The dynamics of black hole binaries from scattering amplitudes

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Gravitational binaries (motivation I)

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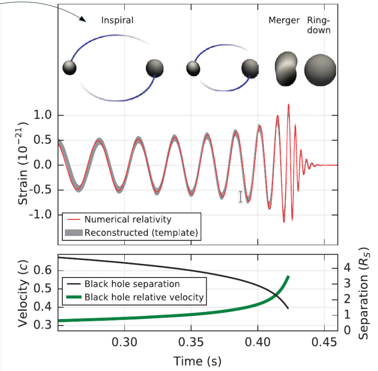
Observation of Gravitational Waves from a Binary Black Hole Merger

1602.03837

B. P. Abbott *et al.**
(LIGO Scientific Collaboration and Virgo Collaboration)
(Received 21 January 2016; published 11 February 2016)

We will focus on the inspiral part



The aim

Use a particle-physicist approach to derive classical observables relevant to gravitational binaries

1710.10599

PHYSICAL REVIEW D **97,** 044038 (2018)

High-energy gravitational scattering and the general relativistic two-body problem

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A technique for translating the classical scattering function of two gravitationally interacting bodies into a corresponding (effective one-body) Hamiltonian description has been recently introduced [Phys. Rev. D **94**, 104015 (2016)]. Using this technique, we derive, for the first time, to second-order in Newton's constant (i.e. one classical loop) the Hamiltonian of two point masses having an arbitrary (possibly relativistic) relative velocity. The resulting (second post-Minkowskian) Hamiltonian is found to have a tame high-energy structure which we relate both to gravitational self-force studies of large mass-ratio binary systems, and to the ultra high-energy quantum scattering results of Amati, Ciafaloni and Veneziano. We derive several consequences of our second post-Minkowskian Hamiltonian: (i) the need to use special phase-space gauges to get a tame high-energy limit; and (ii) predictions about a (rest-mass independent) linear Regge trajectory behavior of high-angular-momenta, high-energy circular orbits. Ways of testing these predictions by dedicated numerical simulations are indicated. We finally indicate a way to connect our classical results to the quantum gravitational scattering amplitude of two particles, and we urge amplitude experts to use their novel techniques to compute the two-loop scattering amplitude of scalar masses, from which one could deduce the third post-Minkowskian effective one-body Hamiltonian.

This is the framework we will consider

The approach

1. Model the celestial bodies as “elementary” particles with known couplings to gravity (massless fields in general): this defines the classical objects in an EFT approach or in UV complete theory
2. Use perturbative amplitudes to describe the large-distance scattering and take the classical limit
3. Export the new information obtained from open to closed orbits

Each of these steps can be tackled technically in several ways. Here I would like to emphasize two general points

- It is a general approach applicable to all perturbative gravitational theories (GR, supergravity, string theory) and various types of objects (Schwarzschild, Kerr, shockwaves, strings . . .)
- Classical physics is obtained by resumming an infinite set of contributions which leads to exponentiation

Motivation (II)

Why is this useful? A new perspective on an old (and difficult!) problem can sometime bring conceptual and practical progress.

A (very partial) list of results emerged over the last four years

- Impressive results at high PM order (3PM solved, 4PM in progress)

  See: Bern, Parra-Martinez, Roiban, Ruf, Shen, Solon, Zeng 2112.10750 and refs therein
- Analytic continuation from open to closed trajectories

  See: Cho, Kälin, Porto 2112.03976 and refs therein
- New results on the radiated energy and angular momentum

  See: Herrmann, Parra-Martinez, Ruf, Zeng 2101.07255, Manohar, Ridgway, Shen 2203.04283
- A new avenue to study spinning objects (Kerr and beyond)

