Quantum Link Models: A Resource Efficient Approach to the Quantum Simulation and Quantum Computation of Gauge Theories

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Phase Transitions
in Particle Physics
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Outline

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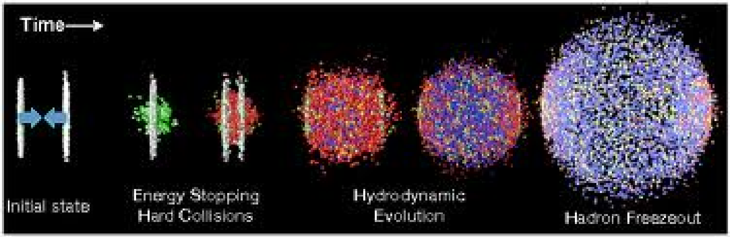
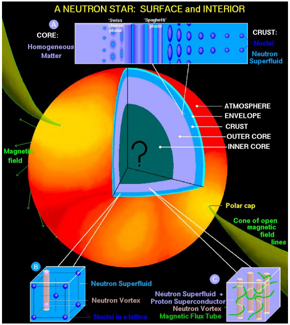
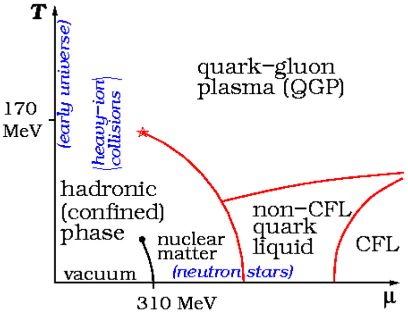
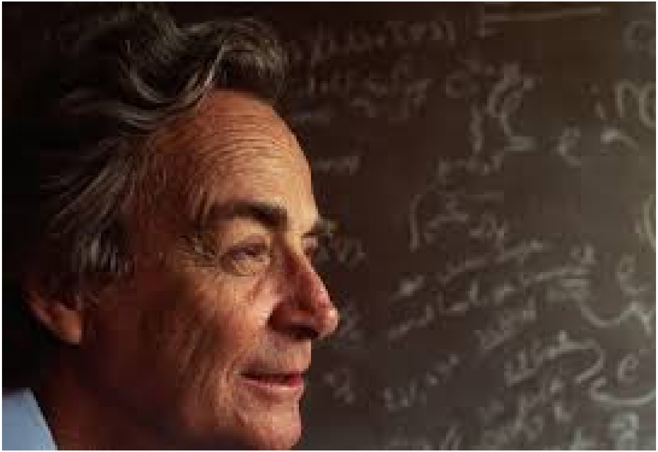
Can heavy-ion collision physics or nuclear astrophysics benefit from quantum simulations in the long run?







Richard Feynman's vision of 1982 (Int. J. Theor. Phys. 21)



"It does seem to be true that all the various field theories have the same kind of behavior, and can be simulated in every way, apparently, with little latticeworks of spins and other things."

Different descriptions of dynamical Abelian gauge fields:

Maxwell's classical electromagnetic gauge fields

$$\vec{\nabla} \cdot \vec{E}(\vec{x}, t) = \rho(\vec{x}, t), \quad \vec{\nabla} \cdot \vec{B}(\vec{x}, t) = 0, \quad \vec{B}(\vec{x}, t) = \vec{\nabla} \times \vec{A}(\vec{x}, t)$$

Quantum Electrodynamics (QED) for perturbative treatment

$$E_i = -i \frac{\partial}{\partial A_i}, \quad [E_i(\vec{x}), A_j(\vec{x}\,')] = i \delta_{ij} \delta(\vec{x} - \vec{x}\,'), \quad \left[\vec{\nabla} \cdot \vec{E} - \rho\right] |\Psi[A]\rangle = 0$$

Wilson's $U(1)$ lattice gauge theory for classical simulation

$$U_{xy} = \exp\left(ie \int_x^y d\vec{l} \cdot \vec{A}\right) = \exp(i\varphi_{xy}) \in U(1), \quad E_{xy} = -i \frac{\partial}{\partial \varphi_{xy}},$$

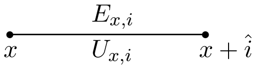
$$[E_{xy}, U_{xy}] = U_{xy}, \quad \left[\sum_i (E_{x,x+\hat{i}} - E_{x-\hat{i},x}) - \rho\right] |\Psi[U]\rangle = 0$$

