tilde{\Delta}_T - \tau'\tilde{\Delta}_T = -\frac{h'}{2} - \tau'\Pi^{(T)}$$

$$\Delta_T(\eta,\vec{k},\mu) \underset{\text{def}}{=} \left[(1-\mu^2)e^{2i\varphi}h_+(\eta,\vec{k}) + (1-\mu^2)e^{-2i\varphi}h_\times(\eta,\vec{k})\right]\tilde{\Delta}_T(\eta,k,\mu)$$

Apply line of sight:

$$\tilde{\Delta}_T(\eta_0,\mu,k) = \int_0^{\eta_0} d\eta\, e^{ix\mu}\left[-\frac{h'}{2}e^{-\tau} + g(\eta)\Pi^{(T)}\right] \overset{\text{by parts}}{=}$$

$$= \int_0^{\eta_0} d\eta\, e^{ix\mu} g(\eta)\left[\frac{h}{2} + \Pi^{(T)}\right] = \int_0^{\eta_0} d\eta\, e^{ix\mu} S_{T,T}(\eta,k)$$

In the instantaneous rec. limit $(g(\eta)\to g(\eta_R)=\delta(\eta-\eta_R))$ and neglecting feedback from $\Pi^{(T)}$:

$$S_{T,T}(k,\eta) \simeq \underbrace{\frac{h(k)}{2}}_{\text{Metric is dominant contribution}} J_R$$

| Note, this must be corrected to account for μ,φ dep. in Δ_T

~~Let's derive the correct expression for Δ_T.~~ Let's go with Δ_P:

$$\Delta_P' + ik\mu\Delta_P - \tau'\Delta_P = \tau'\Pi^{(T)}$$

$$\Delta_P(\eta_0,k,\mu) = -\int_0^{\eta_0} d\eta\, e^{ix\mu}\, g(\eta)\,\Pi^{(T)} = \int_0^{\eta_0} d\eta\, e^{ix\mu}\, S_{P,T}(k,\eta)$$

Note, it depends only on Π!

We now need to derive the expressions for $\Delta_{E,B}$, remembering that:

$$(\Delta_Q \pm i\Delta_U)(\eta_0,\vec{k},\mu) = \left[(1\mp\mu)^2 e^{2i\varphi}h_+ + (1\pm\mu)^2 e^{-2i\varphi}h_\times\right]\int_0^{\eta_0} d\eta\, e^{ix\mu} S_{P,T}(k,\eta)$$

Now, $\Delta_U \neq 0$, so $\bar{\eth}^2 \neq \eth^2$ and $\Delta_E = -\frac{1}{2}\left[\bar{\eth}^2(Q+iU) + \eth^2(Q-iU)\right]$

$$\Delta_B = \frac{i}{2}\left[\bar{\eth}^2(Q+iU) - \eth^2(Q-iU)\right]$$

Let's apply $\eth^2, \bar{\eth}^2$. We do it for the comp with h_+ ($h_\times$ is analogous):

$$\bar{\eth}^2 (Q \pm iU)_+ = h_+ e^{2i\varphi} \int_0^{\eta_0} d\eta \, S_P(k,\eta) \underbrace{\left[-\partial_\mu \pm \frac{2}{1-\mu^2} \right]^2}_{\bar{\eth}^2} \left[(1-\mu^2)(1\mp\mu)^2 e^{ix\mu} \right] =$$

Define $\mathcal{E}(x) = -12 + x^2[1-\partial_x^2] - 8x\partial_x$; $\mathcal{B}(x) = 8x + 2x^2\partial_x$

$x = k(\eta_0 - \eta)$ So that μ-dep exits

$$= h_+ e^{2i\varphi} \int_0^{\eta_0} d\eta \, S_P(k,\eta) \left[-\mathcal{E}(x) \mp i \mathcal{B}(x) \right] \left[(1-\mu^2) e^{ix\mu} \right]$$

(with $h_\times$, select sign in $\mp \mathcal{B}(x)$). So:

$$\Delta_{\tilde{E}}(\eta_0, \vec{k}, \mu) = \left[(1-\mu^2) e^{2i\varphi} h_+ + (1-\mu^2) e^{-2i\varphi} h_\times \right] \mathcal{E}(x) \int_0^{\eta_0} d\eta \, e^{ix\mu} S_P(k,\eta)$$

$$\Delta_{\tilde{B}}(\eta_0, \vec{k}, \mu) = \left[(1-\mu^2) e^{2i\varphi} h_+ - (1-\mu^2) e^{-2i\varphi} h_\times \right] \mathcal{B}(x) \int_0^{\eta_0} d\eta \, e^{ix\mu} S_P(k,\eta)$$

Spectra. Let's use the following properties of $Y_{\ell m}, P_\ell$:

$$Y_{\ell m} = \left[\frac{(2\ell+1)}{4\pi} \frac{(\ell-m)!}{(\ell+m)!} \right]^{1/2} P_\ell^m e^{im\varphi} \quad ; \quad P_\ell^m = (-1)^m (1-\mu^2)^{m/2} \frac{d^m}{d\mu^m} P_\ell$$

$$\text{So } \sum_m Y_{\ell m}^* e^{2i\varphi} = \left[\frac{2\ell+1}{4\pi} \frac{(\ell-2)!}{(\ell+2)!} \right]^{1/2} P_\ell^2 = [\ldots]^{1/2} (1-\mu^2) \partial_\mu^2 P_\ell$$

