Generalized Gibbs ensemble of the Ablowitz-Ladik lattice, and the Circular β-ensemble

Guido, Mazzuca
mazzuca@kth.se

April 20, 2022

Overview

- Background, and motivations
- Ablowitz-Ladik lattice
- Generalized Gibbs Ensemble
- Circular β ensemble
- Glimpses of the proofs

G.M., and T. Grava, *Generalized Gibbs ensemble of the Ablowitz-Ladik lattice, circular β-ensemble and double confluent Heun equation*, arXiv e-print 2107.02303 (2021)

G.M., and R. Memin, *Large Deviations for Ablowitz-Ladik lattice, and the Schur flow*, arXiv e-print 2201.03429 (2022)

Integrable systems

Consider a Poisson manifold $(M, \{\,,\,\})$, such that $\{\,,\,\}$ is non-degenerate. Let $\boldsymbol{x} = (x_1, \ldots, x_{2N})$ be coordinates on M. The evolution $\boldsymbol{x}(0) \to \boldsymbol{x}(t)$ according to Hamilton equations with Hamiltonian $H(\boldsymbol{x})$

$$\frac{dx_j}{dt} = \dot{x}_j = \{x_j, H\}, \;\; j = 1, \ldots, 2N$$

is integrable if there are $H_1 = H, H_2, \ldots H_N$ independent conserved quantities $(\dot{H}_k = 0)$ that Poisson commute: $\{H_j, H_k\} = 0$. (Liouville)

Modern theory of integrable systems

Techniques to detect integrability:

1. Lax pair
2. Bi-Hamiltonian structure

Modern theory of integrable systems

Techniques to detect integrability:

1. Lax pair
2. Bi-Hamiltonian structure

The Hamilton equations

$$\dot{x}_j = \{x_j, H\}, \ \ j = 1, \dots, 2N$$

admits a Lax pair formulation if there exist two matrices $L = L(\boldsymbol{x})$ and $A = A(\boldsymbol{x})$ such that

$$\dot{L} = [A, L] := LA - AL \longleftrightarrow \dot{x}_j = \{x_j, H\}, \ \ j = 1, \dots, 2N$$

Then $\mathrm{Tr} L^k$, k integer, are constant of motions: $\frac{d}{dt}\mathrm{Tr} L^k = 0$

Gibbs measure

Consider the Gibbs measure

$$\mu = \frac{1}{Z(V)} e^{-\mathrm{Tr}(V(L(\mathbf{x})))} \tilde{\mu}\,, \quad \tilde{\mu} = m(x) dx_1, \ldots dx_{2N}\,,$$

here V is a continuous function, and $\tilde{\mu}$ is invariant for the dynamics, thus also μ is invariant.

A classical example is the Gibbs measure for the harmonic oscillator chain:

$$\mu = \frac{\exp\left(-\frac{\beta}{2}\left(\sum_{j=1}^N p_j^2 + r_j^2\right)\right) dr_1, \ldots dr_N dp_1 \ldots dp_N}{\int_{\mathbb{R}^{2N}} \exp\left(-\frac{\beta}{2}\left(\sum_{j=1}^N p_j^2 + r_j^2\right)\right) dr_1, \ldots dr_N dp_1 \ldots dp_N}\,,$$

here $r_j = q_{j+1} - q_j$.

Gibbs measure

Consider the Gibbs measure

$$\mu = \frac{1}{Z(V)} e^{-\mathrm{Tr}(V(L(\boldsymbol{x})))} \tilde{\mu}\,, \quad \tilde{\mu} = m(x) dx_1, \ldots dx_{2N}\,,$$

$$\mu \longrightarrow L$$

thus L becomes a Random Matrix.

Gibbs measure

Consider the Gibbs measure

$$\mu = \frac{1}{Z(V)} e^{-\mathrm{Tr}(V(L(\boldsymbol{x})))} \tilde{\mu}\,, \quad \tilde{\mu} = m(x) dx_1, \ldots dx_{2N}\,,$$

$$\mu \longrightarrow L$$

thus L becomes a Random Matrix.

- Does L can be reduced to a known family of random matrices? Which is the spectrum of L when $N \to \infty$ (density of states) ?
- How do the correlation functions like $S(j,t) = \mathbb{E}(x_j(t)x_\ell(0)) - \mathbb{E}(x_j(t))\mathbb{E}(x_\ell(0))$ behave when $N \to \infty$ and $t \to \infty$?

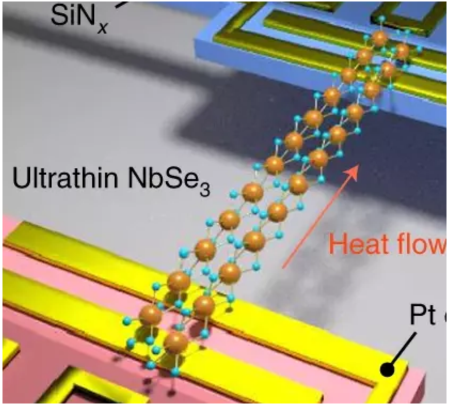
Why:

Correlation functions $\rightarrow$ Transport properties

Why:

Correlation functions $\rightarrow$ Transport properties

Specific 1D phenomenon: conductivity diverges as the length of the chain grows (Anomalous transport).
Surprisingly, this is **measured** experimentally:



(Nature Nanotechnology 2021)

Why:

Correlation functions $\rightarrow$ Transport properties

For a general dynamical system, the computation of a general correlation function $S(j,t)$ as $t, N \rightarrow \infty$ is "utterly out of reach" (Spohn). Rigorous mathematical results in dimension bigger or equal to 3 (Lukkarinen-Spohn 2011).

