From Gumbel to Tracy–Widom via random (ordinary, plane, and cylindric plane) partitions

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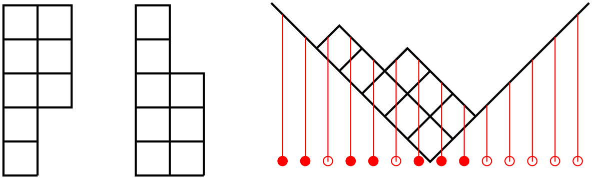
Goal

We aim to show how seemingly simple random combinatorial structures (partitions, plane partitions, cylindric partitions) lead to complex asymptotic probabilistic behavior often encountered (and universal to) the theory of random matrices.

Uniform v Plancherel measure on partitions, Ulam's problem

Partitions

- an *integer partition* (often just partition) $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq 0)$ is a non-increasing sequence of non-negative integers which is eventually 0
- $\lambda = (2, 2, 2, 1, 1)$, pictured below, has *size* $|\lambda| := \sum_i \lambda_i = 8$ and *length* $\ell(\lambda) := \max\{i : \lambda_i > 0\} = 5$
- a partition λ is identified with it's Young diagram



  or Maya diagram/particle ensemble $\{\lambda_i - i + \frac{1}{2}\}$
- partitions are natural discrete models of both fermions (this talk!) and bosons

This talk deals exclusively with extremal statistics on partitions, i.e. the distribution of the largest part λ_1 or the right-most particle position. All results can be extended further to other parts/particles.

Two natural measures on partitions

- (poissonized) Plancherel

$$\mathbb{P}(\lambda) = e^{-\theta^2} \theta^{2|\lambda|} \frac{(\dim \lambda)^2}{(|\lambda|!)^2}$$

  where $\dim \lambda$ = number of fillings of λ with numbers $1, \ldots, |\lambda|$ increasing down rows and columns (standard Young tableaux) and $\theta > 0$ is a parameter

- uniform

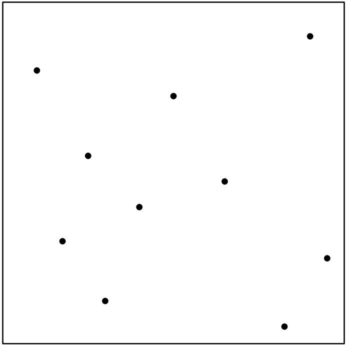
$$\mathbb{P}(\lambda) = (u; u)_\infty u^{|\lambda|}$$

  where $(x; u)_n = \prod_{0 \leq i \leq n-1} (1 - xu^i)$ and $0 \leq u < 1$ is a parameter.

These measures have different extremal statistics asymptotic behavior:

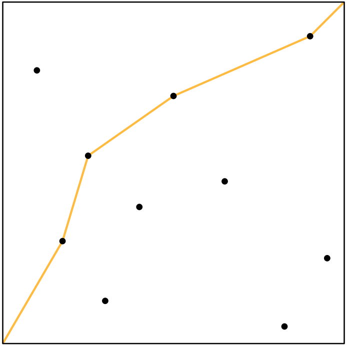
- Plancherel: λ_1 is asymptotically Tracy–Widom GUE in law, a distribution universal for the largest eigenvalue of random hermitian matrices (and other *correlated* systems)
- uniform: λ_1 is asymptotically Gumbel, a distribution universal as maximum of iid random variables

Ulam's problem and Hammersley last passage percolation I



$PPP(\theta^2)$ in the unit square.

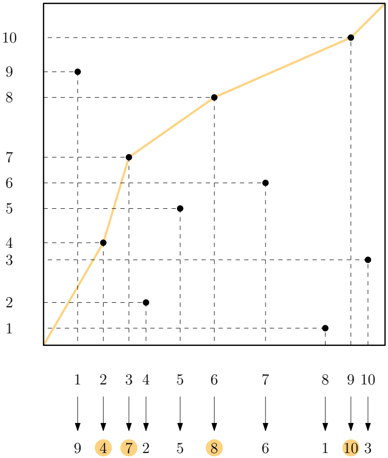
Ulam's problem and Hammersley last passage percolation II



Quantity of interest: L = longest up-right path from $(0,0)$ to $(1,1)$ ($= 4$ here).