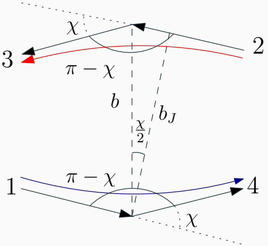
  See: Aoude, Haddad, Helset 2203.06197, Bern, Kosmopoulos, Roiban, Teng 2203.06202 and refs therein
- New insights on and from the high energy regime

  See: Di Vecchia, Heissenberg, RR, Veneziano: 2104.03256, 2204.02378

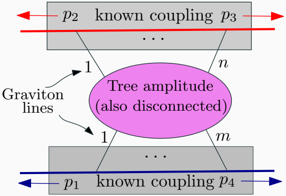
The elastic scattering

An example: scalar particles in GR

Consider the $2 \to 2$ scattering with $p_1^2 = p_4^2 = -m_1^2$, $p_2^2 = p_3^2 = -m_2^2$



A spacetime picture of the scattering



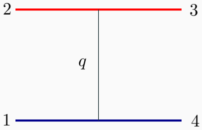
Diagrammatic picture

Key classical quantities:

The centre-of-mass energy E, $E^2 = s = -(p_1 + p_2)^2$, $\sigma = -\frac{p_1 p_2}{m_1 m_2}$.
The angular momentum $J = p\, b_J$, $p = |\vec{p}_i|$, $Ep = m_1 m_2 \sqrt{\sigma^2 - 1}$
The momentum transferred $Q = p_1 + p_4$, $|Q| = 2p \sin\left(\frac{\chi}{2}\right)$

One particle exchange

Let us start from the 1-particle exchange



$$\mathcal{A}_0 = \frac{8\pi G\beta}{q^2} + \dots$$

$$\beta^{\mathcal{N}=8} = 4m_1^2 m_2^2 \sigma^2$$

$$\beta^{GR} = 4m_1^2 m_2^2 \left(\sigma^2 - \frac{1}{D-2}\right)$$

q is quantum and the dots contain analytic terms as $q \to 0$

In terms of classical quantity $b \sim \hbar/q$

$$\widetilde{\mathcal{A}}(s,b) = \int \frac{d^{D-2}q}{(2\pi)^{D-2}} \frac{\mathcal{A}(s,q^2)}{4pE} e^{ib\cdot q}\,.$$

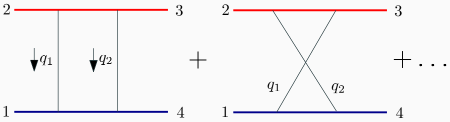
In $D = 4 - 2\epsilon \to 4$ we have

$$i\widetilde{\mathcal{A}}_0^{\mathcal{N}=8} = \frac{2im_1 m_2 G(\pi b^2)^\epsilon \sigma^2 \Gamma(-\epsilon)}{\sqrt{\sigma^2-1}} \to -i\frac{Gm_1 m_2}{\hbar} \log(b) \frac{4\sigma^2}{\sqrt{\sigma^2-1}}$$

No well defined classical limit?!

Two particle exchange

Consider the two particle exchange. The non-analytic contributions are



$$\mathcal{A}_1(s,q^2) = \frac{a_1^{(1)}(s)}{(q^2)^{1+\epsilon}} + \frac{a_1^{(2)}(s)}{(q^2)^{\frac{1}{2}+\epsilon}} + \frac{a_1^{(3)}(s)}{(q^2)^{\epsilon}} + \ldots$$
$$\text{with } q_1 + q_2 = q$$

(I) From $a_1^{(1)}$ we have $\mathcal{O}\left(\frac{1}{\hbar^2}\right)$ term: $i\widetilde{\mathcal{A}}_1^{(1)}(s,b) = \frac{1}{2}(i\widetilde{\mathcal{A}}_0)^2$. Then resumming the leading contributions (as $\hbar \to 0$) we expect $1 + i\widetilde{\mathcal{A}}_0 + i\widetilde{\mathcal{A}}_1^{(1)} + \ldots = e^{i\widetilde{\mathcal{A}}_0}$ (eikonal exponentiation)