Why:

Correlation functions $\rightarrow$ Transport properties

For a general dynamical system, the computation of a general correlation function $S(j,t)$ as $t, N \rightarrow \infty$ is "utterly out of reach" (Spohn). Rigorous mathematical results in dimension bigger or equal to 3 (Lukkarinen-Spohn 2011). Numerical simulations show that:

$$S(j,t) \simeq \frac{1}{\lambda t^{\gamma}} f\left(\frac{j - vt}{\lambda t^{\delta}}\right) .$$

- **Non integrable systems**, such as DNLS, FPUT, etc, $\gamma = \delta = \frac{2}{3}$ and $f = F_{TW}$.
- **Non linear integrable systems**, such as Toda, AL, $\gamma = \delta = 1$ and $f = e^{-x^2}$.

Why:

Correlation functions $\to$ Transport properties

For a general dynamical system, the computation of a general correlation function $S(j,t)$ as $t, N \to \infty$ is "utterly out of reach" (Spohn). Rigorous mathematical results in dimension bigger or equal to 3 (Lukkarinen-Spohn 2011). Numerical simulations show that:

$$S(j,t) \simeq \frac{1}{\lambda t^{\gamma}} f\left(\frac{j - vt}{\lambda t^{\delta}}\right).$$

- **Non integrable systems**, such as DNLS, FPUT, etc, $\gamma = \delta = \frac{2}{3}$ and $f = F_{TW}$.
- **Non linear integrable systems**, such as Toda, AL, $\gamma = \delta = 1$ and $f = e^{-x^2}$.
- **Short range harmonic chain**, we can perfectly describe the behaviour of the correlation functions (Mazur;. . . , M-Grava-McLaughlin-Kriecherbauer).The behaviour can be "wild", for different position-time scales the behaviour is described by Airy, Pearcy integral,. . .

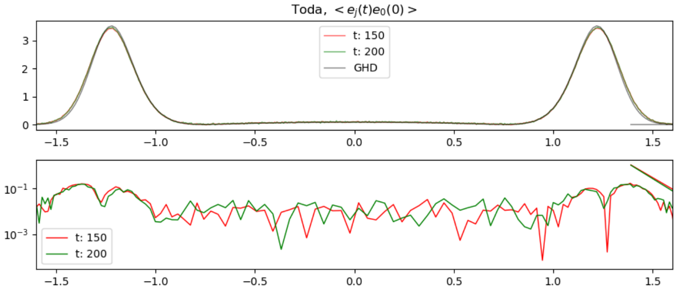
Recent Breakthrough

- H. Spohn was able to characterize the density of states for the GGE of the Toda lattice with polynomial potential in terms of the equilibrium measure of the Gaussian β ensemble at high temperature.

Recent Breakthrough

- H. Spohn was able to characterize the density of states for the GGE of the Toda lattice with polynomial potential in terms of the equilibrium measure of the Gaussian β ensemble at high temperature.

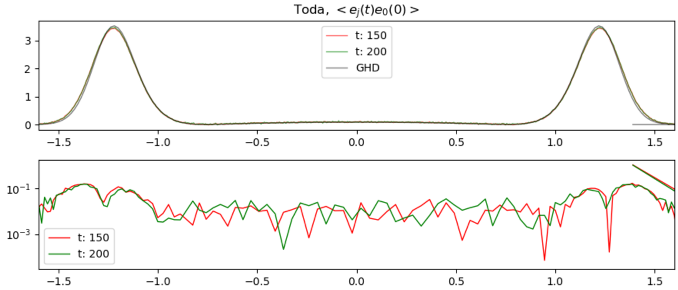
  Applying the theory of Generalized Hydrodynamic, he argued that the decay of correlation functions is ballistic. $(\delta = \gamma = 1)$



Recent Breakthrough

- H. Spohn was able to characterize the density of states for the GGE of the Toda lattice with polynomial potential in terms of the equilibrium measure of the Gaussian β ensemble at high temperature.

  Applying the theory of Generalized Hydrodynamic, he argued that the decay of correlation functions is ballistic. $(\delta = \gamma = 1)$
- A. Guionnet, and R. Memin generalized Spohn results, obtaining a Large deviation principle for the empirical measures with continuous potential.



α-ensemble	Integrable System
Gaussian	Toda lattice (Spohn; Guionnet-Memin)
Circular	Defocusing Ablowitz-Ladik lattice (Spohn, Grava-M.; Memin-M.)
Laguerre	Exponential Toda lattice (Gisonni-Grava-Gubbiotti-M.)
Jacobi	Defocusing Schur flow (Spohn; Memin-M.)
Antisymmetric Gaussian	Volterra lattice (Gisonni-Grava-Gubbiotti-M.)

The α-ensembles are related to the classical β ones in the high temperature regime.