$U(1)$ quantum link models for quantum simulation

$$U_{xy} = S^+_{xy}, \quad U^\dagger_{xy} = S^-_{xy}, \quad E_{xy} = S^3_{xy},$$

$$[E_{xy}, U_{xy}] = U_{xy}, \quad [E_{xy}, U^\dagger_{xy}] = -U^\dagger_{xy}, \quad [U_{xy}, U^\dagger_{xy}] = 2E_{xy}$$

Hamiltonian formulation of Wilson's $U(1)$ lattice gauge theory



$$U = \exp(i\varphi), \; U^\dagger = \exp(-i\varphi) \in U(1)$$

Electric field operator E

$$E = -i\partial_\varphi, \; [E,U] = U, \; [E,U^\dagger] = -U^\dagger, \; [U,U^\dagger] = 0$$

Generator of $U(1)$ gauge transformations

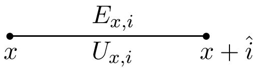
$$G_x = \sum_i (E_{x-\hat{i},i} - E_{x,i}), \quad [H, G_x] = 0$$

$U(1)$ gauge invariant Hamiltonian

$$H = \frac{e^2}{2} \sum_{x,i} E_{x,i}^2 - \frac{1}{2e^2} \sum_{x,i\neq j} (U_{x,i} U_{x+\hat{i},j} U^\dagger_{x+\hat{j},i} U^\dagger_{x,j} + \text{h.c.})$$

operates in an infinite-dimensional Hilbert space per link

Quantum link formulation of $U(1)$ lattice gauge theory



$$U = S^+, \ U^\dagger = S^-$$

Electric field operator E

$$E = S^3, \ [E, U] = U, \ [E, U^\dagger] = -U^\dagger, \ [U, U^\dagger] = 2E$$

Generator of $U(1)$ gauge transformations

$$G_x = \sum_i (E_{x-\hat{i},i} - E_{x,i}), \quad [H, G_x] = 0$$

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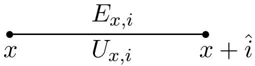
operates in a finite-dimensional Hilbert space per link

Outline

$U(1)$ quantum links from spins $\frac{1}{2}$



$U = S^1 + iS^2 = S^+, \; U^\dagger = S^1 - iS^2 = S^-$

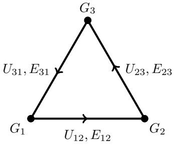
Electric flux operator E

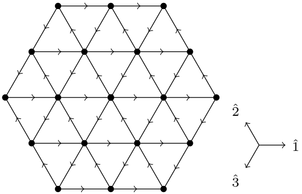
$$E = S^3, \; [E, U] = U, \; [E, U^\dagger] = -U^\dagger, \; [U, U^\dagger] = 2E$$

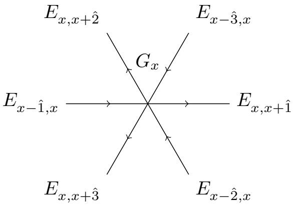
Ring-exchange plaquette Hamiltonian

$$H = -\frac{1}{2e^2} \sum_{\triangle} (U_\triangle + U_\triangle^\dagger) \, , \quad U_\triangle = U_{12} U_{23} U_{31}$$

Triangular lattice and Gauss law





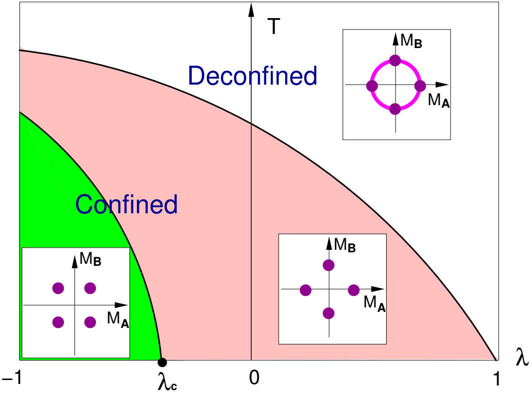


D. Horn, Phys. Lett. B100 (1981) 149
P. Orland, D. Rohrlich, Nucl. Phys. B338 (1990) 647
S. Chandrasekharan, UJW, Nucl. Phys. B492 (1997) 455

Hamiltonian with Rokhsar-Kivelson term

$$H = -\frac{1}{2e^2}\left[\sum_{\triangle}(U_\triangle + U_\triangle^\dagger) - \lambda \sum_{\triangle}(U_\triangle + U_\triangle^\dagger)^2\right]$$