(II) $a_1^{(2)}$ yield a new contribution $\mathcal{O}\left(\frac{1}{\hbar}\right)$ (which is $\mathcal{O}(\epsilon)$ in $\mathcal{N}=8$)

(III) $a_1^{(\mathrm{n}\geq 3)}$ yields a long-range, but quantum terms $\mathcal{O}(\hbar^{n-3})$

Terms with negative powers of $\hbar$ exponentiate

The eikonal

The semiclassical limit requires that the long range part of $\widetilde{\mathcal{A}}$ takes the form

$$\boxed{1 + i\widetilde{\mathcal{A}}(s,b) = \Big(1 + 2i\Delta(s,b)\Big)\, e^{i2\delta(s,b)}}$$

where δ is $\mathcal{O}(\hbar^{-1})$ and Δ encodes the quantum terms $\mathcal{O}(\hbar^m)$ with $m \geq 0$

$\delta = \sum_k \delta_k$ and $\Delta = \sum_k \Delta_k$, $k \geq 0$, are of order G^{k+1} (PM expansion)

$\mathcal{N} = 8$ in $D = 4$: we have $2\delta_0 = -\log(b)\frac{4Gm_1m_2\sigma^2}{\hbar\sqrt{\sigma^2-1}}$, $\delta_1 = 0$ Caron-Huot, Zahraee

Ignoring the quantum terms the inverse FT reads

$$i\,\frac{\mathcal{A}(s,Q^2)}{4pE} = \int d^{D-2}b\left(e^{i2\delta(s,b)} - 1\right)\, e^{-\frac{i}{\hbar}b\cdot Q}$$

A stationary phase approximation yields $Q^\mu = \hbar\frac{\partial\,\mathrm{Re}\,2\delta(s,b)}{\partial b^\mu} = 2p\sin\left(\frac{\chi}{2}\right)$

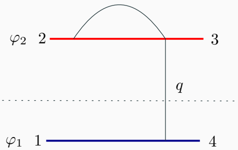
GR in $D = 4$: we have $2\sin\left(\frac{\chi_{1PM}}{2}\right) = \frac{2GE(2\sigma^2-1)}{b\,(\sigma^2-1)}$ and

$$2\sin\left(\frac{\chi_{2PM}}{2}\right) = \frac{3\pi G^2E(m_1+m_2)}{4b^2}\,\frac{5\sigma^2-1}{\sigma^2-1}$$

Regime of validity

In $\mathcal{N} = 8$ sugra, the box diagrams give the full 1-loop amplitude

For scalars minimally coupled to GR $\varphi_{\min}$ there are UV divergent diagrams



These UV divergences can be absorbed in a local redefinition of the action of each particle. The Schwarzschild BHs are "described" by $\varphi_{\min}$

Is this a fine-tuning? Yes, but we are interested in large distance physics $b > R_i \simeq GE_i$. When does this EFT approach break down for BHs?

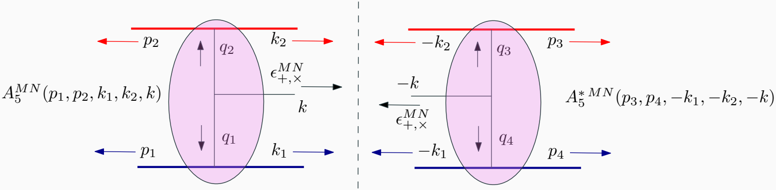
- Orthodox answer: when large curvatures arise
- At $b \sim R_i$: new physics at the horizon scale? Maybe even at larger scales as possible in string theory (due to tidal effects)?