α-**ensemble**	**Integrable System**
Gaussian	Toda lattice (Spohn; Guionnet-Memin)
Circular	Defocusing Ablowitz-Ladik lattice (Spohn, Grava-M.; Memin-M.)
Laguerre	Exponential Toda lattice (Gisonni-Grava-Gubbiotti-M.)
Jacobi	Defocusing Schur flow (Spohn; Memin-M.)
Antisymmetric Gaussian	Volterra lattice (Gisonni-Grava-Gubbiotti-M.)

The α-ensembles are related to the classical β ones in the high temperature regime.

The Ablowitz-Ladik lattice

$$i\dot{\alpha}_j = (2\alpha_j - \alpha_{j-1} - \alpha_{j+1}) + |\alpha_j|^2(\alpha_{j-1} + \alpha_{j+1}), \quad j = 1, \ldots, N$$

where $\alpha_j \in \mathbb{C}$, and we consider periodic boundary condition, thus $\alpha_{j+N} = \alpha_j$.

The Ablowitz-Ladik lattice

$$i\dot{\alpha}_j = (2\alpha_j - \alpha_{j-1} - \alpha_{j+1}) + |\alpha_j|^2(\alpha_{j-1} + \alpha_{j+1}), \quad j = 1, \dots, N$$

where $\alpha_j \in \mathbb{C}$, and we consider periodic boundary condition, thus $\alpha_{j+N} = \alpha_j$.

- The Ablowitz-Ladik (1973–74) system is the integrable discretization of the defocussing cubic NLS:

$$i\partial_t \psi(x,t) = -\frac{1}{2}\partial_x^2 \psi(x,t) + |\psi(x,t)|^2 \psi(x,t).$$

its integrability was discovered by Ablowitz and Ladik (1974) by making spatial discretization of the Zakharov-Shabat Lax pair for NLS;

The Ablowitz-Ladik lattice

$$i\dot{\alpha}_j = (2\alpha_j - \alpha_{j-1} - \alpha_{j+1}) + |\alpha_j|^2(\alpha_{j-1} + \alpha_{j+1}), \quad j = 1, \dots, N$$

where $\alpha_j \in \mathbb{C}$, and we consider periodic boundary condition, thus $\alpha_{j+N} = \alpha_j$.

- The Ablowitz-Ladik (1973–74) system is the integrable discretization of the defocussing cubic NLS:

  $$i\partial_t\psi(x,t) = -\frac{1}{2}\partial_x^2\psi(x,t) + |\psi(x,t)|^2\psi(x,t).$$

  its integrability was discovered by Ablowitz and Ladik (1974) by making spatial discretization of the Zakharov-Shabat Lax pair for NLS;
- For periodic boundary conditions. Finite-gap integration developed by P. Miller, N. Ercolani, I. Krichever and D. Levermore;

The Ablowitz-Ladik lattice

$$i\dot{\alpha}_j = (2\alpha_j - \alpha_{j-1} - \alpha_{j+1}) + |\alpha_j|^2(\alpha_{j-1} + \alpha_{j+1}), \quad j = 1, \ldots, N$$

where $\alpha_j \in \mathbb{C}$, and we consider periodic boundary condition, thus $\alpha_{j+N} = \alpha_j$.

- The Ablowitz-Ladik (1973–74) system is the integrable discretization of the defocussing cubic NLS:

$$i\partial_t\psi(x,t) = -\frac{1}{2}\partial_x^2\psi(x,t) + |\psi(x,t)|^2\psi(x,t).$$

  its integrability was discovered by Ablowitz and Ladik (1974) by making spatial discretization of the Zakharov-Shabat Lax pair for NLS;
- For periodic boundary conditions. Finite-gap integration developed by P. Miller, N. Ercolani, I. Krichever and D. Levermore;
- The DNLS is another discretization, but it is not integrable.

Hamiltonian Structure

There are two conserved quantities:

$$K^{(0)} = \prod_{j=1}^{N} \left(1 - |\alpha_j|^2\right) , \quad K^{(1)} := - \sum_{j=1}^{N} \alpha_j \overline{\alpha}_{j+1} \, .$$

Since $K^{(0)}$ is conserved, this implies that if $|\alpha_j(0)| < 1 \,\forall\, j$, then $|\alpha_j(t)| < 1 \,\forall\, t$. Thus we can consider $\mathbb{D}^N$ as **phase space**, $\mathbb{D} = \{z \in \mathbb{C} \mid |z| < 1\}$.

Hamiltonian Structure

There are two conserved quantities:

$$K^{(0)} = \prod_{j=1}^{N} \left(1 - |\alpha_j|^2\right) \ , \quad K^{(1)} := -\sum_{j=1}^{N} \alpha_j \overline{\alpha}_{j+1} \, .$$

Since $K^{(0)}$ is conserved, this implies that if $|\alpha_j(0)| < 1 \, \forall \, j$, then $|\alpha_j(t)| < 1 \, \forall \, t$. Thus we can consider $\mathbb{D}^N$ as **phase space**, $\mathbb{D} = \{z \in \mathbb{C} \mid |z| < 1\}$.

$$\{f, g\} = i \sum_{j=1}^{N} (1 - |\alpha_j|^2) \left(\frac{\partial f}{\partial \overline{\alpha}_j} \frac{\partial g}{\partial \alpha_j} - \frac{\partial g}{\partial \overline{\alpha}_j} \frac{\partial f}{\partial \alpha_j} \right)$$

