Quasinormal modes and isomonodromy

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VIRTUS IMPAVIDA

Outline

- Fuchsian ODEs and flat holomorphic connections;
- Riemann-Hilbert map and monodromy parameters;
- Application to quasi-normal modes;
- Application to uniformizing maps of polycircular domains;

Various works with J. Barragán-Amado, J. P. Cavalcante, T. Anselmo, F. Novaes
S. Nejad, E. Pallante, R. Nelson, D. Crowdy

Preamble

$$\frac{d}{dz}\Phi(z) = A(z)\Phi(z), \qquad A(z) = \begin{pmatrix} a_{11}(z) & a_{12}(z) \\ a_{21}(z) & a_{22}(z) \end{pmatrix}, \qquad \Phi(z) = \begin{pmatrix} y_1(z) & y_2(z) \\ w_1(z) & w_2(z) \end{pmatrix}$$

$$\Big\downarrow$$

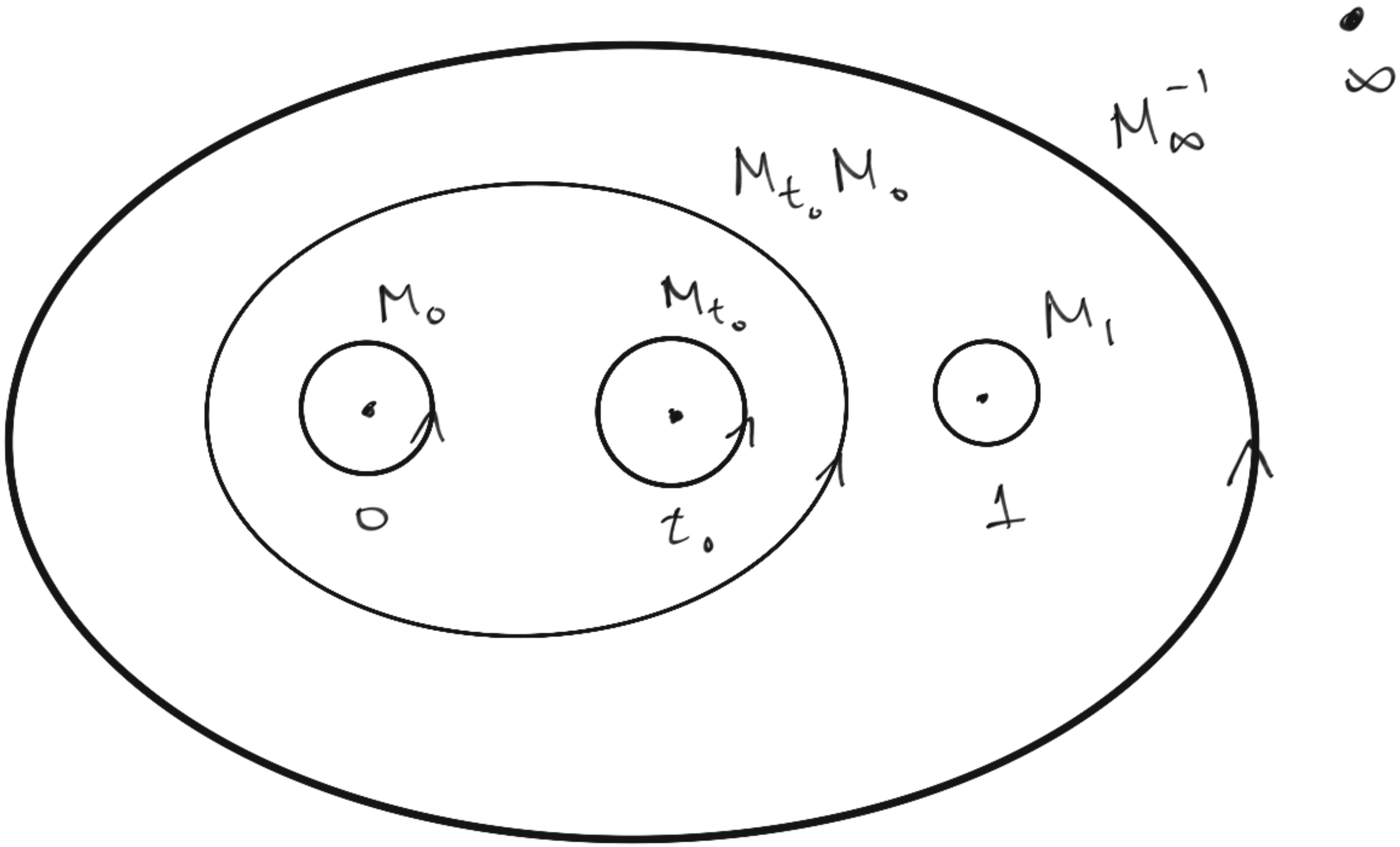
$$\partial_z^2 y - (\mathrm{Tr}\, A + \partial_z \log a_{12})\partial_z y + (\det A - \partial_z a_{11} + a_{11}\partial_z \log a_{12})y = 0.$$

$A(z)$ rational with at most single poles, ODE is Fuchsian. Extra singularity at roots of $a_{12}(z)$.

(Reverse) Riemann-Hilbert problem: how parameters in $A(z)$ affect analytical behavior of solutions?

$$\Phi(z_k + (z - z_k)e^{2i\pi}) = \Phi(z)M_k$$

$$M_k \sim \begin{pmatrix} e^{i\pi\theta_k} & 0 \\ 0 & e^{-i\pi\theta_k} \end{pmatrix}$$



$$M_\infty M_1 M_{t_0} M_0 = \mathbf{1}$$

define Fenchel-Nielsen coordinates

$$\begin{cases} 2\cos\pi\sigma = \operatorname{Tr} M_{t_0}M_0 \\ e^{i\eta} \end{cases}$$

isomonodromic deformations

"Residual gauge symmetry": change parameters of $A(z)$ while keeping monodromy data. Schlesinger equations

$$\frac{\partial A_0}{\partial t} = \frac{1}{t}[A_0, A_t], \qquad \frac{\partial A_1}{\partial t} = \frac{1}{t-1}[A_1, A_t], \qquad \frac{\partial A_t}{\partial t} = -\frac{1}{t}[A_0, A_t] - \frac{1}{t-1}[A_1, A_t],$$

can be translated to the ODE

$$y'' + p(z,t)y' + q(z,t)y = 0,$$

$$p(z,t) = \frac{1-\hat{\theta}_0}{z} + \frac{1-\hat{\theta}_1}{z-1} + \frac{1-\hat{\theta}_t}{z-t} - \frac{1}{z-\lambda}, \qquad q(z,t) = \frac{\kappa_1(\kappa_2+1)}{z(z-1)} - \frac{t(t-1)K}{z(z-1)(z-t)} + \frac{\lambda(\lambda-1)\mu}{z(z-1)(z-\lambda)},$$

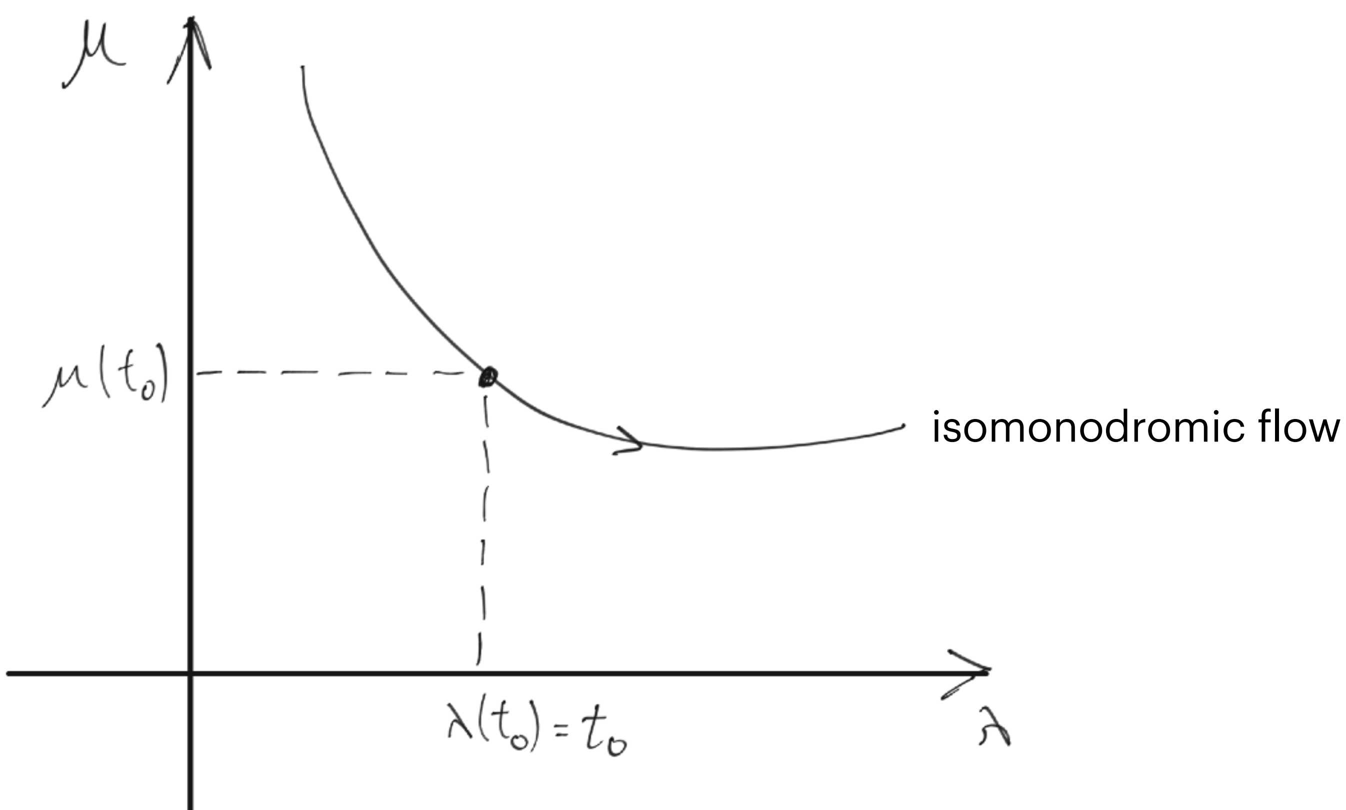
dynamics of apparent singularities maintain monodromy data

The system is Hamiltonian

$$\frac{\partial \lambda}{\partial t} = \frac{\partial K}{\partial \mu}, \qquad \frac{\partial \mu}{\partial t} = -\frac{\partial K}{\partial \lambda},$$

with generating function (Jimbo-Miwa-Ueno tau function)

$$\frac{d}{dt} \log \tau_{JMU}(t) = \frac{1}{t}\mathrm{Tr}(A_0 A_t) + \frac{1}{t-1}\mathrm{Tr}(A_1 A_t),$$



strategy: place initial conditions such that ODE from Schlesinger flow match desired (Heun) ODE

$$\lambda(t_0) = t_0, \qquad \mu(t_0) = \frac{K_0}{\hat{\theta}_{t_0}}$$

Kyiv formula

Full expansion for (isomonodromic, Painlevé VI) tau function is given in terms of monodromy data (GIL2013)

$$\tau(t) = C \sum_m e^{im\tilde{\eta}} \mathcal{B}(\{\theta_k\}, \sigma + 2m; t)$$

$$\mathcal{B}(\{\theta_k\}, \sigma; t) = \mathcal{N}^{\theta_1}_{\theta_\infty, \sigma} \mathcal{N}^{\theta_t}_{\sigma, \theta_0} t^{\frac{1}{4}(\sigma^2 - \theta_0^2 - \theta_t^2)} (1-t)^{\frac{1}{2}\theta_t \theta_1} \sum_{\rho, \nu \in Y} \mathcal{B}(\{\theta_k\}, \sigma) t^{|\rho| + |\nu|},$$

$$\mathcal{B}(\{\theta_k\}, \sigma) = \prod_{(i,j)\in\rho} \frac{((\theta_t + \sigma + 2(i-j))^2 - \theta_0^2)((\theta_1 + \sigma + 2(i-j))^2 - \theta_\infty^2)}{4h_\rho^2(i,j)(\rho'_j - i + \nu_i - j + 1 + \sigma)^2} \prod_{(i,j)\in\nu} \frac{((\theta_t - \sigma + 2(i-j))^2 - \theta_0^2)((\theta_1 - \sigma + 2(i-j))^2 - \theta_\infty^2)}{4h_\nu^2(i,j)(\nu'_j - i + \rho_i - j + 1 - \sigma)^2}$$