D'Appollonio, Di Vecchia, RR, Veneziano 1310.1254 and refs therein

Novelties at 3PM

The 3PM eikonal δ_2 is derived by taking the classical limit of the 2-loop amplitude. It presents several novelties:

- It is the first term (in the scalar case) that cannot be obtained from the probe limit (Damour 1912.02139)
- It has an imaginary part: elastic unitarity is lost. This is due to the existence of a 3-particle cut to the unitary relation



- It is not entirely captured by potential gravitons
- The real part (and thus the deflection angle) has a universal Ultra-Relativistic (UR) limit

Results in $\mathcal{N}=8$ (as an example)

Start from the 2-loop amplitude in $\mathcal{N}=8$ (known in terms of scalar integrals) and extract the first non-analytic terms in the small q expansion

$$\mathcal{A}_2(s,q^2) = \frac{a_2^{\rm sscl}(s)}{(q^2)^{1+2\epsilon}} + \frac{a_2^{\rm scl}(s)}{(q^2)^{\frac{1}{2}+2\epsilon}} + \frac{a_2^{\rm cl}(s)}{(q^2)^{2\epsilon}} + \dots$$

Go to b-space and solve for δ_2. By using also $\delta_{0,1}$ and Δ_1, we get DHRV

radiation reaction

Parra-Martinez, Ruf, Zeng: 2005.04236

$$\begin{aligned}(2\delta_2) = {} & \frac{16m_1^2m_2^2G^3\sigma^6}{b^2(\sigma^2-1)^2} - \frac{16m_1^2m_2^2\sigma^4G^3}{b^2(\sigma^2-1)}\cosh^{-1}(\sigma)\left[1-\frac{\sigma(\sigma^2-2)}{(\sigma^2-1)^{\frac{3}{2}}}\right] \\ & -i\frac{16m_1^2m_2^2G^3}{\pi b^2}\frac{\sigma^4}{(\sigma^2-1)^2}\left\{\frac{1}{\epsilon}\left(\sigma^2+\frac{\sigma(\sigma^2-2)}{(\sigma^2-1)^{\frac{1}{2}}}\cosh^{-1}(\sigma)\right)\right. \\ & -\left(\log(4(\sigma^2-1))-3\log(\pi b^2 e^{\gamma_E})\right)\left[\sigma^2+\frac{\sigma(\sigma^2-2)}{(\sigma^2-1)^{\frac{1}{2}}}\cosh^{-1}(\sigma)\right] \\ & \left.+(\sigma^2-1)\left[1+\frac{\sigma(\sigma^2-2)}{(\sigma^2-1)^{\frac{3}{2}}}\right](\cosh^{-1}(\sigma))^2+\frac{\sigma(\sigma^2-2)}{(\sigma^2-1)^{\frac{1}{2}}}\mathrm{Li}_2(1-z^2)+2\sigma^2\right\}\end{aligned}$$

A consequence of analyticity and crossing – DHRV: 2104.03256

$z=\sigma-\sqrt{\sigma^2-1}$

PN limit $v\to 0$, $\sigma^2-1=v^2(1-v^2)^{-1}\sim v^2$, $\cosh^{-1}(\sigma)\sim v$

In the UR limit ($\sigma \gg 1$), $\mathrm{Re}(2\delta_2) \to \frac{16G^3(m_1m_2\sigma)^2}{b^2}$ which is universal!

Amati, Ciafaloni, Veneziano; Ademollo, Bellini, Ciafaloni; DNRVW 1911.11716; Bern, Ita, Parra-Martinez, Ruf; DHRV: 2008.12743

Radiative effects

Soft eikonal operator

So far we pretended that the elastic scattering exists... but this is not true in GR: $\mathrm{Im}\delta_2$ is divergent! How to define finite observables?