(Ercolani, Lozano)

$$\dot{\alpha}_j = \left\{ \alpha_j, \underbrace{-2 \log \left(K^{(0)} \right) + 2\Re(K^{(1)})}_{:= H_{AL}} \right\}$$

Integrability (N even)

Nenciu, and Simon proved that the AL equations of motion are equivalent to the Lax pair:

$$\dot{\mathcal{E}} = i\,[\mathcal{E}, A(\mathcal{E})]$$

where $\mathcal{E} = \mathcal{L}\mathcal{M}$, such that

$$\mathcal{M} = \begin{pmatrix} -\alpha_1 & & & & \rho_1 \\ & \Xi_3 & & & \\ & & \ddots & & \\ & & & \Xi_{N-1} & \\ \rho_1 & & & & \overline{\alpha}_1 \end{pmatrix}, \qquad \mathcal{L} = \begin{pmatrix} \Xi_2 & & & \\ & \Xi_4 & & \\ & & \ddots & \\ & & & \Xi_N \end{pmatrix},$$

here $\Xi_j = \begin{pmatrix} \overline{\alpha}_j & \rho_j \\ \rho_j & -\alpha_j \end{pmatrix}$ and $\rho_j = \sqrt{1 - |\alpha_j|^2}$.

Structure of periodic CMV matrix

$$
\mathcal{E} = \begin{pmatrix}
* & * & * & & & & & & & * \\
* & * & * & & & & & & & * \\
 & * & * & * & * & & & & & \\
 & * & * & * & * & & & & & \\
 & & & & & \ddots & \ddots & & & \\
 & & & & & & * & * & * & * \\
 & & & & & & * & * & * & * \\
* & & & & & & & * & * & * \\
* & & & & & & & * & * & *
\end{pmatrix}.
$$

- Periodic CMV (Cantero Moral Velazquez) Matrix:
  - unitary $\lambda_j = e^{i\theta_j}$, $\theta_j \in \mathbb{T}$

Generalized Gibbs Ensemble

In view of the Lax pair:

$$\dot{\mathcal{E}} = i\,[\mathcal{E}, A(\mathcal{E})] \ ,$$

then

$$K^{(\ell)} = \mathtt{Tr}\left(\mathcal{E}^{\ell}\right) \ , \quad \ell = 1, \ldots, N-1$$

are conserved.

Generalized Gibbs Ensemble

In view of the Lax pair:

$$\dot{\mathcal{E}} = i\,[\mathcal{E}, A(\mathcal{E})]\ ,$$

then

$$K^{(\ell)} = \mathtt{Tr}\left(\mathcal{E}^{\ell}\right)\ , \quad \ell = 1, \ldots, N-1$$

are conserved.

So we can define the Generalized Gibbs Ensemble as

$$\mu_{AL} = \frac{1}{Z_N^{AL}(V,\beta)} \prod_{j=1}^{N} (1-|\alpha_j|^2)^{\beta-1} \exp(-\mathtt{Tr}(V(\mathcal{E})))\mathrm{d}^2\boldsymbol{\alpha}\,, \quad \alpha_j \in \mathbb{D}$$

here $V(z)$ is a continuous function, $V(z)\ :\ \mathbb{D} \to \mathbb{R}$.

The -1 comes from the Poisson bracket (volume form)

Integrability and Random matrix

$$\mu_{AL} \longrightarrow \mathcal{E}$$

thus $\mathcal{E}$ becomes a Random Matrix.

Integrability and Random matrix

$$\mu_{AL} \longrightarrow \mathcal{E}$$

thus $\mathcal{E}$ becomes a Random Matrix.
Define the empirical measure as

$$\mu_N(\mathcal{E}) = \frac{1}{N}\sum_{j=1}^{N}\delta_{e^{i\theta_j}} \,,$$

where $e^{i\theta_j}$s are the eigenvalues of $\mathcal{E}$.
Study the weak limit of $\mu_N(\mathcal{E})$, or density of states

$$\mu_N(\mathcal{E}) \rightharpoonup \nu_\beta^V$$

The eigenvalues are the fundamental ingredient of the finite-gap integration.

Circular β Ensemble

$$\mathrm{d}\mathbb{P}_C\,(\theta_1,\ldots,\theta_N) = (\mathcal{Z}_N^E(V,\widetilde{\beta}))^{-1}|\Delta(e^{i\theta})|^{\widetilde{\beta}}\exp\left(-\sum_{j=1}^{N} V(e^{i\theta_j})\right)\mathrm{d}\theta_1\ldots\mathrm{d}\theta_N\,,$$

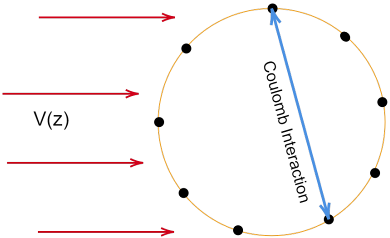
where $\Delta(e^{i\theta}) = \prod_{\ell\neq j}(e^{i\theta_j} - e^{i\theta_\ell})$, $\theta_j \in [-\pi,\pi)$, and $\mathcal{Z}_N^E(V,\widetilde{\beta})$ is the partition function.

