Gravitational waves from compact objects

Lecture V

Lectures for the 2022 GGI APCG School

Leonardo Gualtieri
University of Rome “La Sapienza”

THE POST-NEWTONIAN EXPANSION

Besides the slow-motion approximation, in late inspiral we have to *relax the weak-field approximation*. This can be done with a (very complex) perturbative approach called **post-Newtonian (PN) expansion**.

The first step towards PN expansions is to reformulate Einstein's equations as follows: let's define

$$H^{\mu\nu} \equiv \eta^{\mu\nu} - (-g)^{1/2} g^{\mu\nu} \,,$$

and choose a gauge (harmonic gauge) such that $H^{\mu\nu}_{,\nu} = 0$. Then, Einstein's equations (*with no approximation!*) ca be rewritten as

$$\Box_F H^{\mu\nu} = -\frac{16\pi G}{c^4} \tau^{\mu\nu}$$

where

$$\tau^{\mu\nu} = (-g) T^{\mu\nu} + \text{ terms quadratic in } H^{\alpha\beta}_{,\gamma} \,. \tag{1}$$

Thus,

$$H^{\mu\nu} = \frac{4G}{c^4} \int \frac{\tau^{\mu\nu}\left(t - \frac{|\vec{x}-\vec{x}'|}{c}, \vec{x}'\right)}{|\vec{x}-\vec{x}'|} d^3x' \tag{2}$$

which is an *exact* expression, but it is not a close expression, since $\tau^{\mu\nu}$ depends on $H^{\mu\nu}$. This equation can be solved iteratively, as follows:

$$H^{\mu\nu} = H^{(0)\mu\nu} + H^{(1)\mu\nu} + H^{(2)\mu\nu} + \dots$$
$$\tau^{\mu\nu} = \tau^{(0)\mu\nu} + \tau^{(1)\mu\nu} + \tau^{(2)\mu\nu} + \dots$$

and $\tau^{(0)\mu\nu} = (-g) T^{\mu\nu}$, $\tau^{(1)\mu\nu}$ is given by the quadratic terms in (1) with $H^{\mu\nu}$ replaced with $H^{(0)\mu\nu}$ and so on. Then, solving (2) order by order.

$$H^{(0)\mu\nu} = \frac{4G}{c^4} \int \frac{(-g) T^{\mu\nu}\left(t - \frac{|\vec{x}-\vec{x}'|}{c}, \vec{x}'\right)}{|\vec{x}-\vec{x}'|} \tag{3}$$

$$H^{(1)\mu\nu} = \frac{4G}{c^4} \int \frac{\tau^{(1)\mu\nu}\left(t - \frac{|\vec{x}-\vec{x}'|}{c}, \vec{x}'\right)}{|\vec{x}-\vec{x}'|} \tag{4}$$

and so on. It turns out that each term in this expansion has an higher power of G; thus, it is called **post-Minkowskian expansion**.

IF we want to describe a system of selfgravitating compact bodies, depending only on their masses m_i and their positions r_i, dimensional considerations tell us that the order G^n in this expansion should be proportional to the combination

$$\left(\frac{Gm}{c^2 r}\right)^n .$$

Assuming that the dynamics if the system is dominated by the gravitational interactions among the bodies, the virial theorem guarantees that the velocities of the bodies are of the order

$$v \sim \sqrt{\frac{Gm}{r}} .$$

Therefore, *an expansion in orders of* $\frac{GM}{c^2 r}$ *is an expansion in orders of* $\frac{v^2}{c^2}$.

Let's now consider the first term in this expansion, eq. (3). It has the same structure of the metric perturbation in the weak-field regime, and we have seen that it can be written as a multipole expansion, which is an *expansion in orders of* $\frac{v}{c}$. We can then rearrange the terms of these two expansions, by treating it as an unique expansion: we define a dimensionless parameter

$$\epsilon \sim \frac{Gm}{c^2 r} \sim \frac{v^2}{c^2} ,$$

and expand in ϵ all quantities and equations describing the system (the metric, the equations of motion of the bodies, etc.). We call n-**PN order** the order $O(\epsilon^n)$. Note that since $\epsilon \sim v^2/c^2$, there will be terms with integer and half-integer order.

Note also that ϵ does not need to have a specific value: it is a "book-keeping parameter", which can be used to recognize the PN order of any term. Ofter, conventionally, instead of $O(\epsilon^n)$ one writes $O\left(\frac{1}{c^{2n}}\right)$.