Phase diagram



D. Banerjee, S. Caspar, F.-J. Jiang, J.-H. Peng, UJW, arXiv:2107.01283

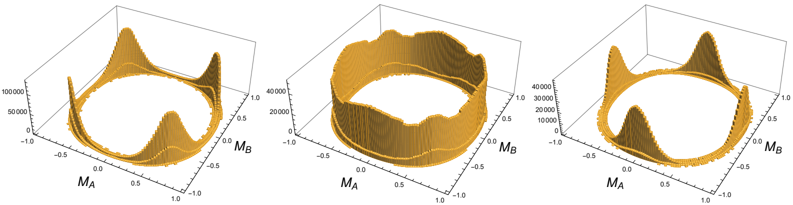
Electric flux via height variables on a dual hexagonal lattice with two sublattices A and B

$$E_{x,x+\hat{i}} = \left(h^A_{\widetilde{x}} - h^B_{\widetilde{x}'}\right) \bmod 2 = \pm\frac{1}{2}\,, \quad h^A_{\widetilde{x}} \in \{0,1\}\,, \quad h^B_{\widetilde{x}} \in \{-\frac{1}{2}, \frac{1}{2}\}\,.$$

Order parameters on the two sublattices

$$M_A = \frac{2}{L^2} \sum_{\widetilde{x}\in A} \left(h^A_{\widetilde{x}} - \frac{1}{2}\right)\,, \quad M_B = \frac{2}{L^2} \sum_{\widetilde{x}\in B} h^B_{\widetilde{x}}\,,$$

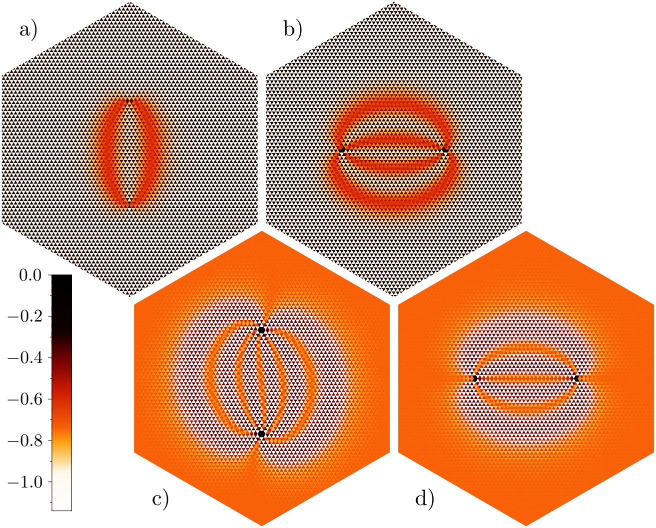
Order parameter distributions in the (M_A, M_B) plane



D. Banerjee, S. Caspar, F.-J. Jiang, J.-H. Peng, UJW, arXiv:2107.01283

Energy distribution for the strings connecting external charges

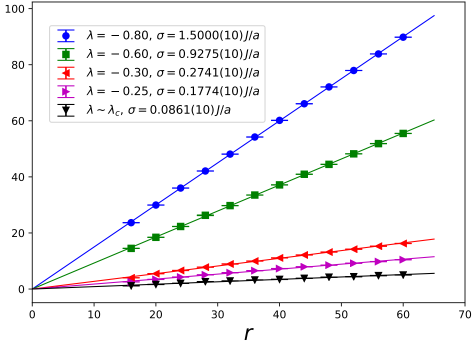
Charges ± 1 (a) and ± 2 (b) for $\lambda > \lambda_c$



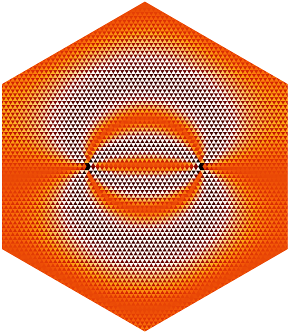
Charges ± 3 (c) and ± 2 (d) for $\lambda < \lambda_c$