Circular β Ensemble

$$\mathrm{d}\mathbb{P}_C(\theta_1,\ldots,\theta_N) = (\mathcal{Z}_N^E(V,\widetilde{\beta}))^{-1}|\Delta(e^{i\theta})|^{\widetilde{\beta}} \exp\left(-\sum_{j=1}^{N} V(e^{i\theta_j})\right) \mathrm{d}\theta_1 \ldots \mathrm{d}\theta_N\,,$$

where $\Delta(e^{i\theta}) = \prod_{\ell\neq j}(e^{i\theta_j} - e^{i\theta_\ell})$, $\theta_j \in [-\pi,\pi)$, and $\mathcal{Z}_N^E(V,\widetilde{\beta})$ is the partition function.

Physical Interpretation: charged particles constrained on the unit circle, subjected to an external potential $V(z)$ at temperature $\widetilde{\beta}^{-1}$



Matrix Representation - Killip, and Nenciu

Matrix Representation - Killip, and Nenciu

Definition

We said that a complex random variable X with values on the unit disk $\mathbb{D}$ is Θ_ν-distributed $(\nu > 1)$ if:

$$\mathbb{E}[f(X)] = \frac{\nu - 1}{2\pi} \int_{\mathbb{D}} f(z)(1 - |z|^2)^{\frac{\nu-3}{2}} \mathrm{d}^2 z \,.$$

if $\nu = 1$ let Θ_1 denote the uniform distribution on the unit circle.

Remark: let $\nu \in \mathbb{N}$, if $\boldsymbol{u}$ is chosen at random according to the surface measure on the unit sphere S^ν in $\mathbb{R}^{\nu+1}$, then $u_1 + iu_2$ is $\Theta_\nu-$distributed.

Theorem (Killip, Nenciu)

Let $\alpha_j \sim \Theta_{\widetilde{\beta}(N-j)+1}$, $\rho_j = \sqrt{1-|\alpha_j|^2}$, and define Ξ_j as

$$\Xi_j = \begin{pmatrix} \overline{\alpha}_j & \rho_j \\ \rho_j & -\alpha_j \end{pmatrix} .$$

for $1 \leq j \leq N-1$ while $\Xi_0 = (1)$ and $\Xi_N = (\overline{\alpha}_N)$ are 1×1 matrices. From these define the $N \times N$ block diagonal matrices as:

$$L = \operatorname{diag}(\Xi_1, \Xi_3, \Xi_5, \ldots) \quad \text{and} \quad M = \operatorname{diag}(\Xi_0, \Xi_2, \Xi_4, \ldots) .$$

The eigenvalues of the two CMV matrices $E = LM$ and $\widetilde{E} = ML$ are distributed according to the Circular Beta Ensemble:

$$\mathrm{d}\mathbb{P}_C(\theta_1, \ldots, \theta_N) = (\mathcal{Z}_N^E(0, \widetilde{\beta}))^{-1} |\Delta(e^{i\theta})|^{\widetilde{\beta}} \mathrm{d}\theta_1 \ldots \mathrm{d}\theta_N , \quad \theta_j \in [-\pi, \pi) .$$

Structure of CMV matrix

$$
E = \begin{pmatrix}
* & * & * & & & & & & \\
* & * & * & & & & & & \\
 & * & * & * & * & & & & \\
 & * & * & * & * & & & & \\
 & & & \ddots & \ddots & \ddots & \ddots & & \\
 & & & & & * & * & * & * \\
 & & & & & * & * & * & * \\
 & & & & & & & * & *
\end{pmatrix}.
$$

- Pentadiagonal
- Unitary
- finite rank perturbation of $\mathcal{E}$

$$\mathrm{d}\mathbb{P}_C\,(\theta_1,\ldots,\theta_N) = (\mathcal{Z}_N^E(0,\widetilde{\beta}))^{-1}|\Delta(e^{i\theta})|^{\widetilde{\beta}}\mathrm{d}\theta_1\ldots\mathrm{d}\theta_N\,,\quad \theta_j\in[-\pi,\pi)\,,$$

$$\mathrm{d}\mathbb{P}_\alpha(\alpha_1,\ldots,\alpha_N) = (Z_N^E(0,\widetilde{\beta}))^{-1}\prod_{j=1}^{N-1}\left(1-|\alpha_j|^2\right)^{\widetilde{\beta}(N-j)/2-1}\mathrm{d}\alpha_j\mathrm{d}\alpha_N\,.$$

$$\mathrm{d}\mathbb{P}_C\,(\theta_1,\ldots,\theta_N) = (\mathcal{Z}_N^E(0,\widetilde{\beta}))^{-1}|\Delta(e^{i\theta})|^{\widetilde{\beta}}\mathrm{d}\theta_1\ldots\mathrm{d}\theta_N\,, \quad \theta_j\in[-\pi,\pi)\,,$$

$$\mathrm{d}\mathbb{P}_\alpha(\alpha_1,\ldots,\alpha_N) = (Z_N^E(0,\widetilde{\beta}))^{-1}\prod_{j=1}^{N-1}\left(1-|\alpha_j|^2\right)^{\widetilde{\beta}(N-j)/2-1}\mathrm{d}\alpha_j\mathrm{d}\alpha_N\,.$$