Dress the elastic scattering it with soft gravitons ($\omega < \omega_* \sim \frac{v}{b}$). The emission of such gravitons exponentiate in momentum space

Bloch-Nordsieck, Weinberg; Laddha, Saha, Sahoo, Sen; Addazi, Bianchi, Veneziano

We know that the exchanged gravitons exponentiate in impact parameter space (eikonal exponentiation). Combining the two we obtain

$$S_{s.r.}(\sigma, b; a, a^\dagger) = \exp\left(\frac{1}{\hbar}\int_{\vec{k}}\sum_j \left[f_j(k)a_j^\dagger(k) - f_j^*(k)a_j(k)\right]\right) \times [1 + 2i\Delta(\sigma, b)]\, e^{i\mathrm{Re}2\delta(\sigma,b)}$$

S-matrix with soft gravitons

with

$$f_j(k) = \varepsilon_j^{*\mu\nu}(k)F_{\mu\nu}(k)\,, \quad F^{\mu\nu}(k) = \sum_n \frac{\kappa\, p_n^\mu p_n^\nu}{p_n \cdot k}$$

$a_j^\dagger$ and a_j are the creation/annihilation operators for soft graviton (with physical polarisations $j = 1, 2$ in $D = 4$)

The f_j's act on δ as $Q^\mu = p_1^\mu + p_4^\mu = \hbar\frac{\partial 2\mathrm{Re}\delta}{\partial b_\mu} = 2p\hat{b}^\mu \sin\frac{\chi}{2} = -(p_2^\mu + p_3^\mu)$

The soft energy spectrum

The elastic amplitude $\langle 0|S_{s.r}|0\rangle$ is suppressed: applying the BCH formula to normal order the exponential one generates the divergent part of $\mathrm{Im}2\delta_2$

The final state $S_{s.r}|0\rangle$ contains a coherent superposition of soft gravitons

We can take the expectation value of an observable $\mathcal{O}$ in the final state

$$\langle \mathcal{O}\rangle = \langle 0|S^\dagger_{s.r.}\mathcal{O}\,S_{s.r.}|0\rangle\,.$$

and derive various classical quantities. The soft energy spectrum is

DHRV 2204.02378

$$\frac{dE^{\mathcal{N}=8}}{d\omega} \simeq \frac{4G}{\pi}\Bigg[2m_1m_2\sigma^2\,\frac{\mathrm{arccosh}\sigma}{\sqrt{\sigma^2-1}} - 2m_1m_2\sigma_Q^2\,\frac{\mathrm{arccosh}\sigma_Q}{\sqrt{\sigma_Q^2-1}}$$

$\omega\to 0$ limit

non-linear memory ignored

$$-\frac{(Q^2)^2}{4m_1^2}\,\frac{\mathrm{arccosh}\left(1+\frac{Q^2}{2m_1^2}\right)}{\sqrt{\left(1+\frac{Q^2}{2m_1^2}\right)^2-1}} - \frac{(Q^2)^2}{4m_2^2}\,\frac{\mathrm{arccosh}\left(1+\frac{Q^2}{2m_2^2}\right)}{\sqrt{\left(1+\frac{Q^2}{2m_2^2}\right)^2-1}}\Bigg]_{Q=2p\sin\frac{\Theta_s}{2}}$$

$$\frac{dE^{\mathrm{gr}}}{d\omega} \simeq \frac{4G}{\pi}\Bigg[2m_1m_2\left(\sigma^2-\tfrac{1}{2}\right)\frac{\mathrm{arccosh}\sigma}{\sqrt{\sigma^2-1}} - 2m_1m_2\left(\sigma_Q^2-\tfrac{1}{2}\right)\frac{\mathrm{arccosh}\sigma_Q}{\sqrt{\sigma_Q^2-1}} \qquad \text{with } \sigma_Q = \sigma - \frac{Q^2}{2m_1m_2}$$

$$+\frac{m_1^2}{2} - m_1^2\left(\left(1+\frac{Q^2}{2m_1^2}\right)^2-\frac{1}{2}\right)\frac{\mathrm{arccosh}\left(1+\frac{Q^2}{2m_1^2}\right)}{\sqrt{\left(1+\frac{Q^2}{2m_1^2}\right)^2-1}} + \frac{m_2^2}{2} - m_2^2\left(\left(1+\frac{Q^2}{2m_2^2}\right)^2-\frac{1}{2}\right)\frac{\mathrm{arccosh}\left(1+\frac{Q^2}{2m_2^2}\right)}{\sqrt{\left(1+\frac{Q^2}{2m_2^2}\right)^2-1}}\Bigg]$$