D. Banerjee, S. Caspar, F.-J. Jiang, J.-H. Peng, UJW, arXiv:2107.01283

Linear confining potential



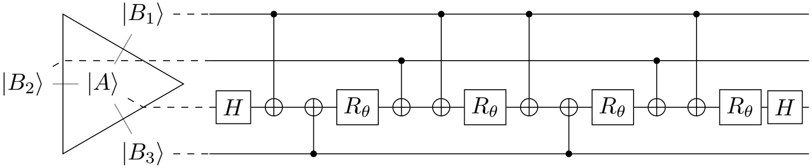
Flux profile at λ_c



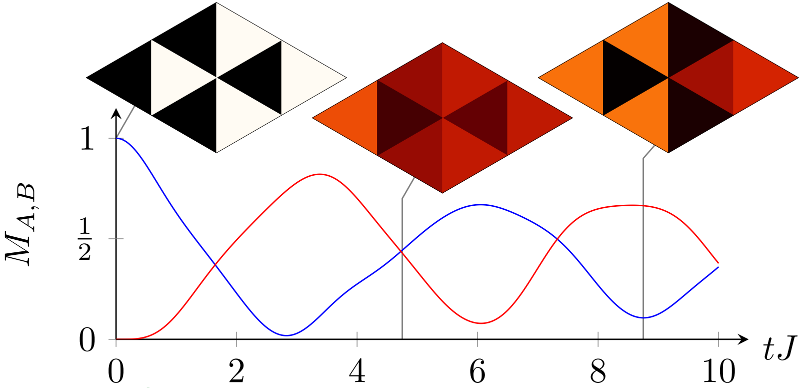
The model can't quite reach a continuum limit.

D-Theory reaches a continuum limit via dimensional reduction of discrete variables, here from a $(3+1)$-d massless Coulomb phase to the confining phase of $(2+1)$-d $U(1)$ gauge theory.

Circuit decomposition of the time-evolution operator

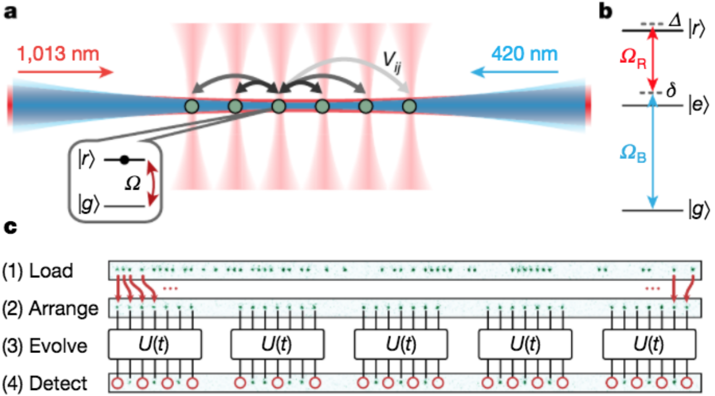


Time-evolution of the order parameters



D. Banerjee, S. Caspar, F.-J. Jiang, J.-H. Peng, UJW, arXiv:2107.01283

A quantum simulator with 51 Rb atoms in optical tweezers



realizes the dynamics of spin $\frac{1}{2}$ quantum link models

H. Bernien, S. Schwartz, A. Keesling, H. Levine, A. Omran, H. Pichler, S. Choi, A. S. Zibrov, M. Endres, M. Greiner, V. Vuletic, M. D. Lukin, Nature 551 (2017) 579.
F. M. Surace, P. P. Mazza, G. Giudici, A. Lerose, A. Gambassi, M. Dalmonte, Phys. Rev. X10 (2020) 021041.
A. Celi, B. Vermersch, O. Viyuela, H. Pichler, M. D. Lukin, P. Zoller, Phys. Rev. X10 (2020) 021057.

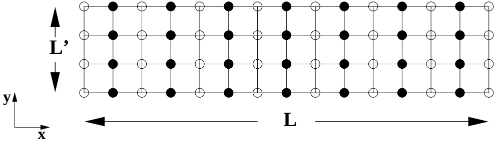
Outline

$\mathbb{C}P(N-1)$ Models from $SU(N)$ quantum spins

$$T_x^a, \quad a \in \{1, 2, ..., N^2 - 1\}, \quad [T_x^a, T_y^b] = i\delta_{xy} f_{abc} T_x^c$$

Spin ladder Hamiltonian

$$H = -J \sum_{x \in A} [T_x^a T_{x+\hat{1}}^{a*} + T_x^a T_{x+\hat{2}}^a] - J \sum_{x \in B} [T_x^{a*} T_{x+\hat{1}}^a + T_x^{a*} T_{x+\hat{2}}^{a*}]$$