Interest: $\theta \to \infty$? (large number of points, large random permutation, ...)

Ulam's problem and Hammersley last passage percolation III



L is the length (any) of the longest increasing subsequence in a random permutation of S_N with $N \sim Poisson(\theta^2)$.

Hammersley LPP and the poissonized Plancherel measure

By the Robinson–Schensted–Knuth correspondence and Schensted's theorem,

$$L = \lambda_1$$

in distribution where λ has the poissonized Plancherel measure:

$$\mathbb{P}(\lambda) = e^{-\theta^2}\theta^{2|\lambda|}\frac{(\dim\lambda)^2}{(|\lambda|!)^2} = e^{-\theta^2}s_\lambda(pl_\theta)s_\lambda(pl_\theta).$$

where $s_\lambda(pl_\theta) = \det_{1\le i,j\le \ell(\lambda)} h_{\lambda_i-i+j}(pl_\theta)$ and $h_k(pl_\theta) = \frac{\theta^k}{k!}\mathbb{1}_{k\ge 0}$.

(repr. theoretically $\dim\lambda := \dim_{S_n}\lambda$ = number of standard Young tableaux of shape λ, λ an irrep. of S_n with $n = |\lambda| \sim Poisson(\theta^2)$, s is a Schur function, pl_θ the Plancherel specialization sending $p_1 \to \theta, p_i \to 0, i \ge 2$).

The Baik–Deift–Johansson theorem and Tracy–Widom

Theorem (BaiDeiJoh99)

We have:

$$\lim_{\theta\to\infty} \mathbb{P}\left(\frac{L-2\theta}{\theta^{1/3}} \leq s\right) = F_{\mathrm{GUE}}(s) := \det(1 - Ai_2)_{L^2(s,\infty)}$$

with

$$Ai_2(x,y) := \int_0^\infty Ai(x+s)Ai(y+s)ds$$

and Ai the Airy function (solution of $y'' = xy$ decaying at ∞).

$F_{\mathrm{GUE}}(s)$ is the Tracy–Widom GUE distribution—the the universal asymptotic distribution of the largest eigenvalue of random (iid entries) Hermitian random matrices.

Unpacking the result: Fredholm determinants

Let $(X, d\mu)$ be a measured space (discrete with counting or continuous with Lebesgue) and A a trace-class operator on $\mathcal{L}^2X$ with kernel $A(x,y)$. That is, A acts on functions f as

$$(Af)(x) = \int_X A(x,y)f(y)d\mu(y).$$

The Fredholm determinant $\det(1+zA)_{\mathcal{L}^2X}$ is defined by:

$$\begin{aligned}\det(1+zA)_{\mathcal{L}^2X} &= 1 + \sum_{N\geq 1}\frac{z^N}{N!}\int_X d\mu(x_1)\cdots\int_X d\mu(x_N)\det_{1\leq a,b\leq N} A(x_a,x_b)\\ &= \exp\sum_{j\geq 1}\frac{(-1)^{j-1}z^j \operatorname{tr} A^j}{j}.\end{aligned}$$

We are usually interested in $X=(s,\infty)$ (and $d\mu$ is Lebesgue) or $X=\{\ell,\ell+1,\ell+2,\dots\}$ (and $d\mu$ is counting), and in $z=-1$. For operators A with kernels as above, $\operatorname{tr} A = \int_X A(x,x)d\mu(x)$.

The Erdős–Lehner theorem and Gumbel

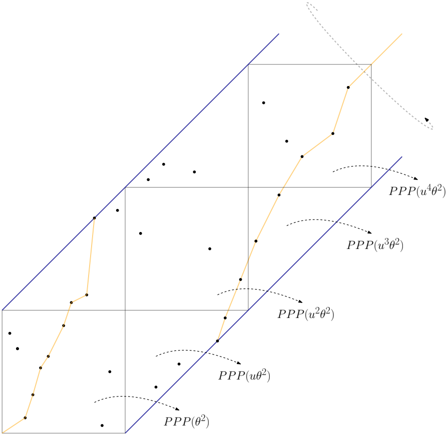
Theorem (ErdLeh41)