$c=1$ conformal blocks, following AGT. Initial conditions are transcendental equations for ODE parameters

$$\tau(\{\theta_k\}; \sigma, \eta; t_0) = 0, \qquad K_0 = \frac{\partial}{\partial t} \log \tau(\{\theta_k\}^-; \sigma - 1, \eta; t_0) - \frac{\theta_t - 1}{2t_0} - \frac{\theta_t - 1}{2(t_0 - 1)}$$

Problem generalizes: any Fuchsian ODE Riemann-Hilbert problem can be solved this way

$$A(z; z_k) = \sum_k \frac{A_k}{z - z_k}, \qquad \frac{\partial A_k}{\partial z_l} = \frac{[A_k, A_l]}{z_k - z_l}, \qquad \frac{\partial A_k}{\partial z_k} = -\sum_{l \neq k} \frac{[A_k, A_l]}{z_k - z_l}$$

(multi-)Hamiltonian flow:

$$\frac{\partial}{\partial z_k} \log \tau(\{\theta_k\}; \{\sigma_k\}, \{\eta_k\}; \{z_k\}) = \sum_{l \neq k} \frac{\operatorname{Tr} A_k A_l}{z_k - z_l}$$

allows transcendental equations that determine accessory parameters from monodromy data

$$\tau_{\mathrm{JMU}}(\hat{\rho}_k^+; \{w_k\}) = 0, \qquad \beta_k = -\frac{\partial}{\partial t_k} \log \tau_{\mathrm{JMU}}(\hat{\rho}; \{w_k\}) + \frac{\hat{\theta}_k}{2w_k} + \frac{\hat{\theta}_k}{2(w_k - 1)}.$$

Why monodromy data?

monodromy data determines (partially) *connection* data

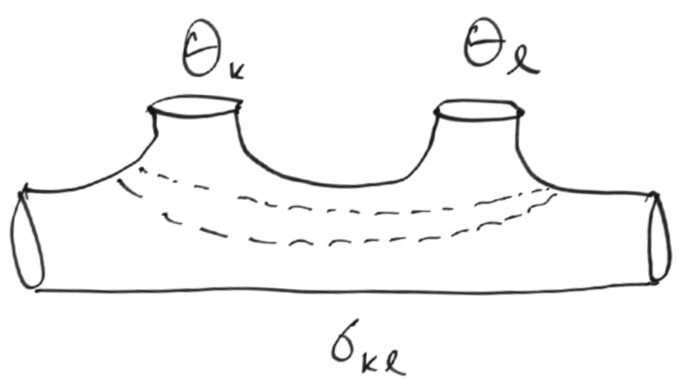
$$\Phi_k(z) = (z - z_k)^{\frac{1}{2}\theta_k\sigma_3}(1 + \mathcal{O}(z - z_k))$$

$$\Phi_k(z) = \Phi_l(z) C_{kl}$$

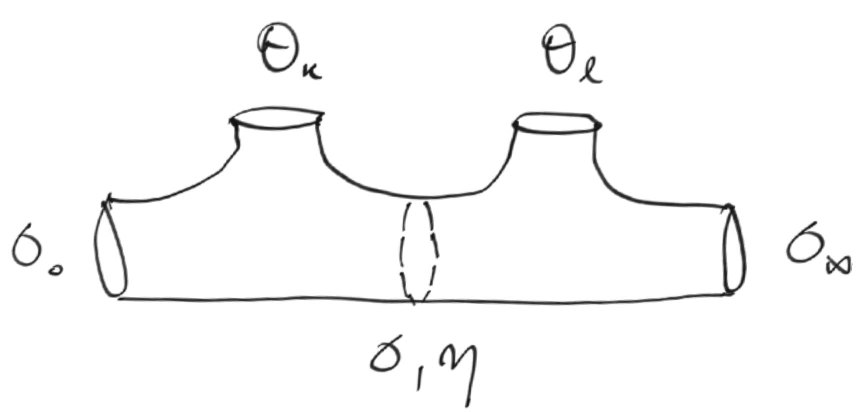
$$M_k = e^{i\pi\theta_k\sigma_3}, \qquad M_l = C_{kl} e^{i\pi\theta_l\sigma_3} C_{kl}^{-1},$$

$$\mathrm{Tr}\, M_k M_l = 2\cos\pi\sigma_{kl}$$

Example: triangular connection



$\sigma_{kl} = \theta_k + \theta_l + 2n, \qquad n$ integer



$$e^{i\pi\eta} = \frac{\sin\frac{\pi}{2}(\theta_l + \sigma_\infty + \sigma)\ \sin\frac{\pi}{2}(\theta_l - \sigma_\infty + \sigma)}{\sin\frac{\pi}{2}(\theta_l + \sigma_\infty - \sigma)\ \sin\frac{\pi}{2}(\theta_l - \sigma_\infty - \sigma)} \frac{\sin\frac{\pi}{2}(\theta_k + \sigma_0 + \sigma)\ \sin\frac{\pi}{2}(\theta_k - \sigma_0 + \sigma)}{\sin\frac{\pi}{2}(\theta_k + \sigma_0 - \sigma)\ \sin\frac{\pi}{2}(\theta_k - \sigma_0 - \sigma)}$$

Conformal blocks realize the Riemann-Hilbert map

$$\{\sigma_i, \eta_j\} = \frac{1}{2\pi}\delta_{ij} \longleftrightarrow \{t_i, K_j\} = \delta_{ij}$$

Semi-classical level 2 null vector condition of Liouville

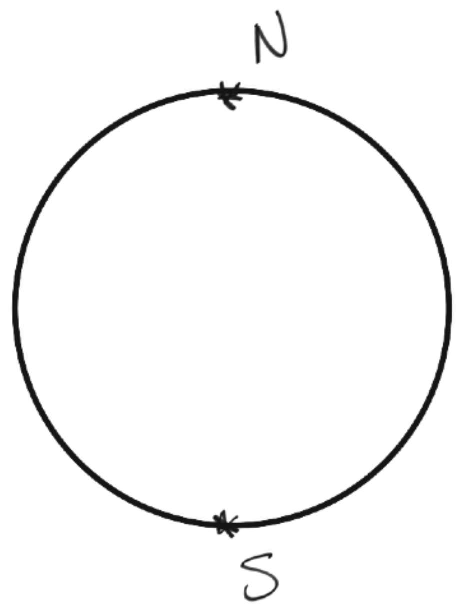
methods of calculation

- Zamolodchikov recursion formula (any b);
- Nekrasov expansions (appearance of $\mathscr{M}(\mathbb{C})$ as moduli of instantons in $\mathscr{N}$=2 SYM) (any b);
- Riemann-Hilbert problem ($b \to 0$).

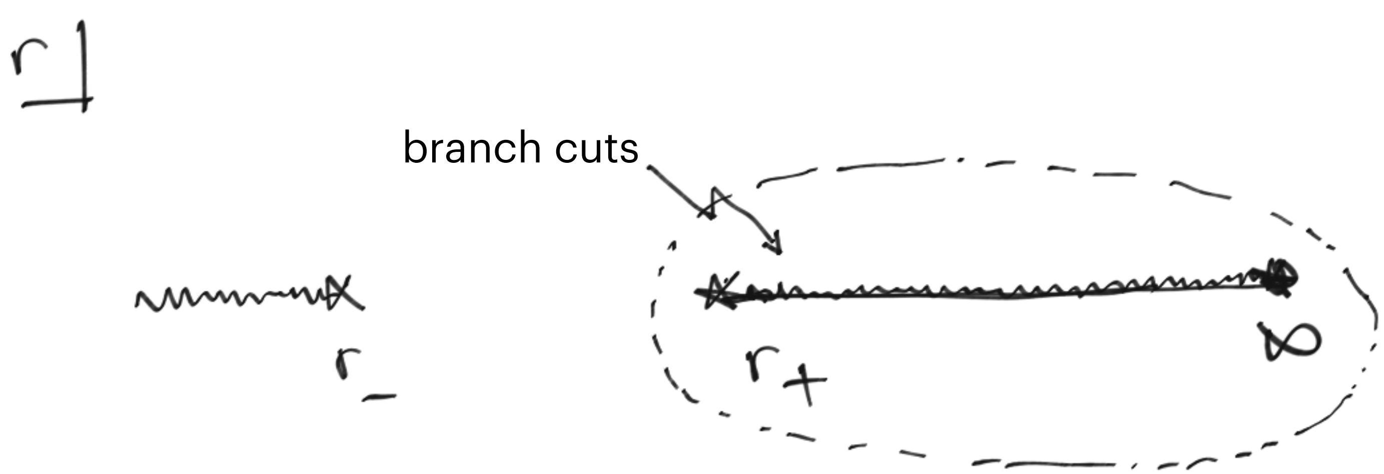
Application: black hole QNMs

Schematically, wave equation is separable in many black hole backgrounds, for different values of spin

$$\nabla^2\Phi = \mu^2\Phi, \qquad \Phi = \sum e^{-i\omega t + im\phi} R_{\omega,\ell,m}(r) S_{\omega,\ell,m}(\theta)$$



angular equation determines ℓ by requiring solution to be regular at North and South poles



radial QNM determined by requiring solution with no flux at outer horizon and infinity

why one should care?

Analytical & numerical methods exist for a number of black holes (ODE falls into Heun class); BUT…

tau permits a formal solution in terms of monodromy data, confluent limit is well-controlled,
Miwa's theorem guarantees analycity and isolated zeros;

Relation to conformal blocks is explicit and (perhaps) hints at underlying integrable or physical structure,
resurgence is clear (for PVI) and allows for choice of expansion point;

Numerically stable methods (Fredholm determinant formulation) exist for tau.
Works for higher number of singular points.