This is called PN expansion because the limit of small velocities, which is also of weak field, is the Newtonian limit: the expressions of the equations of motion of the bodies, at 0-PN order, coincide with the Newtonian equations of motion, in terms of the accelerations of the bodies; higher-order terms are the corrections to the Newtonian expressions.

Strictly speaking, the PN expansion of the metric is only allowed in the "near region" where $r \ll \lambda_{GW}$. In this region, the metric can be written as

$$g_{00} = -1 - 2\frac{\Phi}{c^2} + O\left(\frac{1}{c^4}\right)$$
$$g_{0i} = -O\left(\frac{1}{c^3}\right)$$
$$g_{ij} = \left(1 - 2\frac{\Phi}{c^2}\right)\delta_{ij} + O\left(\frac{1}{c^4}\right)$$

where

$$\Phi(t,\vec{x}) = -\frac{G}{c^2}\int \frac{T^{00}\left(t - \frac{|\vec{x}-\vec{x}'|}{c}, \vec{x}'\right)}{|\vec{x}-\vec{x}'|}\,,$$

i.e. it the Newtonian potential, and give the 1-PN correction to the Minkowski metric.

This is just the first step of a very involved procedure, which requires - order by order - finding the motion of the bodies, inclfinding the metric in the far zone and then the GW emission, include the effect of energy loss in the acceleration of the bodies, and so on. At the end, we find the emitted gravitational waveform at the required PN order.

In order to compare the model of the waveform with the observation of the signal from an inspiralling compact binary, and extract the values of parameters of the binary - the masses, the spins, etc. - it is convenient to consider the *Fourier transform of the waveform*:

$$\tilde{h}(\nu) = \tilde{h}_0(\nu) e^{i\tilde{\phi}(\nu)} .$$

In particular, it is convenient to compare the *phase* of the Fourier transform with the data, since it changes more rapidly.

Note that $\tilde{\phi}(\nu)$ is slightly different from the expressions of the phase we have derived. We have found (in the weak-field, slow motion limit) that

$$\phi(t) = -2 \left[\frac{c^3 (t_C - t)}{5G\mathcal{M}} \right]^{5/8} + \phi^{in}$$

and $\nu_{GW} = const./(t_c - t)^{3/8}$. Using the latter expression to remove the dependence on $t_c - t$ we find

$$\phi(\nu_{GW}) = -\frac{1}{16} \left(\frac{c^3}{\pi G\mathcal{M}\nu_{GW}} \right)^{5/3} + \phi^{in} .$$

By performing the Fourier transform (which also involves h_0), one gets an expression with a different numerical coefficient:

$$\tilde{\phi}(\nu) = \frac{3}{128} \left(\frac{c^3}{\pi G\mathcal{M}\nu_{GW}} \right)^{5/3} + const.$$

This comes from the quadrupole formula; since it is the leading contribution to the phase, we call it the 0-PN term; the PN expansion gives corrections to this formula.

It is useful to express the perturbation parameter $\epsilon = v^2/c^2$ in terms of the frequency:

$$l_0 = \frac{(GM)^{1/3}}{\omega_K^{2/3}} = \frac{(GM)^{1/3}}{(\pi\nu_{GW})^{2/3}} \quad \Rightarrow \quad v = \omega_K l_0 = \pi\nu_{GW} l_0 = (GM\pi\nu_{GW})^{1/3}\,.$$

Therefore, we define the *dimensionless frequency*

$$x \equiv \left(\frac{GM\pi\nu}{c^3}\right)^{2/3} = \frac{v^2}{c^2}\,.$$

A term $\sim x^n$ is the of n-PN order. We have

$$\left(\frac{c^3}{\pi G\mathcal{M}\nu_{GW}}\right)^{5/3} = x^{5/2}\frac{\mathcal{M}^{5/3}}{M^{5/3}} = x^{5/2}\frac{\mu M^{2/3}}{M^{5/3}} = x^{5/2}\frac{\mu}{M}\,.$$

Thus if $\eta = \mu/M$ is the *symmetric mass ratio*,

$$\tilde{\phi}(x) = \frac{3}{128\eta}x^{5/2} + \quad \text{PN corrections}\,.$$

The result of the PN computation is

$$\tilde{\phi}(x) = \frac{3x^{-5/2}}{128\eta}\left[1 + \frac{20}{9}\left(\frac{743}{336} + \frac{11}{4}\eta\right)x - 4(4\pi - \beta)x^{3/2} + \dots\right]\,, \qquad (5)$$

where β is a linear combination of χ_1, χ_2 weigthed by the masses. The expression of $\tilde{\phi}(x)$ is known up to 3.5-PN order, and describes with great accuracy the waveform up to the late inspiral, where $v \sim c$ and the PN expansion becomes poorly convergent.