Conserved $SU(N)$ spin

$$T^a = \sum_{x \in A} T_x^a - \sum_{x \in B} T_x^{a*}, \quad [T^a, H] = 0$$

B. B. Beard, M. Pepe, S. Riederer, UJW, PRL 94 (2005) 010603.

Goldstone boson fields in $\mathbb{C}P(N-1) = SU(N)/U(N-1)$

$$P(x)^\dagger = P(x), \quad \mathrm{Tr}P(x) = 1, \quad P(x)^2 = P(x)$$

Low-energy effective action

$$\begin{aligned} S[P] &= \int_0^\beta dt \int_0^L dx \int_0^{L'} dy \ \mathrm{Tr}\left\{\rho_s' \partial_y P \partial_y P \right. \\ &+ \left. \rho_s \left[\partial_x P \partial_x P + \frac{1}{c^2}\partial_t P \partial_t P\right] - \frac{1}{a} P \partial_x P \partial_t P \right\} \end{aligned}$$

Topological charge

$$Q[P] = \frac{1}{\pi i}\int_0^\beta dt \int_0^L dx \ \mathrm{Tr}[P\partial_x P \partial_t P] \in \Pi_2[SU(N)/U(N-1)] = Z$$

Dimensional reduction to the $(1+1)$-d $\mathbb{C}P(N-1)$ model

$$S[P] = \int_0^\beta dt \int_0^L dx \, \mathrm{Tr} \left\{ \frac{1}{g^2} \left[\partial_x P \partial_x P + \frac{1}{c^2} \partial_t P \partial_t P \right] - n P \partial_x P \partial_t P \right\}$$

Asymptotically free coupling and emergent θ-vacuum angle

$$\frac{1}{g^2} = \frac{\rho_s L'}{c}, \quad \theta = n\pi$$

Very large correlation length

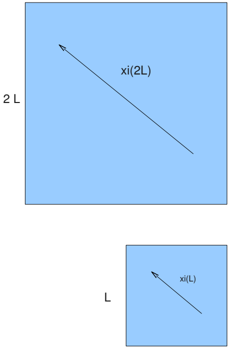
$$\xi \propto \exp\left(\frac{4\pi}{Ng^2}\right) = \exp\left(\frac{4\pi L' \rho_s}{cN}\right) \gg L'$$

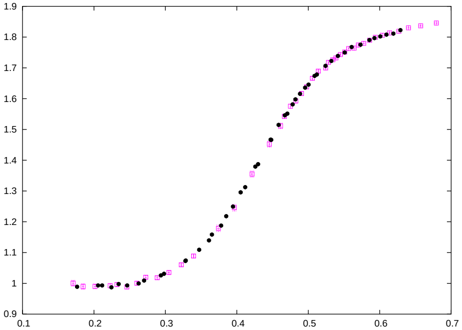
Measured correlation lengths in the $\mathbb{C}P(2)$ model

$$\xi(2) = 4.0(1), \quad \xi(4) = 17.6(2), \quad \xi(6) = 61(2)$$

Step-scaling approach

M. Lüscher, P. Weisz, and U. Wolff, Nucl. Phys. B359 (1991) 221



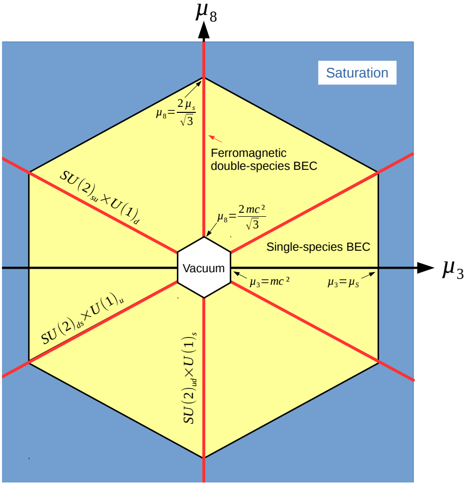


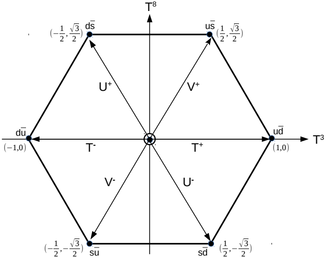
Universal step-scaling function

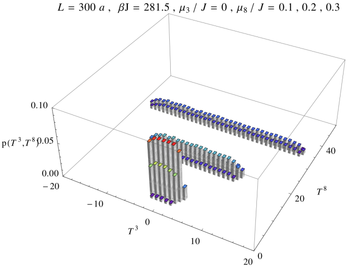
$$F(z) = \xi(2L)/\xi(L) \ , \quad z = \xi(L)/L$$