For the uniform measure $\mathbb{P}(\lambda) \propto u^{|\lambda|}$ *we have, as* $u = 1 - M^{-1} \to 1$ *as* $M \to \infty$*, the following Gumbel limit law:*

$$\lim_{M\to\infty} \mathbb{P}\left(\frac{\lambda_1 - 2M\log M}{M} < \xi\right) = e^{-e^{-\xi}}.$$

Cylindrical geometry and finite temperature systems

Cylindrical geometry



With L the longest up-right path in this cylindric geometry, let

$$\lambda_1 = L + \kappa_1$$

where $\kappa = (\kappa_1 \geq \kappa_2 \geq \cdots \geq 0)$ is an independent uniform partition $\mathbb{P}(\kappa) \propto u^{|\kappa|}$.

Imamura–Mucciconi–Sasamoto 2021 show that L is distributed according to a q-Whittaker measure.

The finite temperature Plancherel measure

λ_1 comes from the following 2-partition model (Borodin 06): on pairs $\mu \subset \lambda$ consider the measure

$$\mathbb{P}(\mu, \lambda) \propto u^{|\mu|} \cdot \frac{\theta^{2(|\lambda|-|\mu|)} \dim^2(\lambda/\mu)}{(|\lambda/\mu|!)^2} \propto u^{|\mu|} s_{\lambda/\mu}(pl_\theta) s_{\lambda/\mu}(pl_\theta)$$

with $u = e^{-\beta}$, $\beta =$ inverse temperature, $\dim(\lambda/\mu) =$ number of SYT of shape λ/μ, and $s_{\lambda/\mu}(pl_\theta) = \det\limits_{1 \le i,j \le \ell(\lambda)} h_{\lambda_i - i - \mu_j + j}(pl_\theta)$ with $h_k(pl_\theta) = \frac{\theta^k}{k!} \mathbb{1}_{k \ge 0}$.

- $u = 0$ yields the poissonized Plancherel measure
- $\theta = 0$ yields the (grand canonical) uniform measure

A finite temperature analogue of the Baik–Deift–Johansson theorem

Theorem (B/Bouttier 2019)

Let $M = \frac{\theta}{1-u} \to \infty$ *and* $u = \exp(-\alpha M^{-1/3}) \to 1$. *Then*

$$\lim_{M\to\infty} \mathbb{P}\left(\frac{\lambda_1 - 2M}{M^{1/3}} \leq s\right) = F^{\alpha}(s) := \det(1 - Ai^{\alpha})_{L^2(s,\infty)}$$

with

$$Ai^{\alpha}(x,y) := \int_{-\infty}^{\infty} \frac{e^{\alpha s}}{1+e^{\alpha s}} \cdot Ai(x+s)Ai(y+s)ds$$

the finite temperature Airy kernel.

A word on the finite temperature Airy kernel and distribution

$F^{\alpha}(s)$ (Johansson 2006) interpolates between:

- the Tracy–Widom GUE $F_{\mathrm{GUE}}(s)$ law (universal max of corr. systems, $\alpha \to \infty$)
- and the Gumbel law $e^{-e^{-s}}$ (universal max of iid rv's, $\alpha \to 0+$)

$$\lim_{\alpha\to\infty} F^{\alpha}(s) = F_{\mathrm{GUE}}(s), \quad \lim_{\alpha\to 0+} \frac{1}{\alpha} F^{\alpha}\left(\frac{s}{\alpha} - \frac{\log(4\pi\alpha^3)}{2\alpha}\right) = e^{-e^{-s}}$$

as the kernel itself interpolates between Airy and diagonal exponential

$$\lim_{\alpha\to\infty} Ai^{\alpha}(x,y) = Ai_2(x,y), \lim_{\alpha\to 0+} \frac{1}{\alpha} Ai^{\alpha}\left(\frac{x}{\alpha} - \frac{\log(4\pi\alpha^3)}{2\alpha}, \frac{y}{\alpha} - \frac{\log(4\pi\alpha^3)}{2\alpha}\right) = e^{-x}\delta_{x,y}.$$

Interlude: more on the finite temperature Airy kernel and distribution

The distribution $F^{\alpha}(s) = \det(1 - Ai^{\alpha})_{L^2(s,\infty)}$ appeared in the edge/long time scaling of:

- finite temperature random matrix models and continuous fermions: Johansson (2006), Dean–Le Doussal–Majumdar–Schehr (2015+), Liechty–Wang (2018), ...
- finite temperature discrete fermionic models: B–Bouttier (2019)
- the KPZ equation and related models: Sasamoto–Spohn (2010), Calabrese–Le Doussal–Rosso (2010), Amir–Corwin–Quastel (2011), Borodin–Corwin–Ferrari (2014), Dimitrov (2018), Imamura–Mucciconi–Sasamoto 2021, ...

and has been studied in its own right by the Riemann–Hilbert community recently (Cafasso–Claeys 2019, Bothner–Cafasso–Tarricone 2021, ...).

Interlude: Johansson's result and the Moshe–Neuberger–Shapiro matrix model

Consider the following (Moshe–Neuberger–Shapiro) 2-matrix distribution (U unitary, H hermitian $N \times N$ matrices):

$$\mathbb{P}(U, H) \propto e^{-(2b+1)\,\mathrm{tr}\,H^2} e^{2b\,\mathrm{tr}\,UHU^{-1}H} dUdH \tag{1}$$

with dU Haar measure and dH Lebesgue measure. Let $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_N$ be the eigenvalues of H.

Theorem (Johansson 2006, Liechty–Wang 2018)

It holds that

$$\lim_{N\to\infty} \mathbb{P}\left(\sqrt{2}N^{1/6}(\lambda_1 - \sqrt{2N}) \leq s\right) = F^{\alpha}(s).$$

Remark: the statement is an oversimplification for many reasons (for one, I haven't told you what happens to b in the $N \to \infty$ limit).

A sketch of the proof

- put everything in (fermionic gl_∞) Fock space
- passage to the grand canonical ensemble: on triples $c \in \mathbb{Z}, \mu \subset \lambda \supset \mu$ consider:

$$\mathbb{P}(\mu, \lambda, c) \propto u^{|\mu| + \frac{c^2}{2}} t^c \cdot \frac{\theta^{2(|\lambda| - |\mu|)} \dim^2(\lambda/\mu)}{(|\lambda/\mu|!)^2} \propto u^{|\mu| + \frac{c^2}{2}} t^c s_{\lambda/\mu}(pl_\theta) s_{\lambda/\mu}(pl_\theta)$$

$u = e^{-\beta}$ with β inverse temperature; $t = e^f$ with f chemical potential; c = charge

- (Borodin 06, B/Bouttier 2019, Wick lemma in finite temp.): the ensemble $\{\lambda_i - i + c + 1/2\}$ is determinantal with Bessel fn. J corr. kernel

$$\mathbb{P}(\{k_1, \ldots, k_n\} \subset \{\lambda_i - i + c + 1/2\}) = \det_{1 \le i,j \le n} K(k_i, k_j)$$

$$K(a, b) = \sum_{\ell \in \mathbb{Z} + 1/2} J_{a-\ell}\left(\tfrac{2\theta}{1-u}\right) J_{b-\ell}\left(\tfrac{2\theta}{1-u}\right) \frac{u^\ell t}{1 + u^\ell t}$$

- dist. of $\lambda_1 + c$ = Fredholm det. of above op. (inclusion-exclusion)
- Nicholson's approximation $M^{1/3} J_{2M + xM^{1/3}}(2M) \to Ai(x), \ M \to \infty$ or steepest descent analysis yields discrete kernel $\to$ fin. temp. Airy kernel
- a few more estimates to conclude convergence of Fredholm det.
- one can remove the charge c in the end

More details: free fermions in finite temperature

Let ψ_k, ψ_k^* for $k \in \mathbb{Z}' := \mathbb{Z} + \frac{1}{2}$ be creation/annihilation operators satisfying CAR:

$$\psi_k \psi_\ell^* + \psi_\ell^* \psi_k = \delta_{k,\ell}, \quad \psi_k \psi_\ell + \psi_\ell \psi_k = 0, \quad \psi_k^* \psi_\ell^* + \psi_\ell^* \psi_k^* = 0$$

acting on Fock space basis vectors $|\lambda, c\rangle$ by placing/removing particles at position k (with signs). Let $\alpha_{\pm 1} = \sum_{k \in \mathbb{Z}'} \psi_k \psi_{k \pm 1}^*$ be Heisenberg bosons ($[\alpha_1, \alpha_{-1}] = 1$).