Scalar QNMs in 5d Kerr-AdS black hole

with J. Barragán-Amado and E. Pallante

angular part: Heun equation

$$\theta_0 = m_1, \quad \theta_1 = 2-\Delta, \quad \theta_{u_0} = m_2, \quad \theta_\infty = \omega + a_1 m_1 + a_2 m_2 \qquad\qquad u=0, \quad u=1, \quad u=u_0=\frac{a_2^2-a_1^2}{a_2^2-1}, \quad u=\infty,$$

$$4u_0(u_0-1)Q_0 = -\frac{\omega^2+a_1^2\mu^2-\lambda}{a_2^2-1} - u_0\left[(m_2-\Delta+1)^2-m_2^2-1\right] - (u_0-1)\left[(1-m_1-m_2)^2-\beta^2-1\right]$$

monodromy condition allows the computation of eigenvalue using Nekrasov expansion

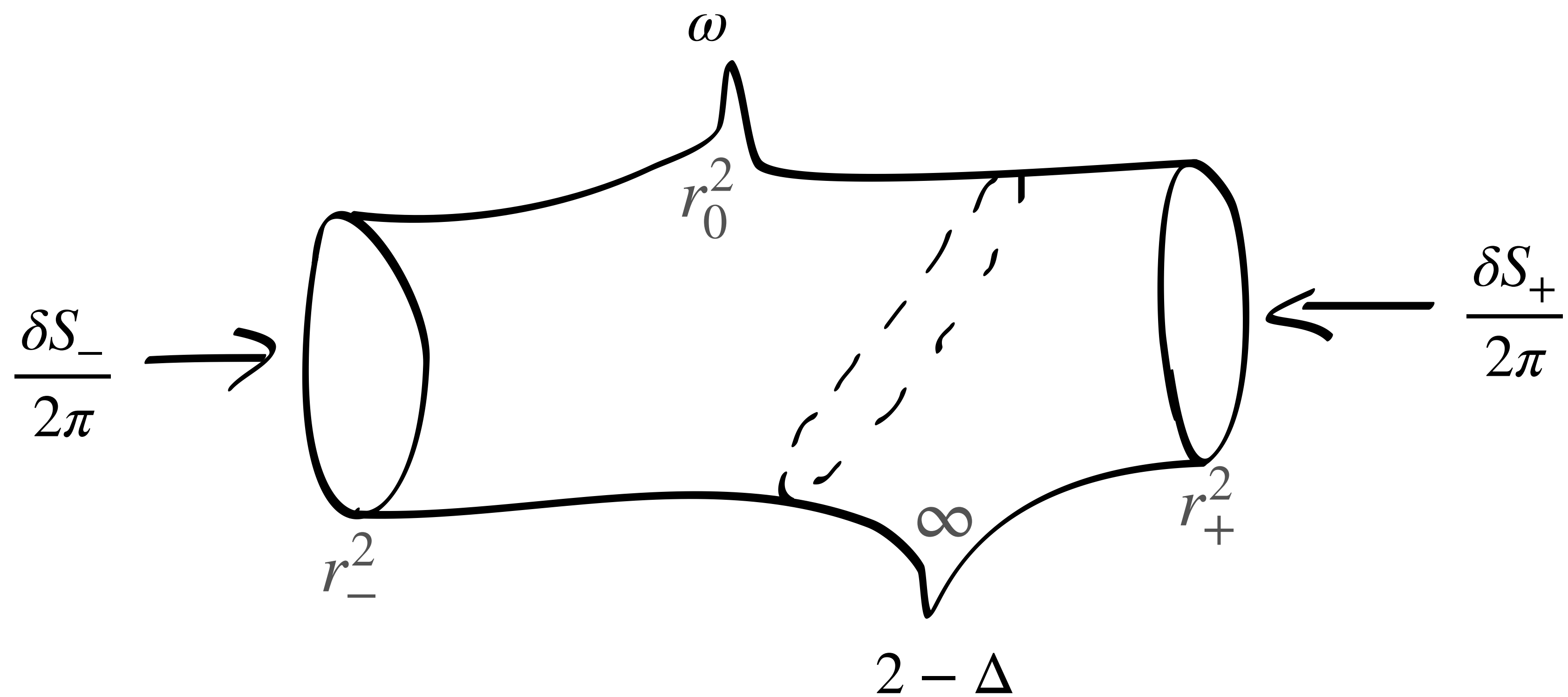
$$\lambda_\ell \simeq \ell(\ell+2) - 2\omega\left(a_1m_1+a_2m_2\right) - \left(a_1m_1+a_2m_2\right)^2 + \frac{a_1^2+a_2^2}{2}\left(\beta^2+\mu^2-\ell(\ell+2)\right)$$
$$+\frac{\left(a_2^2-a_1^2\right)\left(m_2^2-m_1^2\right)}{2\ell(\ell+2)}\left(\beta^2-\mu^2-(\ell^2+2\ell+4)\right) + \mathcal{O}((a_2^2-a_1^2)^2)$$

radial equation: also Heun equation

$$\theta_k = \frac{i}{2\pi}\left(\frac{\omega - m_1\Omega_{k,a} - m_2\Omega_{k,b}}{T_k}\right), \quad \theta_\infty = 2-\Delta, \qquad\qquad z_0 = (r_+^2-r_-^2)/(r_+^2-r_0^2)$$

$$4z_0(z_0-1)K_0 = -\frac{\lambda+\mu^2r_-^2-\omega^2}{r_+^2-r_0^2} - (z_0-1)[(\theta_-+\theta_+-1)^2-\theta_0^2-1] - z_0\left[\left(\theta_+-\Delta+1\right)^2-\theta_+^2-1\right]$$

Liouville representation (entropy intake)



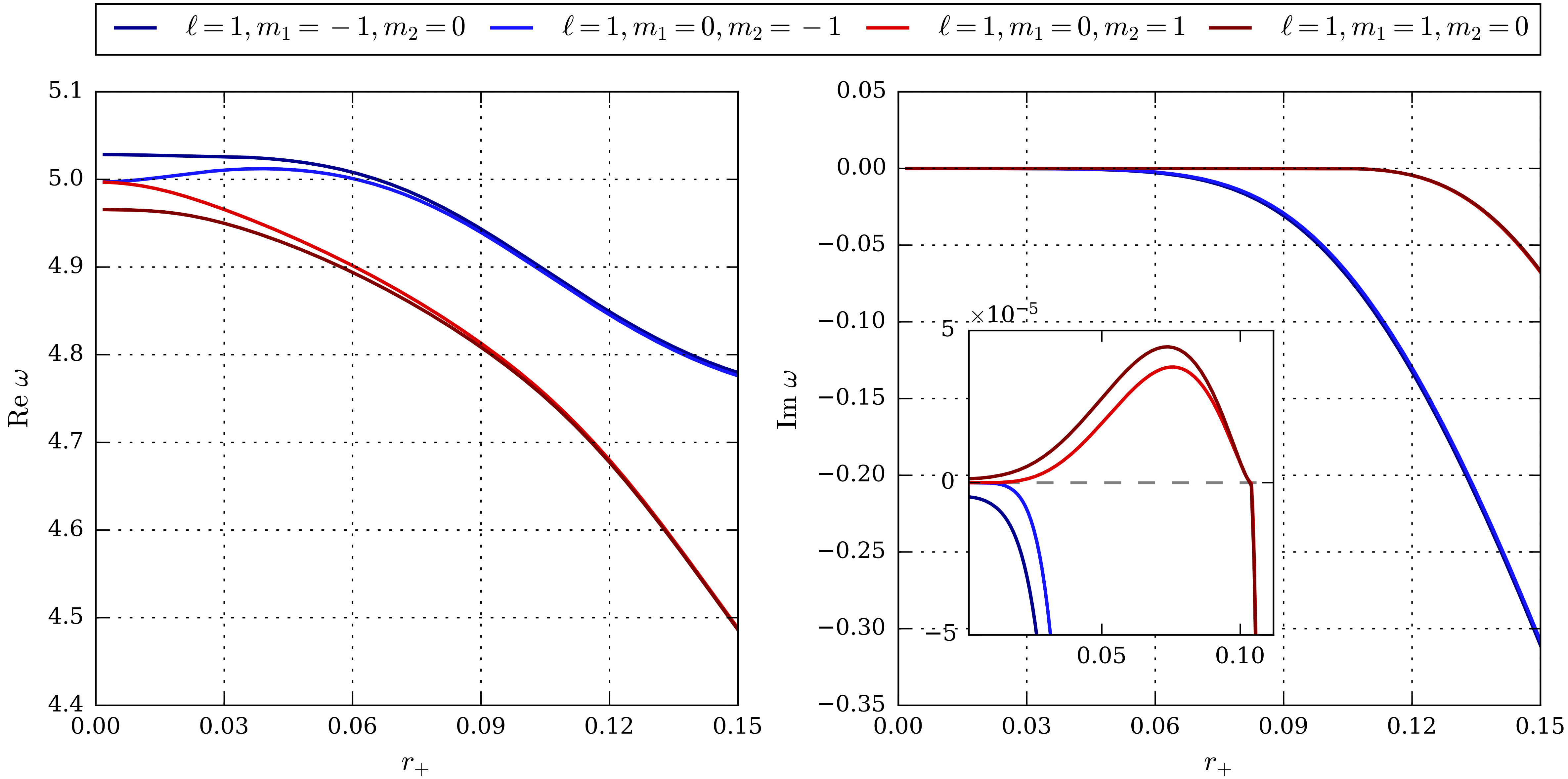
Analytical results for QNMs:

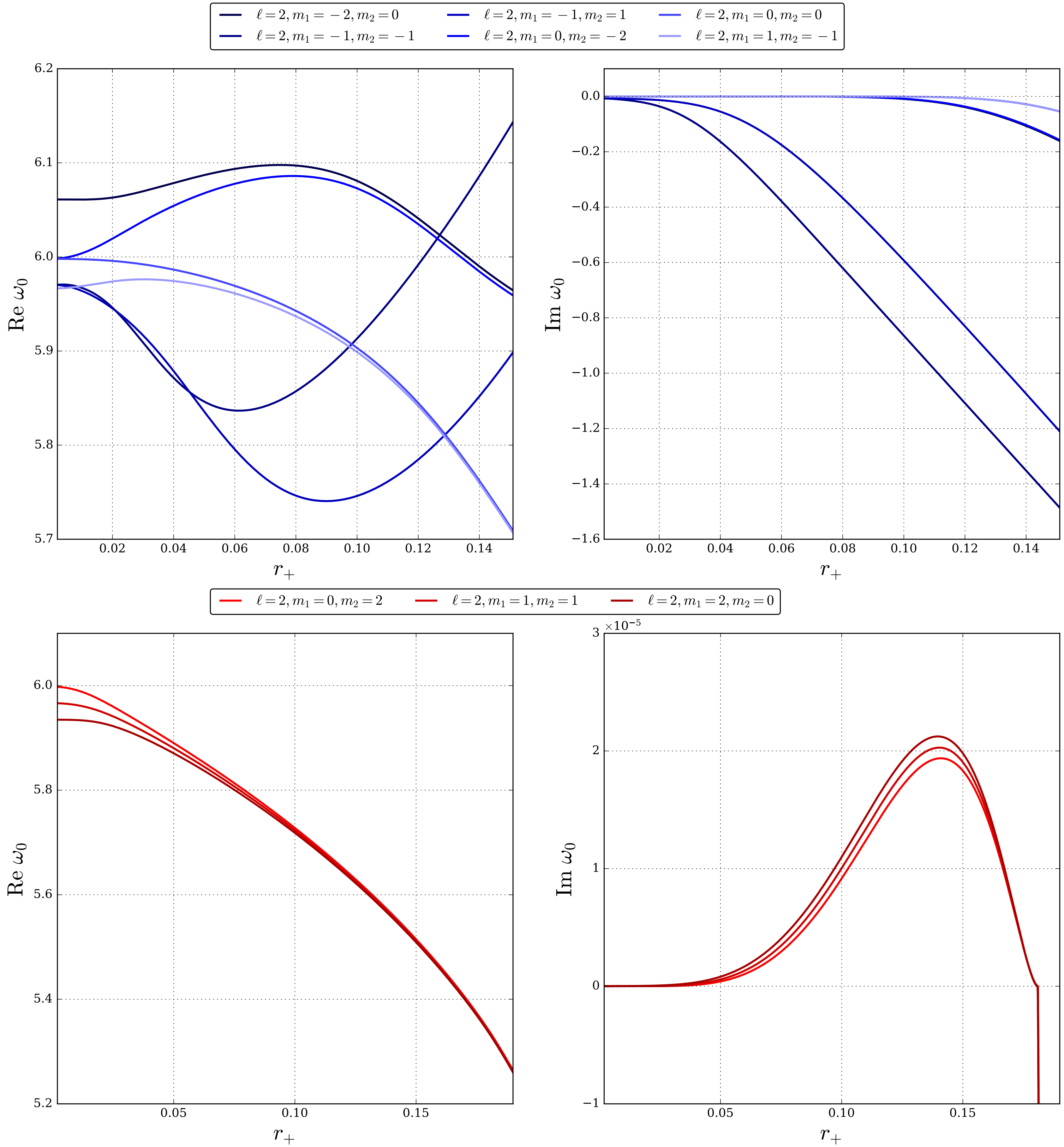
$$\omega_{1,0,0,0} = \Delta - (1+\alpha_+^2)\Delta(\Delta-1)r_+^2 - 2i(1+\alpha_+^2)\Delta(\Delta-1)r_+^3 + \Delta(\Delta-1)\epsilon r_+^2 + i(3+\alpha_+^2)\Delta(\Delta-1)\epsilon r_+^3 + \mathcal{O}(r_+^4, r_+^4\log r_+^2, \epsilon r_+^4, \epsilon^2 r_+^2)$$

$$\epsilon = \frac{r_+^2 - r_-^2}{2r_+^2}, \qquad \alpha_+^2 = \frac{a_1^2 + a_2^2}{2r_+^2}$$