B. B. Beard, M. Pepe, S. Riederer, UJW, PRL 94 (2005) 010603.

Ferromagnetic Double-Species BEC in the $\mathbb{C}P(2)$ Model



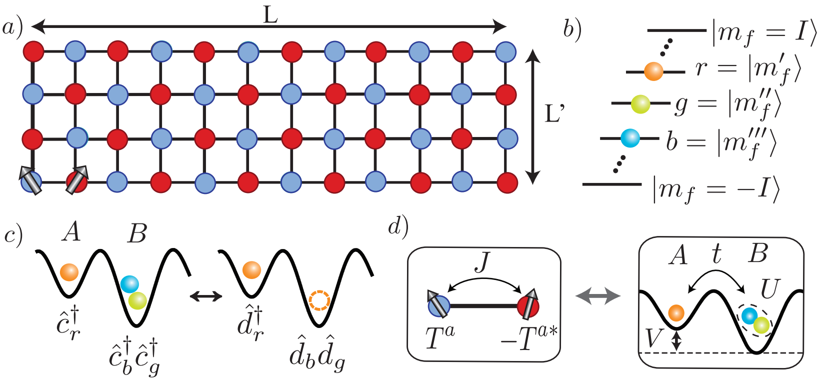




W. Evans, U. Gerber, M. Hornung, UJW, Annals Phys. 398 (2018) 92.

Ladder of $SU(N)$ quantum spins embodied with alkaline-earth atoms

$$H = -J \sum_{\langle xy \rangle} T_x^a T_y^{a*}, \quad [T_x^a, T_y^b] = i\delta_{xy} f_{abc} T_x^c$$



C. Laflamme, W. Evans, M. Dalmonte, U. Gerber, H. Mejia-Diaz, W. Bietenholz, UJW, and P. Zoller, Annals Phys. 360 (2016) 117.

Outline

$U(N)$ quantum link operators

$$U^{ij} = S_1^{ij} + iS_2^{ij},\ U^{ij\dagger} = S_1^{ij} - iS_2^{ij},\ i,j \in \{1,2,\dots,N\},\ [U^{ij},(U^\dagger)^{kl}] \neq 0$$

$SU(N)_L \times SU(N)_R$ gauge transformations of a quantum link

$$[L^a, L^b] = if_{abc}L^c,\ [R^a, R^b] = if_{abc}R^c,\ a,b,c \in \{1,2,\dots,N^2-1\}$$

$$[L^a, R^b] = [L^a, E] = [R^a, E] = 0$$

Infinitesimal gauge transformations of a quantum link

$$[L^a, U] = -\lambda^a U,\ [R^a, U] = U\lambda^a,\ [E, U] = U$$

Algebraic structures of different quantum link models

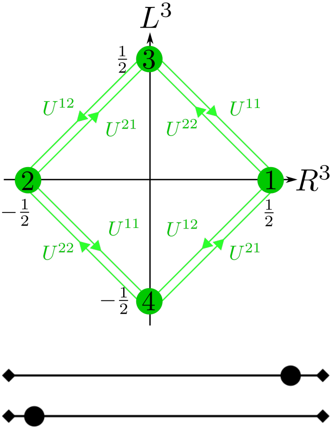
$U(N)$: $U^{ij},\ L^a,\ R^a, E,\ 2N^2+2(N^2-1)+1 = 4N^2-1$ $SU(2N)$ generators

$SO(N)$: $O^{ij},\ L^a,\ R^a,\ N^2+2\dfrac{N(N-1)}{2} = N(2N-1)$ $SO(2N)$ generators

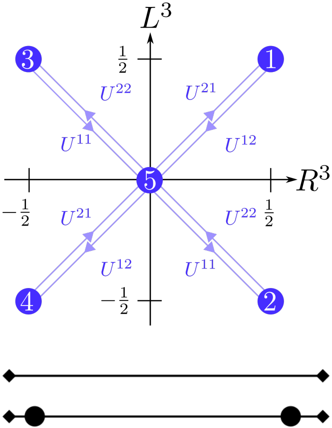
$Sp(N)$: $U^{ij},\ L^a,\ R^a,\ 4N^2+2N(2N+1) = 2N(4N+1)$ $Sp(2N)$ generators