Then $\mathbb{P}(\lambda, \mu, c) \propto \langle \mu, c | u^H t^C e^{\theta \alpha_1} | \lambda, c \rangle \langle \lambda, c | e^{\theta \alpha_{-1}} | \mu, c \rangle$ and moreover

$$\mathbb{P}(\lambda \text{ has particles at pos } k_1, \ldots, k_n) \propto \operatorname{tr} \left(u^H t^C e^{\theta \alpha_1} \prod_{i=1}^{n} \psi_{k_i} \psi_{k_i}^* e^{\theta \alpha_{-1}} \right)$$

$$\propto \det_{1 \le i,j \le n} \operatorname{tr} \left(u^H t^C e^{\theta \alpha_1} \psi_{k_i} \psi_{k_j}^* e^{\theta \alpha_{-1}} \right)$$

(Wick lemma in fin. temp., with $H|\lambda, c\rangle = (|\lambda| + \frac{c^2}{2})|\lambda, c\rangle$, $C|\lambda, c\rangle = c|\lambda, c\rangle$), and placing generating functions $\Psi(z) = \sum_k \psi_k z^k$, $\Psi^*(w) = \sum_k \psi_k^* w^{-k}$, playing with commutation relations, and extracting Fourier coeff. $\to$ kernel above.

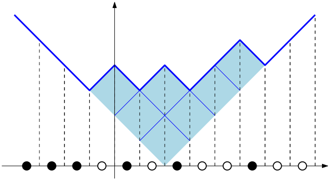


Figure: The Fock basis state $|(4, 2, 1), 2\rangle$.

All three asymptotic regimes into one picture

Theorem (B/Bouttier 2019)

With $u = e^{-\beta} \to 1$ *as* $\beta \to 0+$ *and* $\theta \to \infty$ *(or finite) we have:*

- $\theta\beta^2 \to 0+$ *leads to Gumbel behavior: thermal fluctuations win;*
- $\theta\beta^2 \to \infty$ *leads to Tracy–Widom: quantum fluctuations win;*
- $\theta\beta^2 \to \alpha^3 \in (0, \infty)$ *leads to finite temperature Tracy–Widom* F^{α}*: equilibrium between thermal and quantum.*

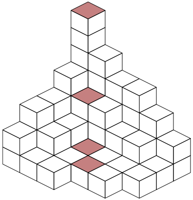
Plane partitions, infinite geometries and the RMT hard-edge

Infinite geometry LPP and (plane) partitions

- let $\Lambda_{1,1}$ = height of peak of a plane partition distributed as

$$\mathbb{P}(\Lambda) \propto q^{\mathrm{vol}(\Lambda)} a^{\,\mathrm{tr}\,(\Lambda)}$$

(= 7 in example)



($\mathrm{tr}\,(\Lambda)$ = number of cubes in the middle = 12 in our example)

- $\Lambda_{1,1} = \lambda_1$ in dist. with λ_1 = first part of random λ from dist.:

$$\mathbb{P}(\lambda) \propto (aq)^{|\lambda|}[s_\lambda(1, q, q^2, \dots)]^2$$

(here the Schur functions s_λ are evaluated in infinite geometric progressions).

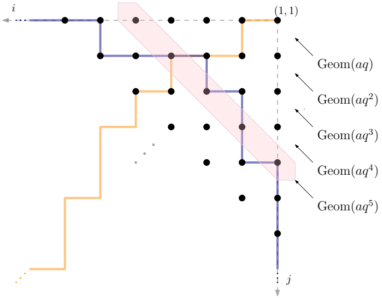
Infinite geometry LPP, equi-distributed diagonals

By the Robinson–Schensted–Knuth and Burge correspondences + Greene and Krattenthaler theorems, we have:

$$L_1 = L_2 = \Lambda_{1,1} = \lambda_1 \text{ in distribution}$$

with L_1, L_2 the longest directed polymers given below:

- equi-distributed-by-diagonal full quarter plane, where on diagonal $i + j = k + 1$ ($k \geq 1$) each variable is iid $Geom(aq^{i+j-1}) = Geom(aq^k)$
- $\mathbb{P}(Geom(u) = k) = u^k(1-u), \; k \in \mathbb{N}$
- L_1 = maximal (sum) path from $(1,1) \to (\infty, \infty)$ using down-left steps (orange)
- L_2 = maximal (sum) path from $(\infty, 1) \to (1, \infty)$ using down-right steps (blue)



Equi-distributed diagonals main result

Theorem (B/Occelli 2020)

Let $q = e^{-\varepsilon}, a = e^{-\alpha\varepsilon}$ *and* $L \in \{L_1, L_2, \Lambda_{1,1}, \lambda_1\}$. *We have:*

$$\lim_{\varepsilon \to 0+} \mathbb{P}\left(\frac{L - 2\varepsilon^{-1} \log \varepsilon^{-1}}{\varepsilon^{-1}} < s \right) = \det(1 - O_\alpha)_{L^2(s,\infty)} := G_\alpha(s)$$

where $\boxed{O_\alpha(x,y) := e^{-\frac{x+y}{2}} Bess_\alpha(e^{-x}, e^{-y}),\ Bess_\alpha(x,y) := \int_0^1 J_\alpha(2\sqrt{tx}) J_\alpha(2\sqrt{ty}) dt}$ *is the RMT Bessel kernel (hard-edge of Laguerre/Jacobi ensembles) and J's are Bessel functions.*

Proposition (Joh 08, B/Occelli 2020)

The distribution G_α *interpolates between Gumbel and Tracy–Widom GUE:*

- $G_0(s) = e^{-e^{-s}}$ *is the Gumbel distribution (see my GitHub);*
- $\lim_{\alpha \to \infty} G_\alpha(-2\log(2\alpha) + \alpha^{-2/3} s) = F_{\mathrm{GUE}}(s)$ *(e.g. BorFor 03).*

All three regimes in one picture

As before but with some important differences, we can see the three different regimes explicitly.

Theorem (B/Occelli 2020)

As $q \to 1-$, L has:

- *Gumbel fluctuations, if $a = 1$ (VerYak 06);*
- *Tracy–Widom fluctuations (on a different scale), if $0 < a < 1$ fixed;*
- *transitional (exponential) hard-edge Bessel fluctuations, if $a \to 1$ critically.*

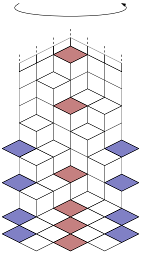
Back to cylinders

Trace-volume-and-seam distributed cylindric plane partitions

Consider the simplest cylindric plane partitions on a circumference-$2N$ cylinder, distributed according to their volume, trace, and seam:

$$\mathbb{P}(\Lambda) = \frac{a^{\operatorname{tr}(\Lambda)} \cdot (a^{-1} q^{-N})^{\operatorname{sm}(\Lambda)} \cdot q^{\operatorname{vol}(\Lambda)}}{Z} \tag{2}$$

where $Z = (q^N; q^N)_\infty^{-1} \prod_{1 \le i,j \le N} (q^{i+j-1}; q^N)_\infty^{-1}$, tr is the trace (number of cubes under dark red, $= 11$ below), and sm is the seam (number of cubes under dark blue, after identification, $= 3$ below)



We again look at the distribution of the peak $\Lambda_{1,1}$.

Some equalities in distribution

From the Robinson–Schensted–Knuth correspondence we again have

$$\Lambda_{1,1} = \lambda_1 = L + \kappa_1 \text{ in distribution}$$

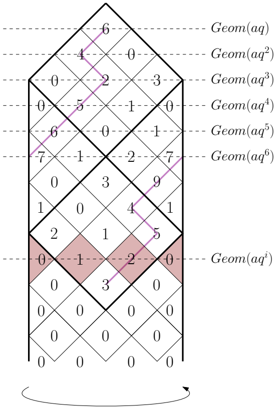
- λ_1 is the first part of a partition coming from the following 2-partition model:

$$\mathbb{P}(\lambda, \mu) \propto q^{N|\mu|}[s_{\lambda/\mu}(\sqrt{a}q^{\frac{1}{2}}, \sqrt{a}q^{\frac{3}{2}}, \ldots, \sqrt{a}q^{N-\frac{1}{2}})]^2$$

- L is the longest downward SE-SW path on a cylinder filled with independent geometric random variables equi-distributed by diagonal
- κ_1 is the first part of a uniform partition independent of everything else, $\mathbb{P}(\kappa) \propto (q^N)^{|\kappa|}$
- $q^N = e^{-\text{inverse temperature}}$ in physical language

The LPP picture

The L ($= 53$ in the example) from the previous slide, the length of the longest (adding the integers encountered) down SE-SW path in a cylindric geometry. Each square on a given horizontal line is iid with the indicated geometric distribution.