What can we learn from the PN inspiral waveform (5)?

- from the 0-PN term we can measure the chirp mass $\mathcal{M}$;
- from the 1-PN term we can measure separately m_1, m_2;
- from the 1.5-PN term we can measure β, and thus a combination of the two spins χ_1, χ_2;
- from the 2-PN term we can measure separately χ_1, χ_2.

Order by order, we can mesure all features of the system, including spin precession.

Note that the frequency increases with time, and thus while in early inspiral only low PN-order terms (i.e., lower powers of $x \sim \nu^{2/3}$) are relevant, as the inspiral proceeds, higher and higher orders become relevant.

In the later part of the inspiral the PN expansion becomes poorly convergent and ill defined; this approach can not give anymore a reliable description of the waveform. One has to introduce corrections to the PN phase, which depend on unknown coefficient; then, these coefficient are obtained from fits with the numerical relativity (NR) waveform describing the merger. This is called *phenomenological approach*; an alternative method is the *effective-one-body* approach, which is more well founded theoretically and which includes a rearrangement of the PN expansion, and free coefficients to be determined by fits of NR waveforms.

Both phenomenological and effective-one-body waveform give very good descriptions of the GW signal up to the late inspiral. These are fundamental tool for the data analysis of interferometric detectors, and have been used, e.g., in the analysis of GW150914.

FINITE SIZE EFFECTS

The late inspiral waveform of NS-NS binaries differs from that of BH-BH binaries due to **finite-size effects**: the tidal field of each body deforms the companion, and this affects the waveform.

In order to understand finite size effects in compact binary systems, let's consider for the moment the multipole moment expansion of the gravitational potential in Newtonian gravity. The gravitational potentual is the solution of Poisson's equation $\nabla^2\Phi = \pi G\rho$ (I will use units $G = c = 1$ in this part), i.e.

$$\Phi(t,\vec{x}) = -\int \frac{\rho(t,\vec{x}')}{|\vec{x}-\vec{x}'|} d^3x' \,. \tag{6}$$

he Taylor expansion of $1/|\vec{x}-\vec{x}'|$ around $x'^i = 0$ gives

$$\frac{1}{|\vec{x}-\vec{x}'|} = \frac{1}{r} + \frac{n^i x'^i}{r^2} + \frac{3}{2}\frac{n^i n^j - \delta^{ij}/3}{r^3} x'^i x'^j + \ldots,.$$

By replacing in Eq. (6), since $n^i n^j - \delta^{ij}/3 = n^{<i} n^{j>}$,

$$\Phi(t,\vec{x}) = -\frac{1}{r}\int \rho d^3x' - \frac{n^i}{r^2}\int \rho x'^i d^3x' - \frac{3}{2}\frac{n^{<i}n^{j>}}{r^3}\int \rho x'^{<i} x'^{j>} d^3x' \,.$$

Since $\int \rho d^3x' = M$, $\int \rho x'^i d^3x'$ is proportional to the position of the center of mass, which vanishes by choosing there the origin, and $\int \rho x'^{<i} x'^{j>} d^3x' = Q^{ij}$,

$$\Phi(t,\vec{x}) = -\frac{M}{r} - \frac{3}{2}\frac{1}{r^3} Q^{ij} n^{<i} n^{j>} + \ldots$$

Let us now consider a static, body with mass M, placed in a static quadrupolar external field. The externail tidal field, expanded around the origin, is

$$\Phi_{ext} = const. + \frac{\partial\Phi_{ext}}{\partial x^i}_{|O} x^i + \frac{1}{2} E_{ij} x^i + x^j + O(r^2) \,,$$

where

$$E_{ij} = \frac{\partial^2 \Phi_{ext}}{\partial x^i \partial x^j}_{|O}$$

is the **tidal field**. We can set to zero the constant, and a tidal field does not exert force on the origin. Moreover, since the external field has no source at the origin, $\nabla^2\Phi_{ext} = \delta^{ij}E_{ij} = 0$. We can conclude that

$$\Phi_{ext} = \frac{1}{2}E_{ij}x^{<i}x^{j>} = \frac{1}{2}E_{ij}r^2n^{<i}n^{j>}\,.$$

The total gravitational potential is then

$$\Phi = -\frac{M}{r} - \frac{3}{2}\frac{1}{r^3}Q_{ij}n^{<i}n^{j>} + O\left(\frac{1}{r^4}\right) + \frac{1}{2}E_{ij}r^2n^{<i}n^{j>} + O(r^3)\,.$$

Note that in this expansion there are terms divergent at infinity: it holds in a *buffer region*, not too close to the body, but not too far away, where there are the sources of the tidal field.