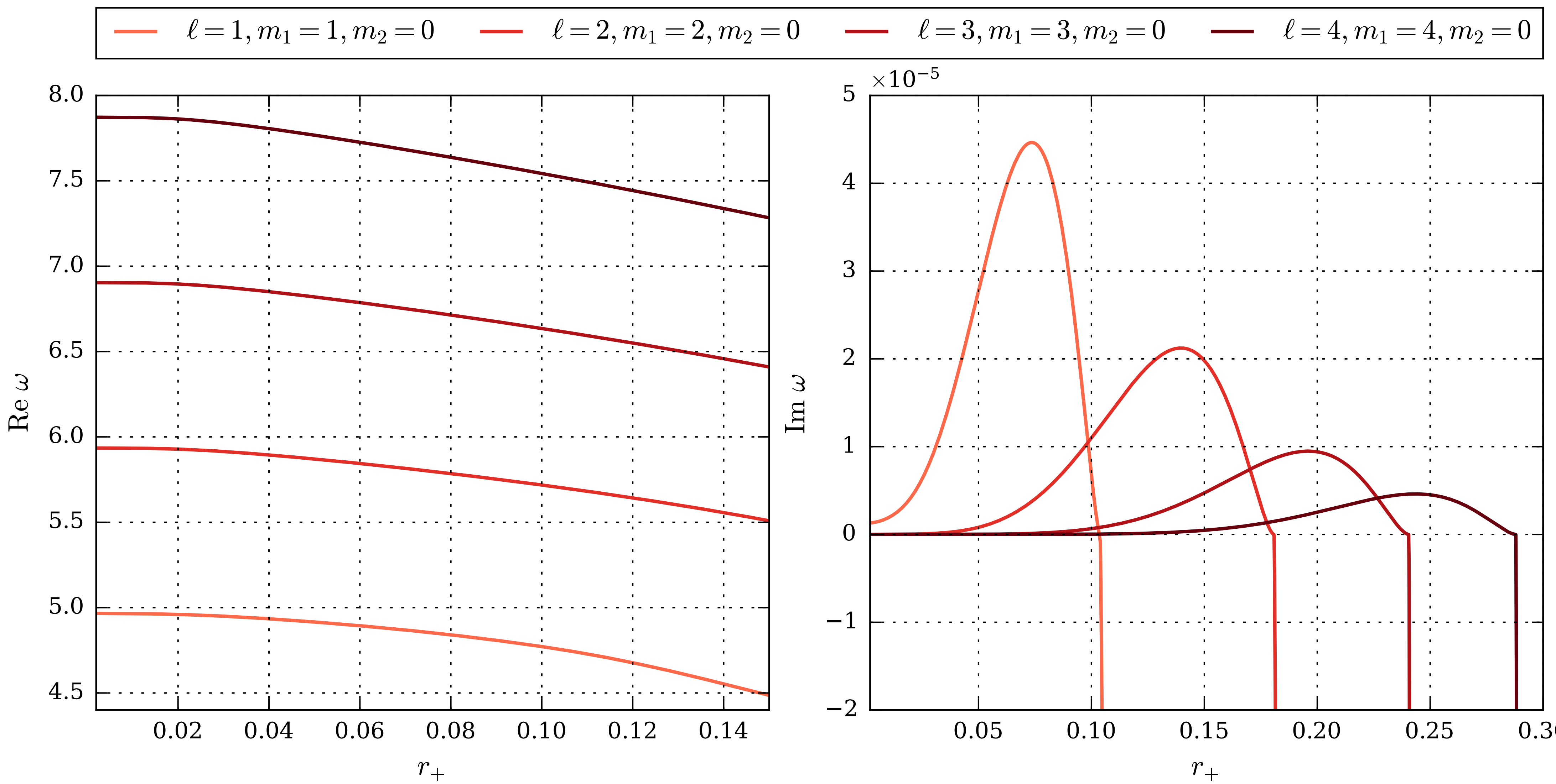
structure of a transseries

numerical results: instabilities

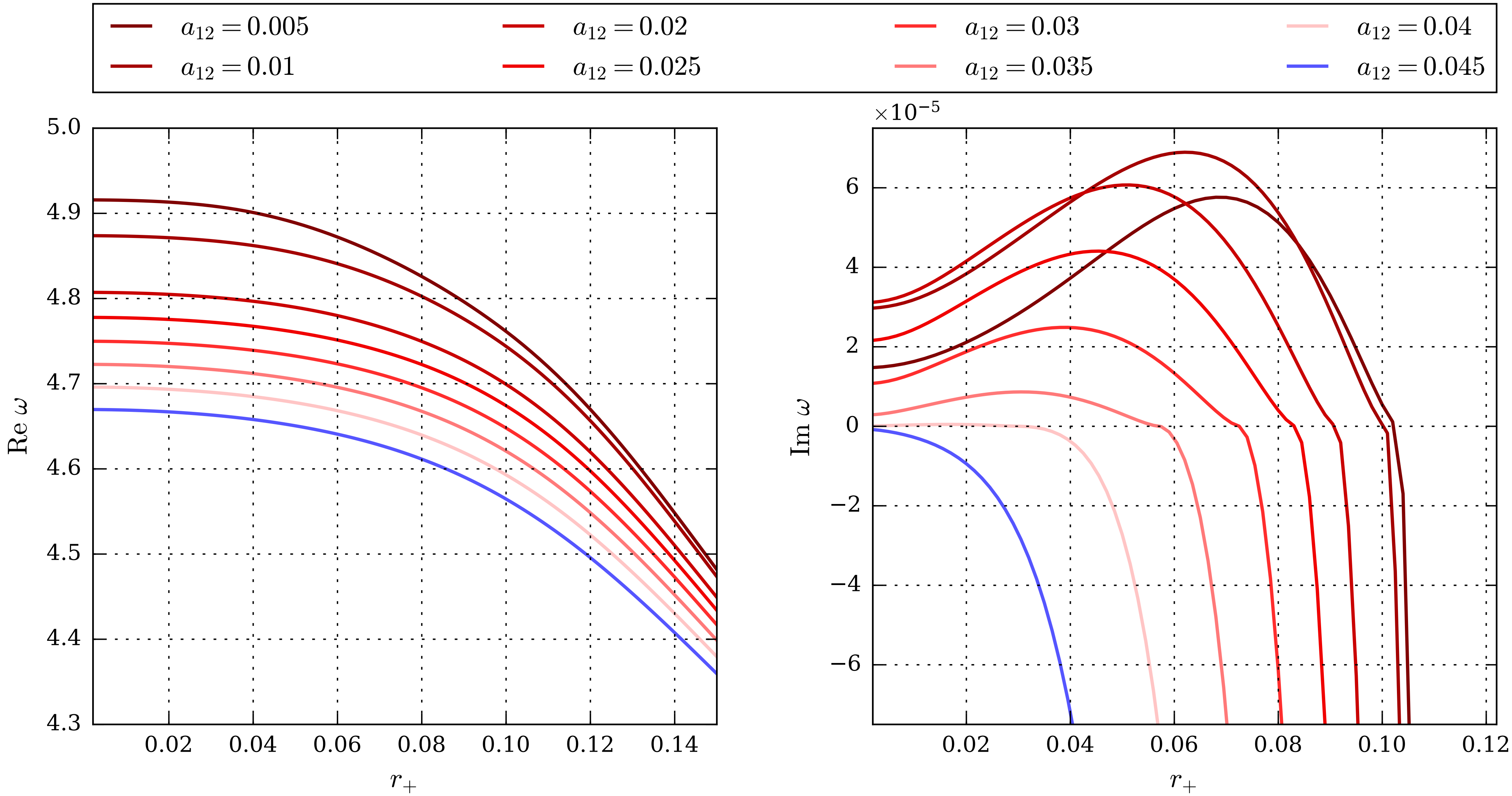




$\ell=2, m_1=-2, m_2=0$
$\ell=2, m_1=-1, m_2=-1$
$\ell=2, m_1=-1, m_2=1$
$\ell=2, m_1=0, m_2=-2$
$\ell=2, m_1=0, m_2=0$
$\ell=2, m_1=1, m_2=-1$
$\mathrm{Re}\,\omega_0$
$\mathrm{Im}\,\omega_0$
r_+
$\ell=2, m_1=0, m_2=2$
$\ell=2, m_1=1, m_2=1$
$\ell=2, m_1=2, m_2=0$
$\mathrm{Re}\,\omega_0$
$\mathrm{Im}\,\omega_0$
$\times 10^{-5}$
r_+



$\ell = 1, m_1 = 1, m_2 = 0$
$\ell = 2, m_1 = 2, m_2 = 0$
$\ell = 3, m_1 = 3, m_2 = 0$
$\ell = 4, m_1 = 4, m_2 = 0$
Re ω
r_+
Im ω
$\times 10^{-5}$
r_+

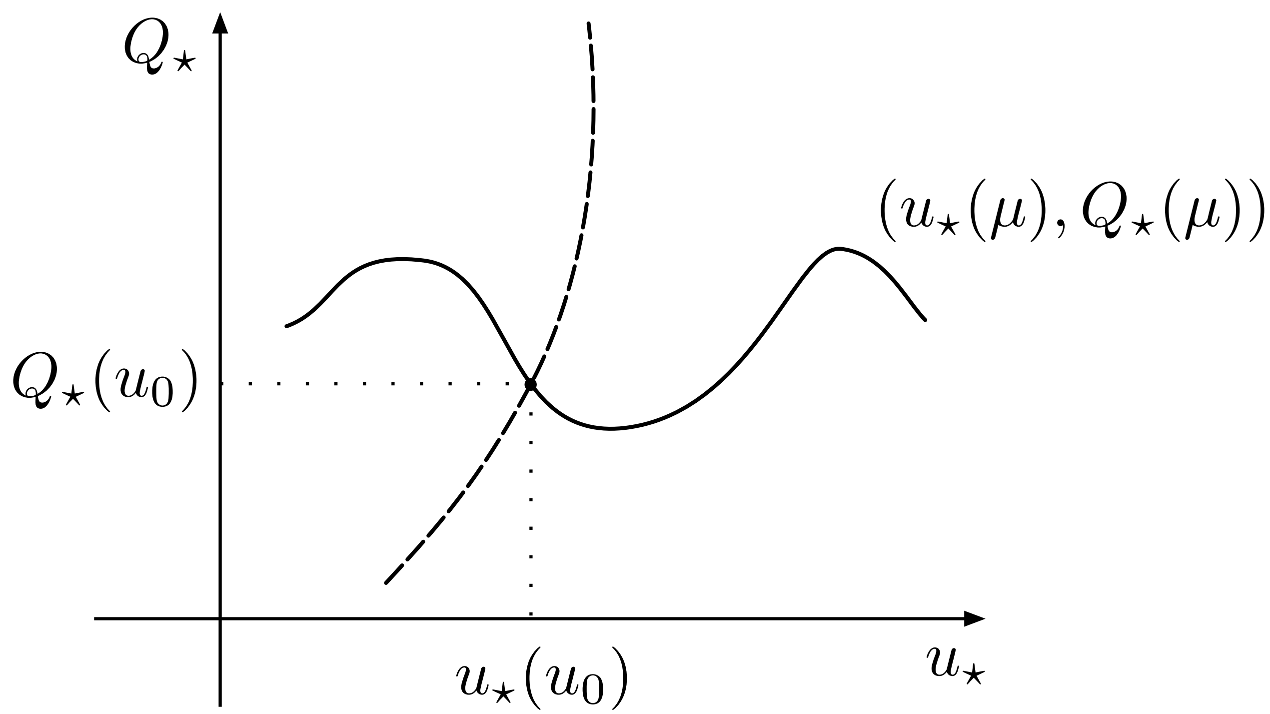


$a_{12} = 0.005$
$a_{12} = 0.01$
$a_{12} = 0.02$
$a_{12} = 0.025$
$a_{12} = 0.03$
$a_{12} = 0.035$
$a_{12} = 0.04$
$a_{12} = 0.045$
Re ω
Im ω
r_+
r_+
$\times 10^{-5}$