R. Brower, S. Chandrasekharan, UJW, Phys. Rev. D60 (1999) 094502

$Sp(1) = SU(2)$ quantum link model with an $Sp(2) = SO(5)$ embedding algebra



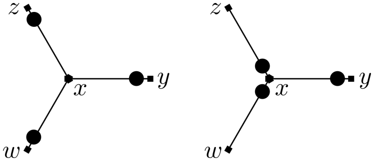
Spinor representation $\{4\}$



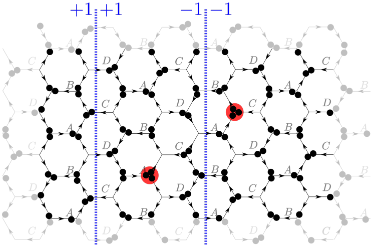
Vector representation $\{5\}$

D. Banerjee, F.-J. Jiang, T. Z. Olesen, P. Orland, UJW, Phys. Rev. B97 (2018) 205108.

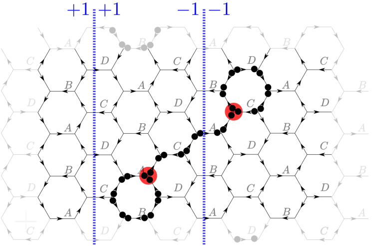
Gauss law for the $Sp(1) = SU(2)$ quantum link model on the honeycomb lattice



Flux strings in the $Sp(1) = SU(2)$ quantum link model



Spinor representation $\{4\}$



Vector representation $\{5\}$

Outline

From Wilson's Lattice Gauge Theory to Quantum Link Models

$U(1)$ Quantum Link Model on a Triangular Lattice

D-Theory: $(1+1)$-d $\mathbb{C}P(N-1)$ Models from the Dimensional Reduction of $(2+1)$-d $SU(N)$ Quantum Spin Ladders

Non-Abelian Quantum Link Models

D-Theory Formulation of QCD

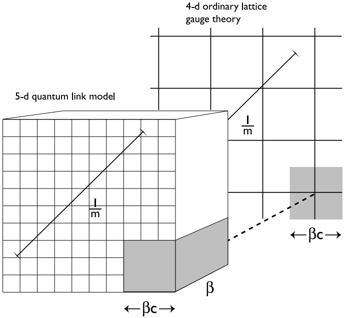
Conclusions

Low-energy effective action for a quantum link model in a $(4+1)$-d massless non-Abelian Coulomb phase

$$S[G_\mu] = \int_0^\beta dx_5 \int d^4x \, \frac{1}{2e^2} \left(\text{Tr } G_{\mu\nu} G_{\mu\nu} + \frac{1}{c^2} \text{Tr } G_{\mu 5} G_{\mu 5} \right),$$

undergoes dimensional reduction from $4+1$ to 4 dimensions

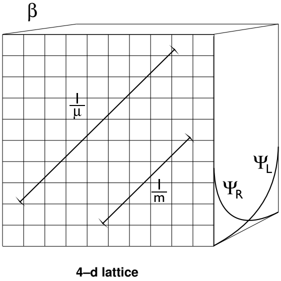
$$S[G_\mu] \to \int d^4x \, \frac{1}{2g^2} \text{Tr } G_{\mu\nu} G_{\mu\nu}, \; \frac{1}{g^2} = \frac{\beta}{e^2}, \; \frac{1}{m} \sim \exp\left(\frac{24\pi^2\beta}{11Ne^2}\right)$$



R. Brower, S. Chandrasekharan, UJW, Phys. Rev. D60 (1999) 094502

Quarks as Domain Wall Fermions

$$
\begin{aligned}
H &= J \sum_{x,\mu\neq\nu} \mathrm{Tr}[U_{x,\mu} U_{x+\hat{\mu},\nu} U^\dagger_{x+\hat{\nu},\mu} U^\dagger_{x,\nu}] + J' \sum_{x,\mu} [\det U_{x,\mu} + \det U^\dagger_{x,\mu}] \\
&+ \frac{1}{2} \sum_{x,\mu} [\Psi^\dagger_x \gamma_0 \gamma_\mu U_{x,\mu} \Psi_{x+\hat{\mu}} - \Psi^\dagger_{x+\hat{\mu}} \gamma_0 \gamma_\mu U^\dagger_{x,\mu} \Psi_x] + M \sum_x \Psi^\dagger_x \gamma_0 \Psi_x \\
&+ \frac{r}{2} \sum_{x,\mu} [2\Psi^\dagger_x \gamma_0 \Psi_x - \Psi^\dagger_x \gamma_0 U_{x,\mu} \Psi_{x+\hat{\mu}} - \Psi^\dagger_{x+\hat{\mu}} \gamma_0 U^\dagger_{x,\mu} \Psi_x].
\end{aligned}
$$