If the body is spherically symmetric by itself, once it is placed in the tidal field it is deformed: the quadrupole moment Q_{ij} is a *consequence* of the tidal field E_{ij}. Indeed, solving Newton's equations one finds that

$$Q_{ij} = -\lambda E_{ij}$$

where λ is called **tidal deformability** of the body. It goes like $\sim R^5$, thus it is also defined the dimensionless **Love number** $k_2 = \frac{3}{2}\frac{\lambda}{R^5}$.

This derivation can be extended to GR, finding that placing a static, spherically symmetric star in an external tidal field

$$g_{00} = -1 + \frac{2M}{r} + \frac{3}{r^3}Q_{ij}n^{<i}n^{j>} + O\left(\frac{1}{r^3}\right) - E_{ij}r^2n^{<i}n^{j>} + O(r^3)$$

where now the tidal tensor is defined in terms of the Riemann tensor:

$$E_{ij} = u^\mu u^\nu R_{\mu i\nu j}\,.$$

By solving the perturbed Einstein's equations one finds that, as in the Newtonian case,

$$Q_{ij} = -\lambda E_{ij}\,.$$

The tidal deformability appears in the tidal waveform (5):

$$\tilde{\phi}(x) = \frac{3x^{-5/2}}{128\eta}\left[1 + \frac{20}{9}\left(\frac{743}{336} + \frac{11}{4}\eta\right)x - 4(4\pi - \beta)x^{3/2} + \cdots - \frac{39}{2}\tilde{\Lambda}x^5\right], \tag{7}$$

where $\tilde{\Lambda}$ is a linear combination of the tidal deformabilities of the two bodies, λ_1, λ_2, weighted by a combination of the masses.

Note that this is a term of 5-PN order. Normally, such term would be negligible; however, NSs have another scale beside the mass: the radius R. Since $\lambda \sim R^5$, the contribution to the PN waveform is enhanced of a factor

$$\sim \left(\frac{R_i}{m_i}\right)^5$$

which, being $R \sim 5m$, gives a factor ~ 3000, which makes the 5-PN correction comparable to lower-order corrections in the expansion.

Indeed, this term *has been measured* in GR170817, finding

$$\tilde{\Lambda} \lesssim 800\,.$$

This is not a precise measurement, still it gives valuable information on the NS structure, allowing to exclude some possible equations of state of the NS matter.

THE MERGER

As mentioned before, to study the merger of the binary we need **numerical relativity** (NR) simulations, i.e. solving numerically Einstein's equations

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 0 \quad \text{(no matter source for BH binaries)}$$

without any approximation.

To this aim, it is necessary to perform a $3+1$ *decomposition of Einstein's equation*: we foliate the spacetime in a sequence of 3-dimensional hypersurfaces Σ_t, each corresponding to a value t of a time coordinate. Then, it is possible to write the metric as

$$ds^2 = (-\alpha^2 + \beta^i\beta_i)dt^2 + 2\beta_i dt dx^i + \gamma_{ij}dx^i dx^j$$

where:

- γ_{ij} is the metric of the three-dimensional surface Σ_t;
- αdt is the *lapse*, i.e. the proper time of an observer moving along the normal of the hypersurfaces;
- β^i is the *shift vetor*, i.e. the relative velocity between the observer along the normal of the hypersurfaces, and the observer at constant coordinates $\{x^i\}$.

The choice of the lapse and of the shift is just a gauge choise; the three-metric γ_{ij}, instead, is a dynamical variable.

Another dynamical variable is the *extrinsic curvature*,

$$K_{ij} = -(\delta_i^\alpha + n^\alpha n_i)n_{j;\alpha}\,,$$

which describes how the hypersurfaces Σ_t are embedded in the spacetime.

Einstein's equations can be reformulated as equations for γ_{ij} and for K_{ij}. In particular:

- Some components of Einstein's equations give elliptic equations, whose solution provides a consistent value of γ_{ij}, K_{ij} on a single hypersurface. These are the so-called **constraints**.
- The other components of Einstein's equations give hyperbolic equations, the **evolution equations**.