vector perturbations of 5d Kerr-AdS

μ-separability (Lunin, Frolov-Krtouš): resulting ODEs are Heun plus apparent singularity

single monodromy parameters for radial, angular variables now depend on separation constant



initial value problem now sets value for μ, related to polarization of QNM modes

only numerical study, also evidence for superradiance

4d Kerr and Reissner-Nordström black hole

with J. P. Cavalcante

angular equation:

$$ {}_s\lambda_{\ell,m}(a\omega) = (\ell - s)(\ell + s + 1) - \frac{2ms^2}{\ell(\ell+1)}a\omega + \left(\frac{2((\ell+1)^2 - m^2)((\ell+1)^2 - s^2)^2}{(2\ell+1)(\ell+1)^3(2\ell+3)} - \frac{2(\ell^2 - m^2)(\ell^2 - s^2)^2}{(2\ell-1)\ell^3(2\ell+1)} - 1 \right) a^2\omega^2 + \mathcal{O}(a^3\omega^3) $$

radial equation:

$$ \theta_{\mathrm{Rad},0} = \theta_- = s - i\frac{\omega - m\Omega_-}{2\pi T_-}, \qquad \theta_{\mathrm{Rad},t} = \theta_+ = s + i\frac{\omega - m\Omega_+}{2\pi T_+}, \qquad \theta_{\mathrm{Rad},\star} = \theta_* = -2s + 4iM\omega, \qquad 2\pi T_\pm = \frac{r_+ - r_-}{4Mr_\pm}, \qquad \Omega_\pm = \frac{a}{2Mr_\pm} $$

$$ t_{\mathrm{Rad}} = z_0 = 2i(r_+ - r_-)\omega, \qquad t_{\mathrm{Rad}}c_{\mathrm{Rad},t} = z_0 c_0 = {}_s\lambda_{\ell,m} + 2s + 2i(1-2s)M\omega - is(r_+ - r_-)\omega + (M^2a^2 - 4Mr_+)\omega^2 $$

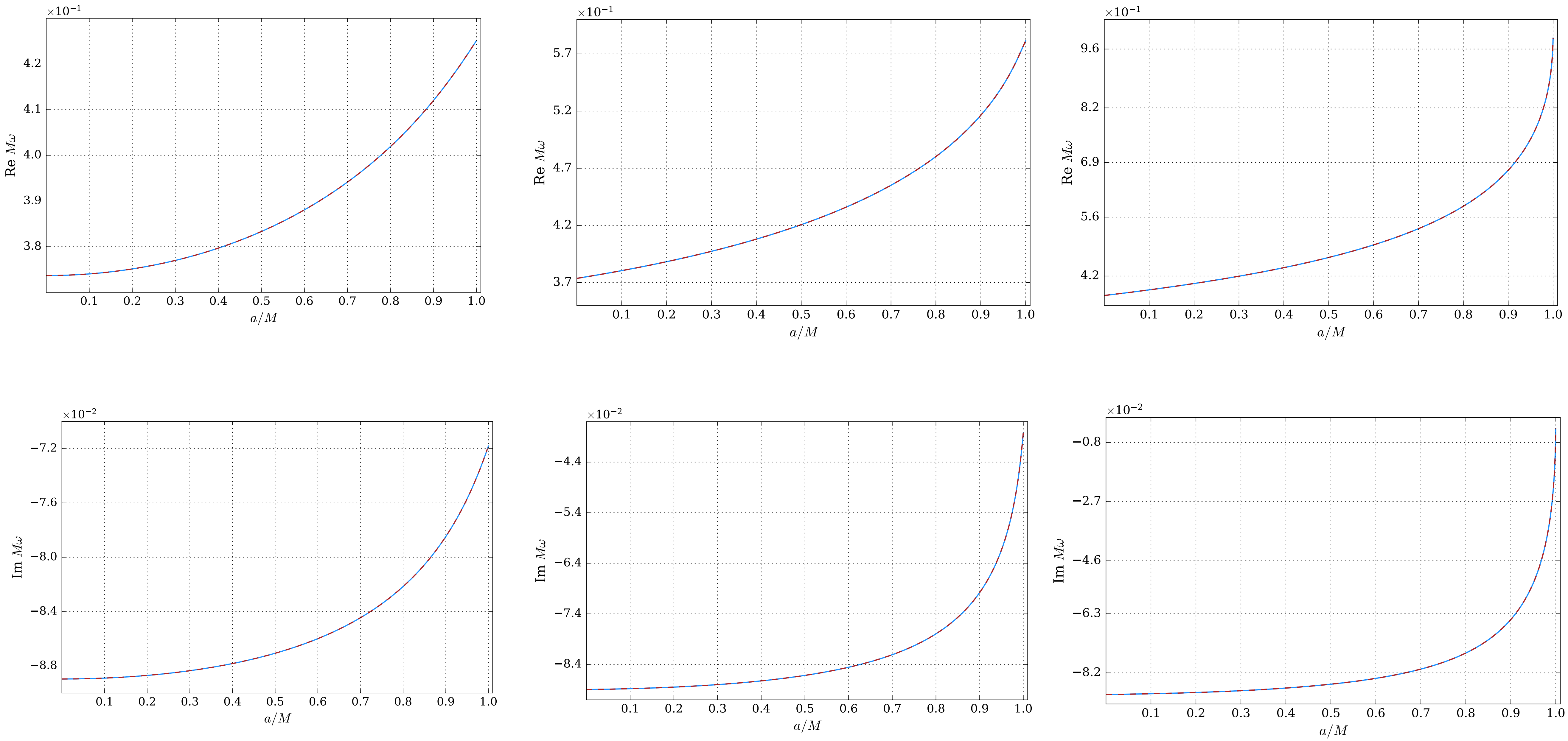
ODE is now confluent Heun, complicated analytical structure near infinity...

$$ e^{i\pi\eta_0} = e^{-i\pi\sigma}\frac{\sin\frac{\pi}{2}(\theta_\star + \sigma)}{\sin\frac{\pi}{2}(\theta_\star - \sigma)}\frac{\sin\frac{\pi}{2}(\theta_t + \theta_0 + \sigma)\sin\frac{\pi}{2}(\theta_t - \theta_0 + \sigma)}{\sin\frac{\pi}{2}(\theta_t + \theta_0 - \sigma)\sin\frac{\pi}{2}(\theta_t - \theta_0 - \sigma)} $$