The main result, two regimes, Airy and Bessel

Theorem (B/Occelli 2021)

We have (recall $\lambda_1 = \Lambda_{1,1}$ in dist.), for $q = e^{-\epsilon} \to 1$ as $\epsilon \to 0+$

- *(i) (N fixed, a grows critically) if $a = e^{-\alpha\epsilon} \to 1$ and for N fixed:*

$$\lim_{\epsilon \to 0+} \mathbb{P}\left(\frac{\lambda_1 - 2\epsilon^{-1}\log \epsilon^{-1}}{\epsilon^{-1}} < s \right) = G_\alpha^N(s) := \det(1 - O_\alpha^N)_{L^2(s,\infty)}$$

  where the determinant on the right is a Fredholm determinant of the following finite temperature Bessel kernel (operator) in exponential coordinates

$$O_\alpha^N(x,y) = e^{-\frac{x}{2}-\frac{y}{2}} \int_0^\infty \frac{1}{1+u^N} J_\alpha(2\sqrt{ue^{-x}}) J_\alpha(2\sqrt{ue^{-y}}) du$$

  with J the Bessel function of the first kind;

- *(ii) (N grows slowly, a fixed) if $N = \beta\epsilon^{-2/3} \to \infty$ and for $0 < a < 1$ fixed (below $c_1 = -2\log(1-\sqrt{a})$, $c_2 = 2^{-1/3}a^{1/6}(1-\sqrt{a})^{-2/3}$):*

$$\lim_{\epsilon \to 0+} \mathbb{P}\left(\frac{\lambda_1 - c_1\epsilon^{-1}}{c_2\epsilon^{-1/3}} < s \right) = F^{\beta c_2}(s) := \det(1 - Ai^{\beta c_2})_{L^2(s,\infty)}$$

  where Ai^β is the finite temperature Airy kernel from before.

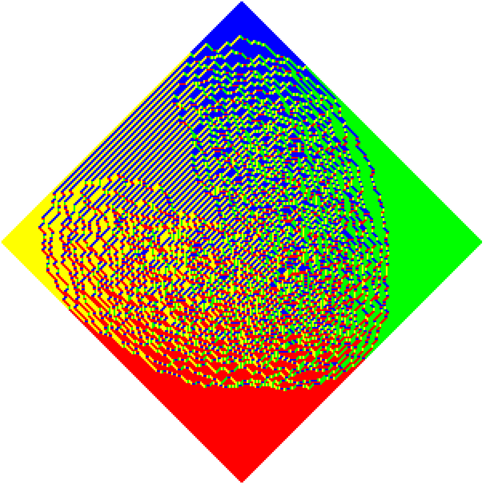
Some remarks

Some remarks on the previous result:

- as $N \to \infty$ in part (i), we recover the exponentiated Bessel kernel from the previous section

  $$O_\alpha^N(x, y) \to O_\alpha(x, y) = e^{-\frac{x}{2} - \frac{y}{2}} Bess_\alpha(e^{-x}, e^{-y})$$

  which interpolates between Gumbel and Tracy–Widom
- a different interpolation between the same two limit laws happens in part (ii) via the finite temperature Airy kernel, as explained above
- there is probably a a more general limiting regime, involving all parameters, where one sees the above two cases transparently
- other limiting regimes are possible, and perhaps even interesting
- both the Bessel and Airy (Tracy–Widom) finite temperature kernels and distributions appeared before, in a unified way, in the physics literature (Le Doussal–Majumdar–Schehr–et al.), as manifestations of limiting behavior of continuous free fermions in certain potentials and at finite temperature
- our work gives a discrete mathematical physical, combinatorial and representation theoretic counterpart to the continuous systems from above



Thank you!