$$\mathrm{d}\mathbb{P}_C\,(\theta_1,\ldots,\theta_N) = (\mathcal{Z}_N^E(V,\widetilde{\beta}))^{-1}|\Delta(e^{i\theta})|^{\widetilde{\beta}}\exp\left(-\sum_{j=1}^N V(e^{i\theta_j})\right)\mathrm{d}\theta_1\ldots\mathrm{d}\theta_N\,,$$

$$\mathrm{d}\mathbb{P}_\alpha(\alpha_1,\ldots,\alpha_N) = (Z_N^E(V,\widetilde{\beta}))^{-1}\prod_{j=1}^{N-1}\left(1-|\alpha_j|^2\right)^{\widetilde{\beta}(N-j)/2-1}\exp\left(-\mathtt{Tr}(V(E))\right)\mathrm{d}\alpha_j\mathrm{d}\alpha_N.$$

$$\mathrm{d}\mathbb{P}_C\,(\theta_1,\dots,\theta_N) = (\mathcal{Z}_N^E(0,\widetilde{\beta}))^{-1}|\Delta(e^{i\theta})|^{\widetilde{\beta}}\mathrm{d}\theta_1\dots\mathrm{d}\theta_N\,,\quad \theta_j\in[-\pi,\pi)\,,$$

$$\mathrm{d}\mathbb{P}_\alpha(\alpha_1,\dots,\alpha_N) = (Z_N^E(0,\widetilde{\beta}))^{-1}\prod_{j=1}^{N-1}\left(1-|\alpha_j|^2\right)^{\widetilde{\beta}(N-j)/2-1}\mathrm{d}\alpha_j\mathrm{d}\alpha_N\,.$$

$$\mathrm{d}\mathbb{P}_C\,(\theta_1,\dots,\theta_N) = (\mathcal{Z}_N^E(V,\widetilde{\beta}))^{-1}|\Delta(e^{i\theta})|^{\widetilde{\beta}}\exp\left(-\sum_{j=1}^N V(e^{i\theta_j})\right)\mathrm{d}\theta_1\dots\mathrm{d}\theta_N\,,$$

$$\mathrm{d}\mathbb{P}_\alpha(\alpha_1,\dots,\alpha_N) = (Z_N^E(V,\widetilde{\beta}))^{-1}\prod_{j=1}^{N-1}\left(1-|\alpha_j|^2\right)^{\color{red}{\widetilde{\beta}(N-j)/2}-1}\exp\left(-\mathtt{Tr}(V(E))\right)\mathrm{d}\alpha_j\mathrm{d}\alpha_N.$$

The last one looks similar to

$$\mu_{AL} = Z_N^{AL}(V,\beta))^{-1}\prod_{j=1}^N(1-|\alpha_j|^2)^{\color{red}{\beta}-1}\exp(-\mathtt{Tr}(V(\mathcal{E})))\mathrm{d}^2\boldsymbol{\alpha}\,,\quad \alpha_j\in\mathbb{D}\,,$$

High temperature regime - $\widetilde{\beta} = \frac{2\beta}{N}$

$$\mathrm{d}\mathbb{P}_\alpha(\alpha_1, \ldots, \alpha_N) = \frac{\prod_{j=1}^{N-1} \left(1 - |\alpha_j|^2\right)^{\beta(1-j/N)-1} \exp\left(-\mathtt{Tr}(V(E))\right) \mathrm{d}\alpha_j \mathrm{d}\alpha_N}{Z_N^E\left(V, \frac{2\beta}{N}\right)}.$$

High temperature regime - $\widetilde{\beta} = \frac{2\beta}{N}$

$$\mathrm{d}\mathbb{P}_\alpha(\alpha_1,\dots,\alpha_N) = \frac{\prod_{j=1}^{N-1}\left(1-|\alpha_j|^2\right)^{\beta(1-j/N)-1}\exp\left(-\mathtt{Tr}(V(E))\right)\mathrm{d}\alpha_j\mathrm{d}\alpha_N}{Z_N^E\left(V,\frac{2\beta}{N}\right)}.$$

Theorem (Hardy, and Lambert)

Let $\beta > 0$, and $V : \mathbb{T} \to \mathbb{R}$ continuous. Then

- the sequence $\mu_N(E) = \frac{1}{N}\sum_{j=1}^N \delta_{e^{i\theta_j}}$ satisfies a large deviation principle, and in particular

$$\mu_N(E) \xrightarrow{a.s.} \mu_\beta^V\,,$$

- $\mu_\beta^V \in \mathcal{P}(\mathbb{T})$, and it is the unique minimizer of the functional

$$f^{(V,\beta)}(\rho) = \int_{\mathbb{T}} V(\theta)\rho(\theta)\mathrm{d}\theta - \beta\int\int_{\mathbb{T}\times\mathbb{T}} \log\sin\left(\left|e^{i\theta}-e^{i\phi}\right|\right)\rho(\theta)\rho(\phi)\mathrm{d}\theta\mathrm{d}\phi$$
$$+\int_{\mathbb{T}}\log\left(\rho(\theta)\right)\rho(\theta)\mathrm{d}\theta + \log(2\pi)\,.$$