quantization condition for radial equation
involves Painlevé V tau function

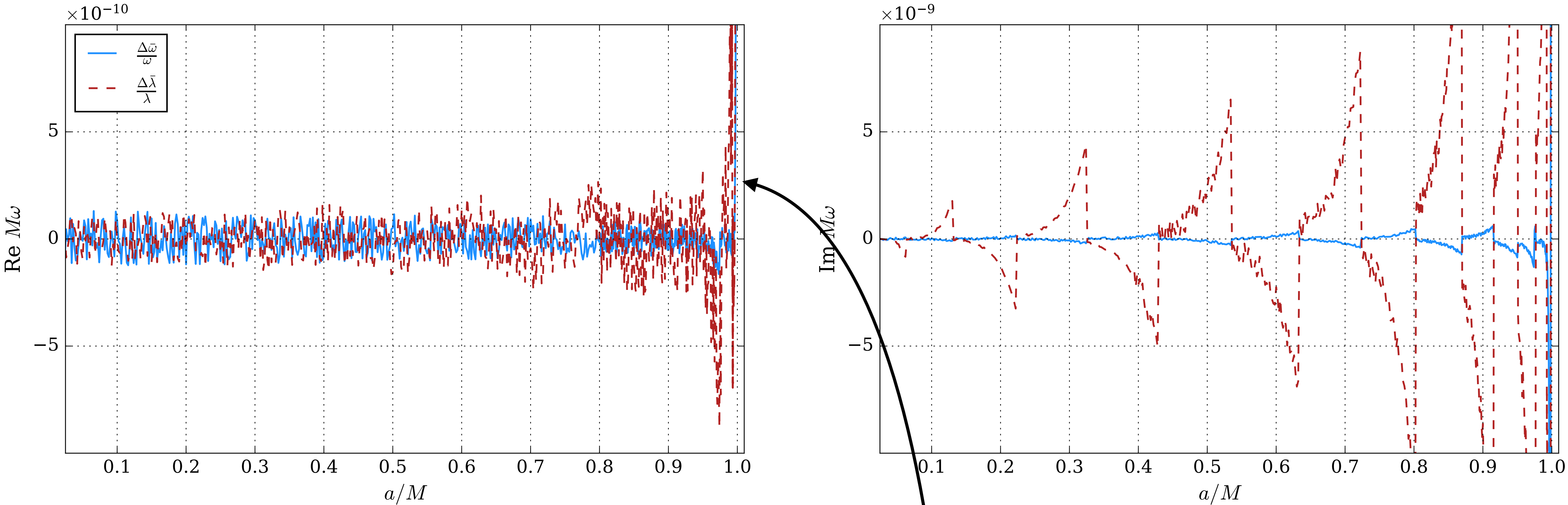
Numerics match

$$s = -2, \ell = 2, m = 0,1,2$$



BCS 0905.2975
CdC-C 2105.08790

relative error



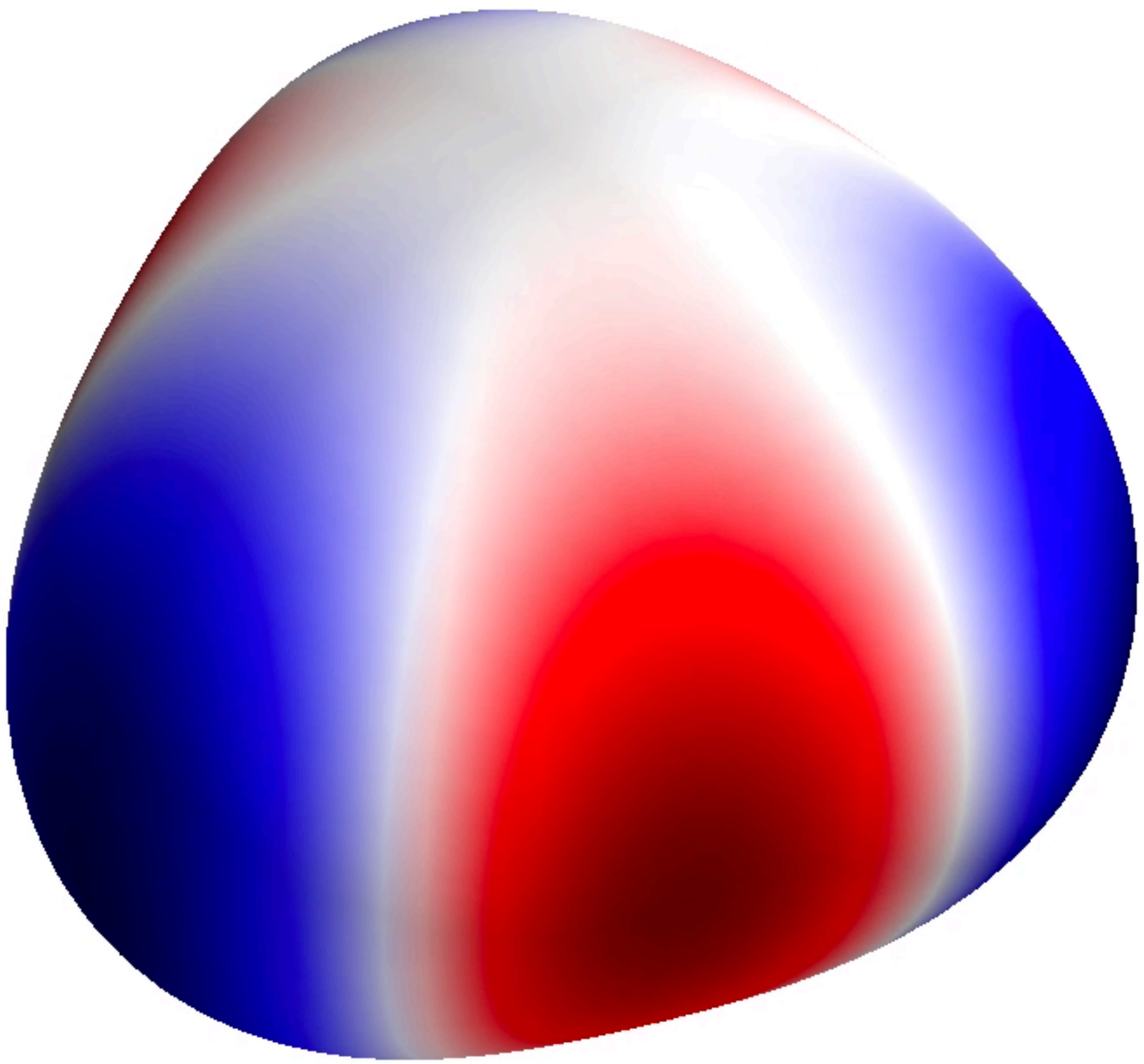
Zamolodchikov recursion
(continuous fraction method)
fails at extremal point

Extremal limit

two types of behavior: $\begin{cases} M\omega \to m/2 & \text{co-rotating} \\ M\omega \nrightarrow m/2 & \text{confluent limit} \end{cases}$

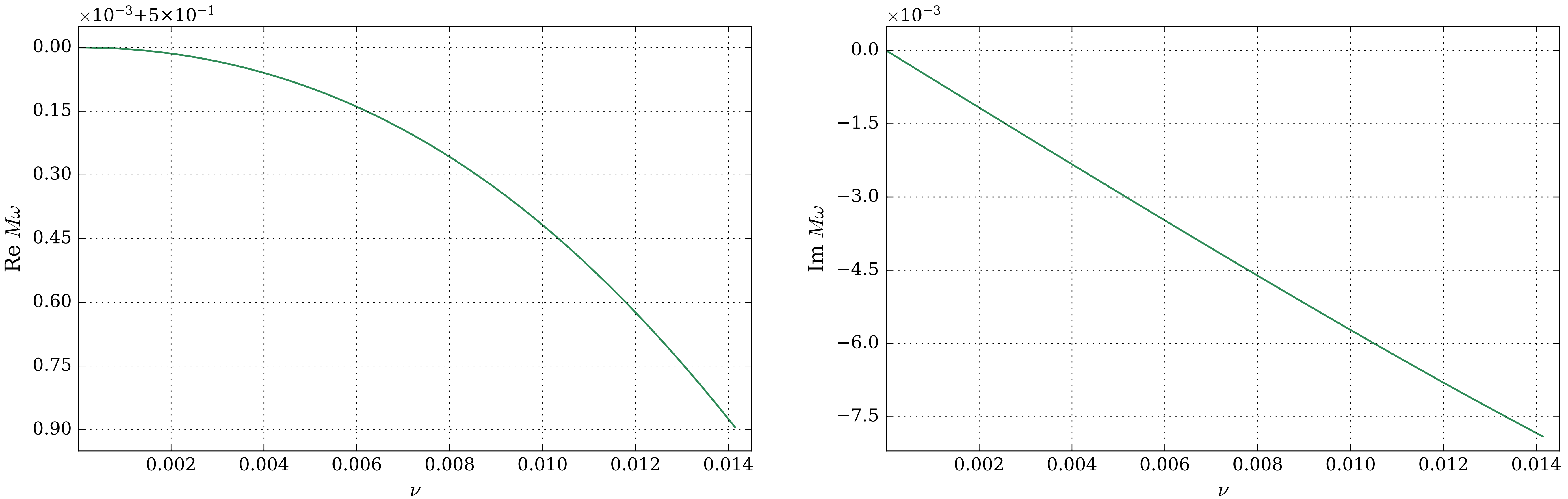
Co-rotating limit:

(most) $m = \ell$ cases



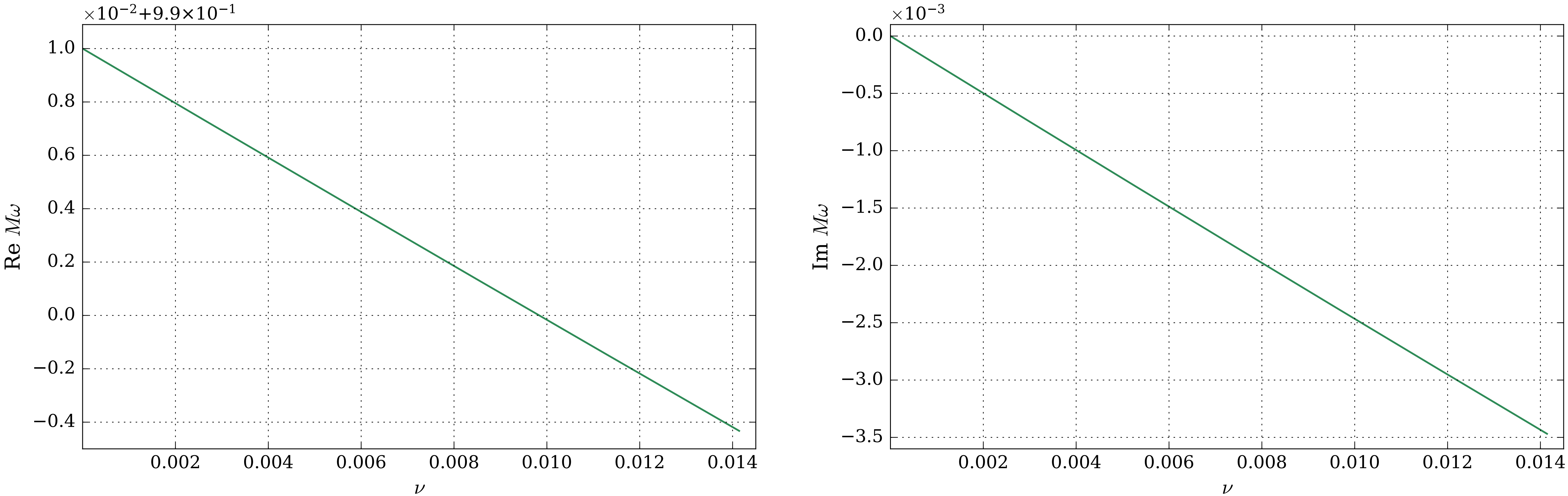
$$M\omega = \frac{m}{2} + \beta_1 \nu + \beta_2 \nu^2 + \ldots, \qquad \sigma = 1 + \alpha_0 + \alpha_1 \nu + \alpha_2 \nu^2 + \ldots$$

$s = 0, \ell = 1, m = 1$



real α_0

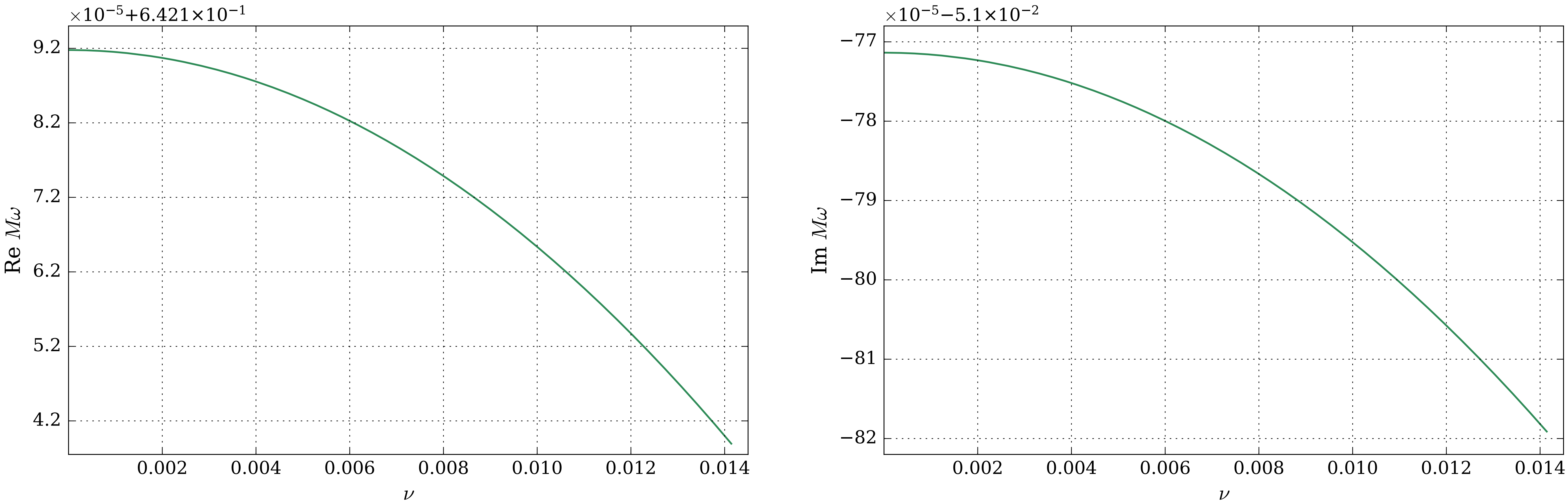
$s = -2, \ell = 2, m = 2$



imaginary α_0

Confluent limit

$$s = -1, \ell = 2, m = 1$$



Limit given by Painlevé III tau function

$$\tau_{III}(\overrightarrow{\theta}_{\text{ext}};\sigma,\eta;z_{\text{ext}}) = 0, \qquad z_{\text{ext}}\frac{d}{dt}\log\tau_{III}(\overrightarrow{\theta}_{\text{ext},-};\sigma-1,\eta;z_{\text{ext}}) - \frac{(\theta_{\text{ext},\circ}-1)^2}{2} = z_{\text{ext}}c_{\text{ext},z}$$

The Reissner-Nordström black hole

$$s = 0, \frac{1}{2}$$

$$\Delta^{-s}\frac{d}{dr}\left(\Delta^{1+s}\frac{dR_s(r)}{dr}\right) + \left(\frac{K(r)^2 - 2is(r-M)K(r)}{\Delta} + 4is\omega r - 2isqQ - {}_s\lambda_\ell\right)R_s(r) = 0$$