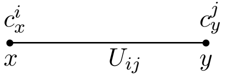


$$
\mu = 2M \exp(-M\beta), \ \frac{1}{m} \propto \exp\left(\frac{24\pi^2 \beta}{(11N - 2N_f)e^2}\right), \ M > \frac{24\pi^2}{(11N - 2N_f)e^2}
$$

Fermionic rishons at the two ends of a link

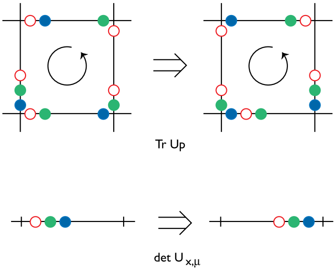
$$\{c_x^i, c_y^{j\dagger}\} = \delta_{xy}\delta_{ij}, \ \{c_x^i, c_y^j\} = \{c_x^{i\dagger}, c_y^{j\dagger}\} = 0$$

Rishon representation of link algebra



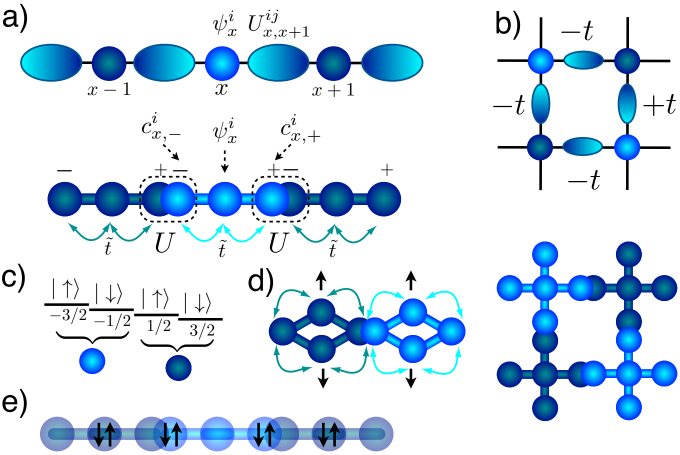
$$U_{xy}^{ij} = c_x^i c_y^{j\dagger}, \ L_{xy}^a = c_x^{i\dagger}\lambda_{ij}^a c_x^j, \ R_{xy}^a = c_y^{i\dagger}\lambda_{ij}^a c_y^j, \ E_{xy} = \frac{1}{2}(c_y^{i\dagger}c_y^i - c_x^{i\dagger}c_x^i)$$

Can a "rishon abacus" implemented with ultra-cold atoms be used as a quantum simulator?



R. Brower, S. Chandrasekharan, UJW, Phys. Rev. D60 (1999) 094502

Optical lattice with ultra-cold alkaline-earth atoms (^{87}Sr or ^{173}Yb) with color encoded in nuclear spin



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Outline

Conclusions

• Quantum link models provide a generalization of Wilson's lattice gauge theory with a finite-dimensional Hilbert space per link, which allows their embodiment in ultra-cold quantum matter.

• Quantum simulator constructions have already been realized for Wilson's lattice gauge theory as well as for quantum link models. The $U(1)$ quantum link model can be encoded very efficiently in a quantum circuit. $\mathbb{C}P(N-1)$ models as well as non-Abelian $U(N)$ and $SU(N)$ quantum link models can be embodied by alkaline-earth atoms.

• This allows the quantum simulation of the real-time evolution of string breaking as well as of vacuum tunneling. Accessible effects also include simplified variants of chiral symmetry restoration at high baryon density or the expansion of a hot quark-gluon plasma.

• In quantum spin and quantum link models regularizing asymptotically free theories, including $(1+1)$-d $\mathbb{C}P(N-1)$ models and $(3+1)$-d QCD, the continuum limit is taken in the D-theory framework by the dimensional reduction of discrete variables.

• The path towards quantum simulation of QCD will be a long one.

However, with a lot of interesting physics along the way.