Recap

$$\mu_{AL} = (Z_N^{AL}(V,\beta))^{-1}\prod_{j=1}^{N}(1-|\alpha_j|^2)^{\beta-1}\exp(-\mathtt{Tr}(V(\mathcal{E})))\mathrm{d}^2\boldsymbol{\alpha}\,.$$

$$\mathrm{d}\mathbb{P}_\alpha = \left(Z_N^E\left(V,\frac{2\beta}{N}\right)\right)^{-1}\prod_{j=1}^{N-1}\left(1-|\alpha_j|^2\right)^{\beta(1-j/N)-1}\exp\left(-\mathtt{Tr}(V(E))\right)\mathrm{d}\alpha_j\mathrm{d}\alpha_N\,,$$

$$\mu_N(E)\xrightarrow{a.s.}\mu_\beta^V$$

$\mu_\beta^V \in \mathcal{P}(\mathbb{T})$, and it is the unique minimizer of $f^{(V,\beta)}(\rho)$.
The structure of $E,\mathcal{E}$ is similar.

Recap

$$\mu_{AL} = (Z_N^{AL}(V,\beta))^{-1}\prod_{j=1}^{N}(1-|\alpha_j|^2)^{\beta-1}\exp(-\mathtt{Tr}(V(\mathcal{E})))\mathrm{d}^2\boldsymbol{\alpha}\,.$$

$$\mathrm{d}\mathbb{P}_\alpha = \left(Z_N^E\left(V,\frac{2\beta}{N}\right)\right)^{-1}\prod_{j=1}^{N-1}\left(1-|\alpha_j|^2\right)^{\beta(1-j/N)-1}\exp\left(-\mathtt{Tr}(V(E))\right)\mathrm{d}\alpha_j\mathrm{d}\alpha_N\,,$$

$$\mu_N(E)\xrightarrow{a.s.}\mu_\beta^V$$

$\mu_\beta^V \in \mathcal{P}(\mathbb{T})$, and it is the unique minimizer of $f^{(V,\beta)}(\rho)$.
The structure of $E,\mathcal{E}$ is similar.

Question

Can we recover, or at least characterize, the density of states ν_β^V, in terms of μ_β^V?

First result

Theorem G.M., and T. Grava

Let $\beta > 0$, $V : \mathbb{T} \to \mathbb{R}$ a Laurent polynomial. Then the mean density of states of the Ablowitz-Ladik lattice ν_β^V can be computed explicitly as

$$\nu_\beta^V = \partial_\beta(\beta \mu_\beta^V)\,,$$

where μ_β^V is the unique minimizer of the functional

$$f^{(V,\beta)}(\rho) = \int_{\mathbb{T}} V(\theta)\rho(\theta)\mathrm{d}\theta - \beta \int\int_{\mathbb{T}\times\mathbb{T}} \log\sin\left(|e^{i\theta} - e^{i\phi}|\right)\rho(\theta)\rho(\phi)\mathrm{d}\theta\mathrm{d}\phi + \int_{\mathbb{T}} \log\left(\rho(\theta)\right)\rho(\theta)\mathrm{d}\theta + \log(2\pi)\,.$$

- Independently, Spohn obtained the same result.

Generalization

Theorem G.M., and R. Memin

Let $\beta > 0$, $V : \mathbb{T} \to \mathbb{R}$ a continuous and bounded function. Then the mean density of states of the Ablowitz-Ladik lattice ν_β^V can be computed explicitly as

$$\nu_\beta^V = \partial_\beta(\beta\mu_\beta^V)\,,$$

where μ_β^V is the unique minimizer of the functional

$$f^{(V,\beta)}(\rho) = \int_{\mathbb{T}} V(\theta)\rho(\theta)\mathrm{d}\theta - \beta \int\int_{\mathbb{T}\times\mathbb{T}} \log\sin\left(\left|e^{i\theta} - e^{i\phi}\right|\right)\rho(\theta)\rho(\phi)\mathrm{d}\theta\mathrm{d}\phi$$
$$+ \int_{\mathbb{T}} \log\left(\rho(\theta)\right)\rho(\theta)\mathrm{d}\theta + \log(2\pi)\,.$$

M-Grava	M-Memin
Transfer operator technique Moment method (It is not the only result of the paper)	Large deviations principles makes use of some ideas of M-Grava

Ideas of the proof M.-Grava

Define the free energies as

$$\mathcal{F}_{AL}(V,\beta) = -\lim_{N\to\infty} \frac{1}{N} \ln(Z_N^{AL}(V,\beta)), \quad \mathcal{F}_C(V,\beta) = -\lim_{N\to\infty} \frac{1}{N} \ln\left(Z_N^E\left(V, \frac{2\beta}{N}\right)\right),$$

where

$$Z_N^{AL}(V,\beta) = \int_{\mathbb{D}^N} \prod_{j=1}^{N} (1-|\alpha_j|^2)^{\beta-1} \exp(-\mathtt{Tr}(V(\mathcal{E})))\mathrm{d}^2\boldsymbol{\alpha}$$

$$Z_N^E\left(V, \frac{2\beta}{N}\right) = \int_{\mathbb{D}^{N-1}\times\mathbb{T}} \prod_{j=1}^{N-1} \left(1-|\alpha_j|^2\right)^{\beta(1-j/N)-1} \exp\left(-\mathtt{Tr}(V(E))\right) \mathrm{d}^2\alpha_j \mathrm{d}\alpha_N$$