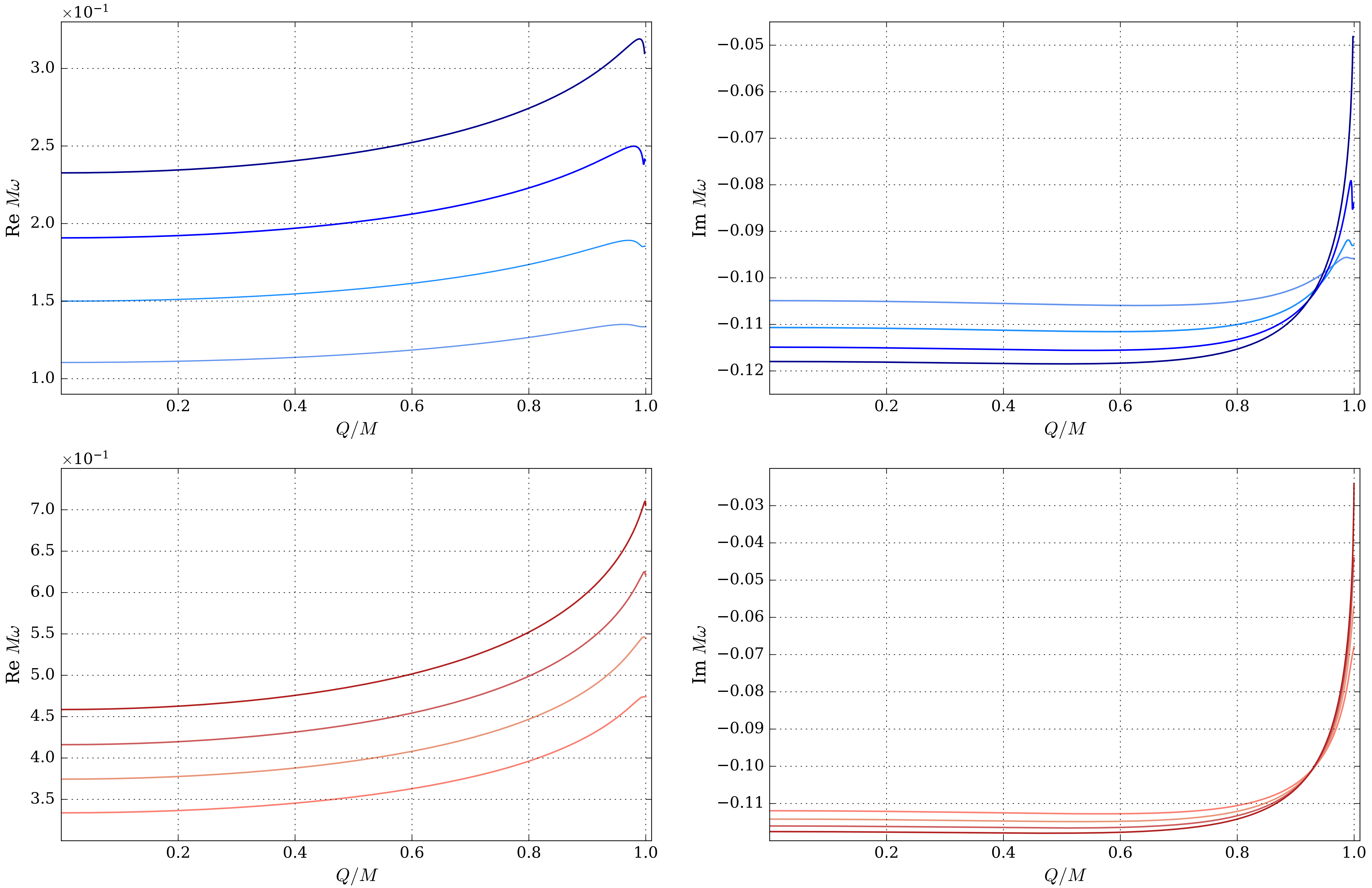
$$K(r) = \omega r^2 - qQr \qquad {}_s\lambda_{\ell,m} = (s-\ell)(s+\ell+1)$$

$$\theta_- = s + \frac{i}{2\pi T_-}\left(\omega - \frac{qQ}{r_-}\right), \qquad \theta_+ = s + \frac{i}{2\pi T_+}\left(\omega - \frac{qQ}{r_+}\right), \qquad \theta_\star = -2s + 2i(2M\omega - qQ),$$

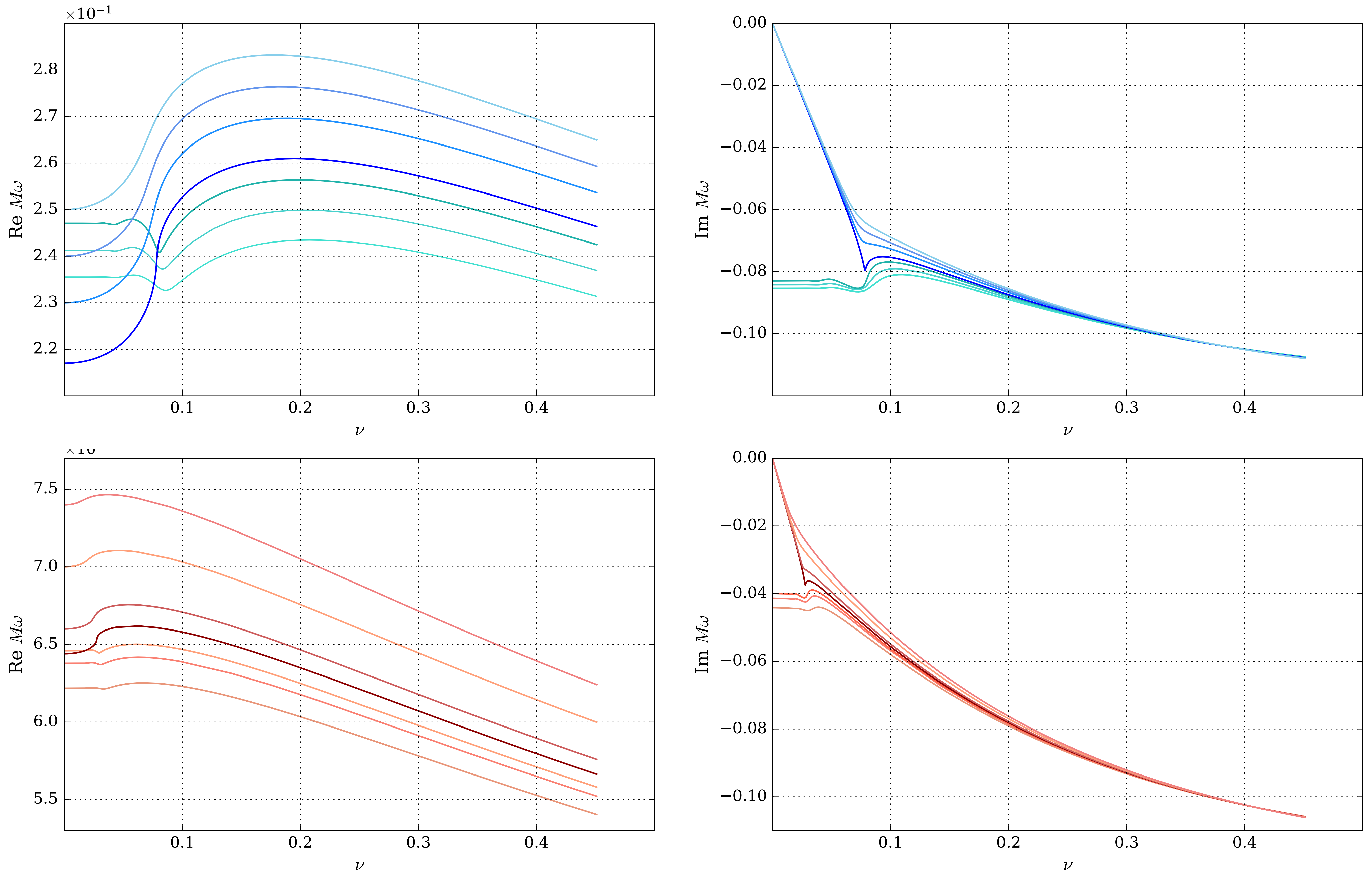
$$2\pi T_\pm = \frac{r_\pm - r_\mp}{2r_\pm^2}, \qquad r_\pm = M \pm \sqrt{M^2 - Q^2},$$

$$z_0 c_{z_0} = {}_s\lambda_{l,m} + 2s - i(1-2s)qQ + (2qQ + i(1-3s))\omega r_+ + i(1-s)\omega r_- - 2\omega^2 r_+^2, \qquad z_0 = 2i\omega(r_+ - r_-)\,.$$

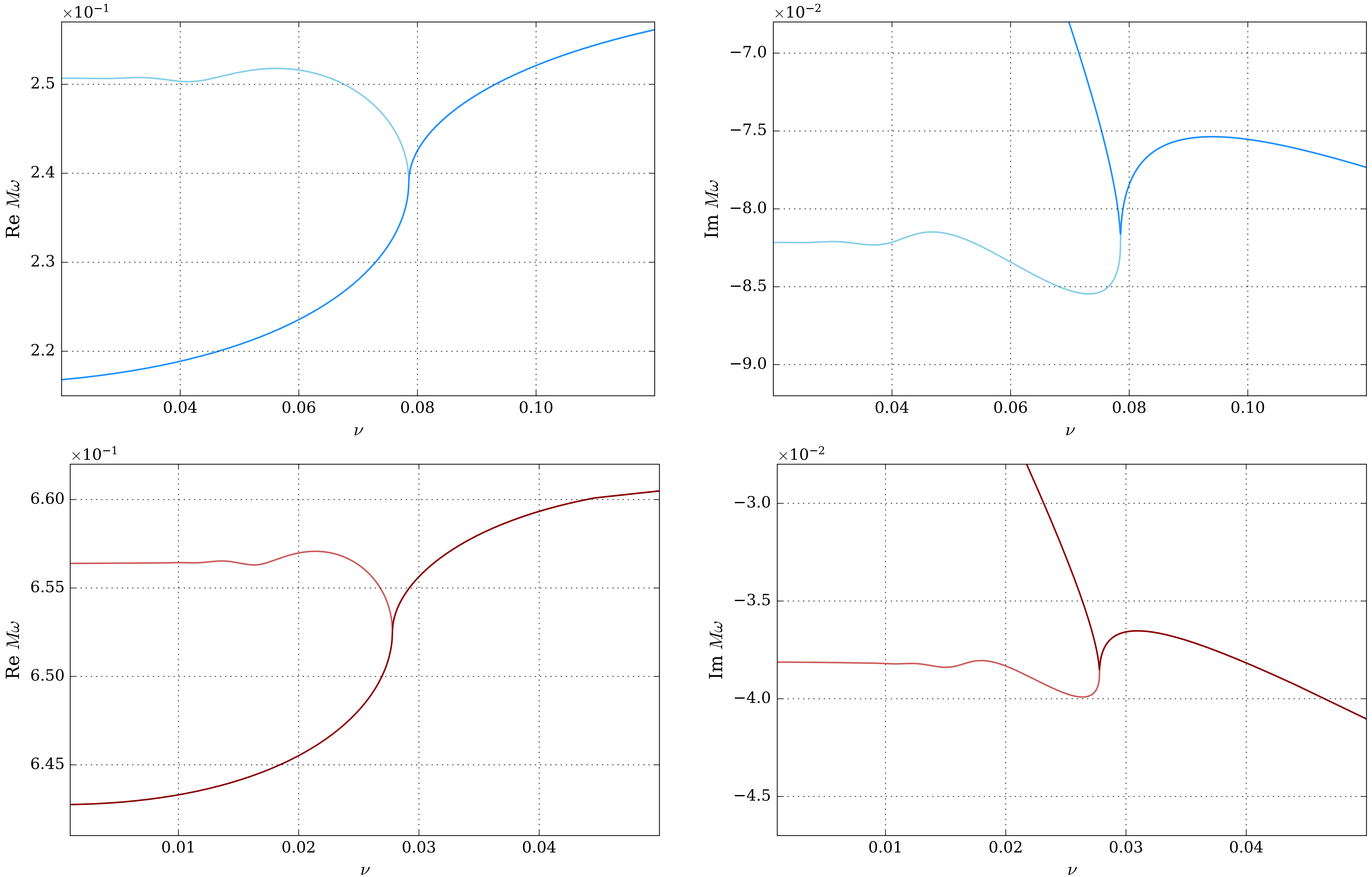
$$Q/M = \cos\nu$$



×10^−1
Re Mω
Q/M
Im Mω
0.2
0.4
0.6
0.8
1.0
3.0
2.5
2.0
1.5
1.0
−0.05
−0.06
−0.07
−0.08
−0.09
−0.10
−0.11
−0.12
7.0
6.5
6.0
5.5
5.0
4.5
4.0
3.5
−0.03
−0.04