And (similar to scalar E):

$$\partial_\mu^2 \, P_\ell(\mu) (1-\mu^2)^2 e^{ix\mu} \longrightarrow (1+\partial_x^2)^2 (x^2 e^{ix\mu} P_\ell(\mu))$$

$$C_\ell^{TT} = \frac{1}{2\ell+1} \sum_m \left| \int d\Omega \, Y_{\ell m}^* \, T(n) \right|^2 = \frac{4\pi}{2\ell+1} \int d\ln k \, P_T(k) \sum_m \left| \int d\Omega \, Y_{\ell m}^* \int d\eta \, S_{TT} (1-\mu^2) e^{2i\varphi} e^{ix\mu} \right|^2$$

$$= \frac{(\ell-2)!}{(\ell+2)!} \int d\ln k \, P_T(k) \left| \int d\eta \, S_{TT}(k,\eta) \int d\mu \, P_\ell^2 (1-\mu^2) e^{ix\mu} \right|^2 =$$

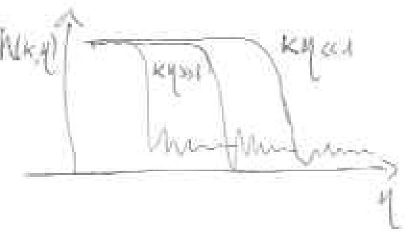
$$= \ldots \left| \int d\eta \, S_{TT} \int d\mu \, \partial_\mu^2 P_\ell (1-\mu^2)^2 e^{ix\mu} \right|^2 =$$

$$= \ldots \int d\mu \, P_\ell(\mu) (1+\partial_x^2)^2 (x^2 e^{ix\mu}) \Big|^2 =$$

$$= (4\pi)^2 \frac{(\ell+2)!}{(\ell-2)!} \int d\ln k \, P_T(k) \left| \int_0^{\eta_0} d\eta \, S_{TT}(k,\eta) \frac{j_\ell(x)}{x^2} \right|^2 = (4\pi)^2 \int d\ln k \, P_T(k) \left| \Delta_T^{(T)}(k,\eta_0) \right|^2$$

→ Main source is $h(k,\eta)$ ⟹ $h'' + 2\frac{a'}{a}h' + k^2 h = 0$

Large modes ($k \ll aH$) remain constant ⟹ $h(k_L,\eta) \sim \text{const}$ until $k\eta < 1$.
Sub-horizon modes decay quickly and oscillate (oscillation seen from static eq. $h''+k^2h=0$; decay due to Hubble friction!)



So we expect C_ℓ^{TT} to exhibit same behaviour:

⟹ $C_\ell^{TT} \sim$ const until ℓ-scale corresponding to scale entering the horizon at η_R ($\ell \sim 100$)

C_ℓ^{TT} quickly decaying and oscillating at larger ℓs

$$C_\ell^{EE} = \frac{4\pi}{2\ell+1}\left|\int d\Omega\, Y_{\ell m}^* E(n)\right|^2 = \frac{4\pi}{2\ell+1}\int d\ln k\, P_T(k) \sum_m \left|\int d\Omega\, Y_{\ell m}^* \int d\eta\, S_{P_T}(k,\eta)(1-\mu^2)e^{2i\varphi}e^{ix\mu}\right|^2 =$$

Same as C_ℓ^{TT}?

$$\overset{?}{=} (4\pi)^2 \int d\ln k\, P_T(k) \left|\int d\eta\, \underbrace{S_{P_T}(k,\eta)\, \mathcal{E}(x)}_{S_{E_T}} \frac{J_\ell(x)}{x^2}\right|^2 = (4\pi)^2 \int d\ln k\, P_T(k) \left|\Delta_E(k,\eta_0)\right|^2$$

$$C_\ell^{BB} = (4\pi)^2 \int d\ln k\, P_T(k) \left|\int d\eta\, \underbrace{S_{P_T}(k,\eta)\, \mathcal{B}(x)}_{S_{B_T}} \frac{J_\ell(x)}{x^2}\right|^2 = (4\pi)^2 \int d\ln k\, P_T(k) \left|\Delta_B(k,\eta_0)\right|^2$$

$$C_\ell^{TE} = (4\pi)^2 \int d\ln k\, P_T(k) \int d\eta\, \Delta_T^{(T)}(k,\eta_0)\, \Delta_E^{(T)}(k,\eta_0)$$

⟹ $S_{P_T} \sim \Pi^{(T)}$ ⟹ We expect the signal to be peaked at the scale entering the horizon at η_R (RECOMBINATION BUMP), and quickly decay
⟹ Due to reionization, we have a peak at $\ell \sim$ a few

⟹ $\langle h_+ h_\times^* \rangle = 0$ ⇏ $a_{B\ell m} = -a_{B\ell m}^*$ ⟹ $C_\ell^{TB} = C_\ell^{EB} = \emptyset$