The ultrarelativistic threshold

The standard PM approach is equivalent to a Taylor expansion in $\frac{Q^2}{2m_i^2}$

$$\frac{dE^{\mathcal{N}=8}}{d\omega} \simeq \frac{4G\,Q^2}{\pi}\left[\frac{\sigma^2}{\sigma^2-1} + \frac{(\sigma^2-2)\sigma}{(\sigma^2-1)^{3/2}}\mathrm{arccosh}\sigma\right]$$

$$\frac{dE^{\mathrm{gr}}}{d\omega} \simeq \frac{2G}{\pi}Q^2\left[\frac{8-5\sigma^2}{3(\sigma^2-1)} + \frac{(2\sigma^2-3)\sigma}{(\sigma^2-1)^{3/2}}\mathrm{arccosh}\sigma\right]$$

Notice the relation $\frac{dE^{\mathcal{N}=8}}{d\omega} \simeq \lim_{\epsilon\to 0}[-4\epsilon \mathrm{Im}\delta_2]$ DHRV 2101.05772

An energy crisis at $\sigma \gg 1$?! The soft spectrum is reliable till $\omega_* \sim \frac{1}{b}$

The total energy emitted by soft gravitons is $E^{\mathrm{rad}}_{\mathrm{soft}} \simeq E(c_1 \log(\sigma) + c_2)$ as $\sigma \to \infty$ (c_i are constant $\sim \chi^3$, c_2 is not universal)

However, when $\frac{Q^2}{2m_i^2} \gtrsim 1$, the standard PM expansion breaks down: this happens in the extreme UR regime ($\sigma\,\chi^2 \gtrsim 1$ for $m_j \sim m_i$) D'Eath; Kovacs, Thorne

In the UR limit, the exact formula yield a universal, finite result:
$\frac{dE^{\mathrm{rad}}_{\mathrm{soft}}}{d\omega} \simeq \frac{Gs\chi^2}{\pi}\left[1 + \log\frac{4}{\chi^2}\right]$ Gruzinov,Veneziano; Ciafaloni, Colferai, Veneziano

Beyond the ultrarelativistic threshold

What about the full spectrum? In the regime $1 \ll \sigma < \frac{1}{\chi^2}$, the apparent energy crisis becomes worse $E^{\rm rad} \sim E\chi^3\sqrt{\sigma}$ Herrmann, Parra-Martinez, Ruf, Zeng

The region $\frac{1}{b} < \omega < \frac{\sqrt{\sigma}}{b}$ is the dominant one

In the extreme UR regime, a natural guess is that the $\sqrt{\sigma}$ singularities are replaced by $\frac{1}{\chi}$ (for instance $\omega < \frac{\sqrt{\sigma}}{b} \to \omega < \frac{1}{R}$)

However there might an extra $\log(1/\chi)$ enhancement in $E^{\rm rad}$ (due to the "high frequency" region $\frac{1}{R} < \omega < \frac{1}{R\chi^2}$)

Gruzinov, Veneziano; Colferai, Ciafaloni, Veneziano

There are still open questions:

- How does E^{rad} changes as we move from $\sigma\chi^2 < 1$ to $\sigma\chi^2 \gg 1$?
- Does E^{rad} become universal when $\sigma\chi^2 \gg 1$ (as for the soft spectrum)?

Work in progress

Conclusion

We can use amplitudes based techniques to extract the theoretical information needed to analyse the inspiral/scattering of binary systems

The approach is flexible and can be applied to different theories/objects

It captures all aspects: conservative, radiation-reaction and real radiation

I didn't discuss many interesting technical (construction of the integrands, integration, . . .) and conceptual (KMOC, analytic continuation to the bound case) developments

I focused on the question: is it consistent to model BHs as "elementary" particles when describing the inspiral/scattering phase?