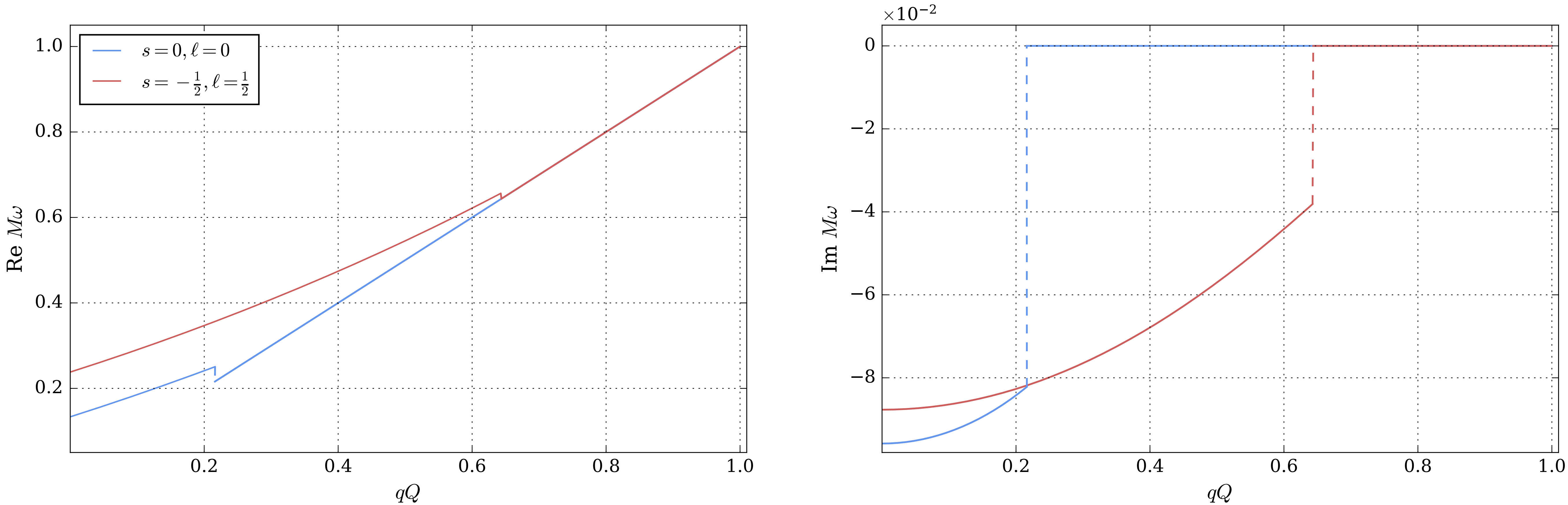


×10⁻¹
2.8
2.7
2.6
2.5
2.4
2.3
2.2
Re $M\omega$
0.1
0.2
0.3
0.4
ν
0.00
−0.02
−0.04
−0.06
−0.08
−0.10
Im $M\omega$
7.5
7.0
6.5
6.0
5.5



$qQ_c(s=0) \simeq 0.216228,$ and $qQ_c(s=-1/2) \simeq 0.642745$

Decoupling at the extremal limit



Still don’t know why, but at least know how.

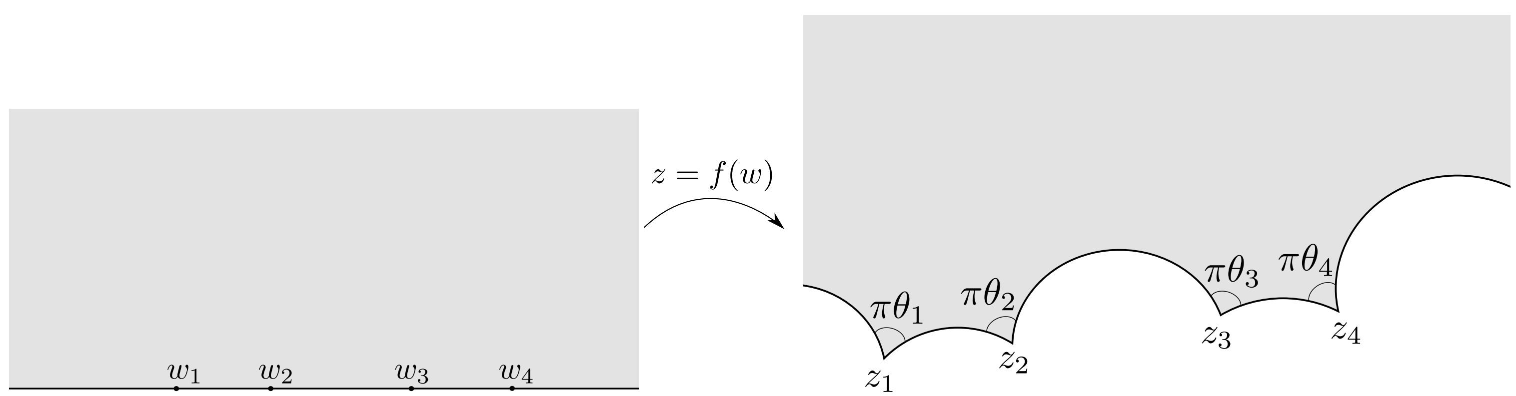
constructive conformal mapping

with T. Anselmo, S. Nejad, R. Nelson and D. Crowdy

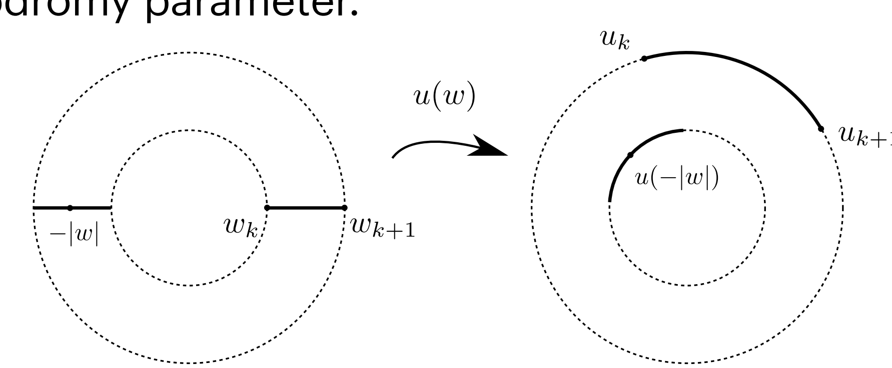
$$\{f(w);w\} \equiv \frac{\partial_w^3 f}{\partial_w f} - \frac{3}{2}\left(\frac{\partial_w^2 f}{\partial_w f}\right) = 2T(w), \qquad T(w) = \sum_{k=0}^{n-1} \frac{\alpha_k}{(w-w_k)^2} + \frac{\beta_k}{w-w_k}$$

can be transformed to Fuchsian equation by writing $f(w) = \dfrac{Y_1(w)}{Y_2(w)}$

solution analytic outside $w = w_k$. can read single monodromy from deficit angle



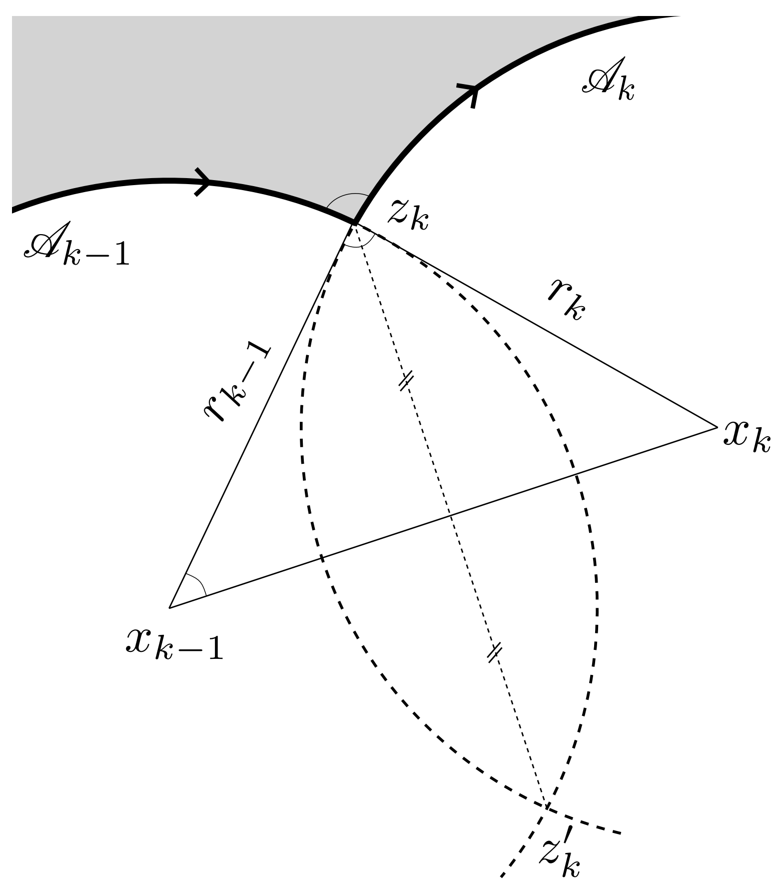
generally complex composite monodromy parameter.



$$\sigma_k = 1 + i\lambda_k$$

Explicit monodromy matrices can be constructed from geometrical representation of polycircular arc domain

$$\mathbf{S}_k = \frac{1}{r_k}\begin{pmatrix} x_k^* & 1 \\ r_k^2 - |x_k|^2 & -x_k \end{pmatrix} \qquad \widehat{\mathbf{M}}_k = \mathbf{S}_k \mathbf{S}_{k-1}^{-1}$$



accessory parameters of $T(w)$ can be obtained by the transcendental equations

$$\tau_{\mathrm{JMU}}(\hat{\rho}_k^+; \{w_k\}) = 0, \qquad \beta_k = -\frac{\partial}{\partial t_k} \log \tau_{\mathrm{JMU}}(\hat{\rho}; \{w_k\}) + \frac{\hat{\theta}_k}{2w_k} + \frac{\hat{\theta}_k}{2(w_k - 1)}.$$

what we learned

- not only a formal solution to the connection problem, but useful and effective way of computing (numerically or analytically);
- advantage to think about monodromy even in usual schemes of calculation (Hill's determinant, continuous fraction);
- any "separable" black hole?
- "factorization" of space-time conformal blocks into 2d chiral ones; CFT interpretation (is it just a trick?);
- pays to consider all monodromy parameters; (extremal) limits better behaved; Relevant physical quantities already translated to monodromy data;

Thank you!