Then, once one finds *initial data*, i.e. γ_{ij}, K_{ij} on a single hypersurface, by solving the constraints, the evolution equations give the evolution of these quantities to the other hypersurfaces.

In this way it is possible to describe, starting from the late inspiral, the merger of the binary into a single BH. It took decades to develop this approach, but finally in 2006 the problem was solved, and it was possible to perform NR simulation lf the BH-BH merger, finding the corresponding waveform. Once the GW were detected, with GW150914, it was confirmed that the NR waveform perfectly matched the observed signal emitted during the merger.

The NR simulations also give the values of the mass and spin of the final BH, M_{fin}, χ_{fin}. Then, NR simulations allowed to find analytical fits of M_{fin} and χ_{fin} in terms of the masses and spins of the binary.

THE RINGDOWN SIGNAL

The last part of the signal is the **ringdown**, in which the final BH oscillates at is proper frequencies, called **quasi-normal modes** (QNMs). The frequencies of the oscillation are also the frequencies of the emitted GW signal, which can be directly measured.

The "quasi" is due to the fact that, at variance with normal modes, the QNMs are necessarily *damped* due to energy loss by GW emission: each mode ($j = 0, 1, \dots$) has *complex frequency*

$$\omega^{(j)} = \omega_R^{(j)} + i\omega_I^{(j)}$$

with $\omega_I^{(j)} > 0$. Then, when a BH oscillates with its j-th mode, its time dependence is

$$e^{i\omega^{(j)}t} = e^{i\omega_R^{(j)}t}e^{-\omega_I^{(j)}t} = e^{i\omega_R^{(j)}t}e^{-t/\tau^{(j)}}$$

where $\tau^{(j)} \equiv 1/\omega_I^{(j)}$ is the **damping time** of the j-th mode.

Soon the QNMs are damped and the BH becomes stationary. I recall that, due to the so-called "no-hair theorems", stationary, asymptotically flat BHs without electric charge are described by the Kerr metric. Thus, during the ringdown the final body is a perturbed Kerr BH; soon the perturbations disappear, and a Kerr BH remains.

In the ringdown stage, the waveform has the form

$$h(t) \sim \sum_j A^{(j)} \sin[\omega_R^{(j)}t + \phi^{(j)}]e^{-t/\tau^{(j)}} \ :$$

it is a combination of oscillations of the QNMs of the final, Kerr BH.

The QNMS are found using *perturbation theory* around a curved background:

$$g_{\mu\nu} = g_{\mu\nu}^{(0)} + h_{\mu\nu} \quad |h_{\mu\nu}| \ll |g_{\mu\nu}^{(0)}|, .$$

The background metric is not Minskowski (the weak field approximation is not satisfied), it is the metric of the final, stationary Kerr BH, which is only characterized by two parameters: the mass M and the angular momentum J. Solving Einstein's equations linearized around the Kerr background, it is possible to find the frequencies and damping times of the Kerr QNMs as functions of M and J:

$$\omega_R^{(j)}(M,J)\,, \qquad \tau^{(j)}(M,J)\,.$$

In principle, by measuring these quantities from the ringwdown waveform, it is possible to find mass and angular momentum of the final BH. In practice, there is one mode with a larger excitation amplitude, and it is easier to extract M, J from this mode: this is the **fundamental mode** $j = 0$.

Let us consider, for simplicity, a non-rotating BH. In this case, the fundamental QNM has

$$\frac{G}{c^3}M\omega^{(0)} \simeq 0.3736 + i\,0.0890\,.$$

In more common units:

$$\nu^{(0)} = \frac{\omega^{(0)}}{2\pi} \simeq \frac{12}{M/M_\odot}\,\text{KHz}\,, \qquad \tau^{(0)} \simeq \left(\frac{M}{M_\odot}\right) 5.5 \cdot 10^{-5}\,s\,.$$

Thus if, for instance, $M = 60\,M_\odot$ (as for GW150914) then $\nu^{(0)} \simeq 200$ Hz (well within LIGO-Virgo bandwidth) and $\tau^{(0)} \simeq 3.3 \cdot 10^{-3}$ s. If, instead, $M \sim 10^6 M_\odot$, as for the BH at the center of our Galaxy, then $\nu^{(0)} \sim 1.2{\cdot}10^{-2}$ Hz and $\tau \sim 60$ s.

In the case of GW150914, only the fundamental mode has been observed; this observation had a limited accuracy, but it was consistent with the theoretical value corresponding to the values of M_{fin}, J_{fin} obtained from NR.