It is a very concrete question and for Schwarzschild BHs we didn't find any problem up to 3PM order. . . but I doubt that this is the whole story (in particular for Kerr)!

Extra slides

Connection to bound orbits

The derivatives of the eikonal give standard observables

$$\text{Time delay} \quad \Delta T = \frac{\partial \operatorname{Re} 2\delta}{\partial E}\,, \qquad \text{Scatt. angle} \quad \chi = \frac{\partial \operatorname{Re} 2\delta}{\partial J}$$

An analytic continuation to $\sigma < 1$ describes bound states ($E < m1 + m2$).
This implies $\sqrt{\sigma^2 - 1} \to i\sqrt{1 - \sigma^2}$, $b \to \pm ib$ so as to have $J \to \pm J$

Kälin, Porto

By using the eikonal $\tilde{\delta}$ after analytic continuation, we can introduce the periastron advance K and the period P

$$P = \left[\frac{\partial \operatorname{Re} 2\tilde{\delta}}{\partial E} - (J \to -J)\right]\,, \quad K - 1 = \frac{1}{2\pi}\left[\frac{\partial \operatorname{Re} 2\tilde{\delta}}{\partial J} + (J \to -J)\right]$$

From $\tilde{\delta}_{0,1}$ we can derive Eqs. (347) for K and $n = \frac{2\pi}{P}$ of Blanchet's review at all orders in ϵ and first subleading order in $j = \frac{J^2}{G^2}\frac{\epsilon}{(m_1 m_2)^2}$

Bound orbits: data

From Blanchet's review 1310.1528

$$n = \frac{\varepsilon^{3/2}\,c^3}{G\,m}\Bigg\{1 + \frac{\varepsilon}{8}\,(-15+\nu) + \frac{\varepsilon^2}{128}\left[555 + 30\,\nu + 11\,\nu^2 + \frac{192}{j^{1/2}}(-5+2\,\nu)\right] +$$

$$+ \frac{\varepsilon^3}{3072}\Bigg[-29385 - 4995\,\nu - 315\,\nu^2 + 135\,\nu^3 + \frac{5760}{j^{1/2}}\left(17 - 9\,\nu + 2\,\nu^2\right)$$

$$+ \frac{16}{j^{3/2}}\left(-10080 + (13952 - 123\pi^2)\nu - 1440\nu^2\right)\Bigg] + \mathcal{O}\left(\frac{1}{c^8}\right)\Bigg\}. \tag{347a}$$

$$K = 1 + \frac{3\varepsilon}{j} + \frac{\varepsilon^2}{4}\left[\frac{3}{j}(-5+2\,\nu) + \frac{15}{j^2}(7-2\,\nu)\right]$$

$$+ \frac{\varepsilon^3}{128}\Bigg[\frac{24}{j}(5 - 5\nu + 4\nu^2) + \frac{1}{j^2}\left(-10080 + (13952 - 123\pi^2)\nu - 1440\nu^2\right)$$

$$+ \frac{5}{j^3}\left(7392 + (-8000 + 123\pi^2)\nu + 336\nu^2\right)\Bigg] + \mathcal{O}\left(\frac{1}{c^8}\right). \tag{347b}$$

$$\nu = \frac{m_1 m_2}{(m_1+m_2)^2}\ , \quad \frac{\sqrt{1-2(1-\sigma)\nu}-1}{\nu} = -\frac{\epsilon}{2}\quad , \quad j = \frac{J^2}{G^2}\frac{\epsilon}{(m_1 m_2)^2}$$

The 2PM approximation ($\tilde{\delta}_{0,1}$) reproduce the terms in the boxes plus all the ϵ corrections at the same order of $1/j$.