Ideas of the proof M.-Grava

Define the free energies as

$$\mathcal{F}_{AL}(V,\beta) = -\lim_{N\to\infty}\frac{1}{N}\ln(Z_N^{AL}(V,\beta)), \quad \mathcal{F}_C(V,\beta) = -\lim_{N\to\infty}\frac{1}{N}\ln\left(Z_N^E\left(V,\frac{2\beta}{N}\right)\right),$$

where

$$Z_N^{AL}(V,\beta) = \int_{\mathbb{D}^N}\prod_{j=1}^N(1-|\alpha_j|^2)^{\beta-1}\exp(-\mathtt{Tr}(V(\mathcal{E})))\mathrm{d}^2\boldsymbol{\alpha}$$

$$Z_N^E\left(V,\frac{2\beta}{N}\right) = \int_{\mathbb{D}^{N-1}\times\mathbb{T}}\prod_{j=1}^{N-1}\left(1-|\alpha_j|^2\right)^{\beta(1-j/N)-1}\exp\left(-\mathtt{Tr}(V(E))\right)\mathrm{d}^2\alpha_j\mathrm{d}\alpha_N$$

It is **rather technical** to prove that

$$\mathcal{F}_{AL}(V,\beta) = \partial_\beta(\beta\mathcal{F}_C(V,\beta))$$

$$\mathcal{F}_{AL}(V,\beta) = \partial_\beta(\beta \mathcal{F}_C(V,\beta))$$

Consider the case $V = 0$. Then, it is possible to compute explicitly $Z_N^E\left(0, \frac{2\beta}{N}\right), Z_N^{AL}(0,\beta)$:

$$Z_N^E\left(0, \frac{2\beta}{N}\right) = 2\frac{\pi^N}{\beta^{N-1}} \prod_{j=1}^{N-1} \frac{1}{1-\frac{j}{N}}$$

$$Z_N^{AL}(0,\beta) = \frac{\pi^N}{\beta^N}$$

$$\mathcal{F}_{AL}(V,\beta) = \partial_\beta(\beta \mathcal{F}_C(V,\beta))$$

Consider the case $V = 0$. Then, it is possible to compute explicitly $Z_N^E\left(0, \frac{2\beta}{N}\right), Z_N^{AL}(0,\beta)$:

$$Z_N^E\left(0, \frac{2\beta}{N}\right) = 2\frac{\pi^N}{\beta^{N-1}} \prod_{j=1}^{N-1} \frac{1}{1-\frac{j}{N}}$$

$$Z_N^{AL}(0,\beta) = \frac{\pi^N}{\beta^N}$$

This implies that

$$\mathcal{F}_C(0,\beta) = \int_0^1 \ln\left(\frac{\beta x}{\pi}\right) dx\,, \quad \mathcal{F}_{AL}(0,\beta) = \ln\left(\frac{\beta}{\pi}\right)$$

$$\mathcal{F}_{AL}(V,\beta) = \partial_\beta(\beta \mathcal{F}_C(V,\beta))$$

Consider the case $V = 0$. Then, it is possible to compute explicitly $Z_N^E\left(0, \frac{2\beta}{N}\right), Z_N^{AL}(0,\beta)$:

$$Z_N^E\left(0, \frac{2\beta}{N}\right) = 2\frac{\pi^N}{\beta^{N-1}} \prod_{j=1}^{N-1} \frac{1}{1-\frac{j}{N}} = \prod_{j=1}^{N-1} F\left(\beta\left(1-\frac{j}{N}\right)\right)$$

$$Z_N^{AL}(0,\beta) = \frac{\pi^N}{\beta^N} = \prod_{j=1}^{N} F\left(\beta\right)$$

This implies that

$$\mathcal{F}_C(0,\beta) = \int_0^1 \ln\left(\frac{\beta x}{\pi}\right) dx\,, \quad \mathcal{F}_{AL}(0,\beta) = \ln\left(\frac{\beta}{\pi}\right)$$

It is possible to generalize this result applying the transfer operator technique.

$$\mathcal{F}_{AL}(V,\beta) = \partial_\beta(\beta\mathcal{F}_C(V,\beta))\,.$$

Moreover, it holds true that

$$\partial_h\mathcal{F}_{AL}(V+hz^k,\beta)\Big|_{h=0} = \int_{\mathbb{T}} e^{ik\theta}\nu_\beta^V(\theta)\mathrm{d}\theta\,, \quad \partial_h\mathcal{F}_C(V+hz^k,\beta)\Big|_{h=0} = \int_{\mathbb{T}} e^{ik\theta}\mu_\beta^V(\theta)\mathrm{d}\theta\,.$$

$$\mathcal{F}_{AL}(V,\beta) = \partial_\beta(\beta\mathcal{F}_C(V,\beta))\,.$$

Moreover, it holds true that

$$\partial_h \mathcal{F}_{AL}(V+hz^k,\beta)\big|_{h=0} = \int_{\mathbb{T}} e^{ik\theta}\nu_\beta^V(\theta)\mathrm{d}\theta\,, \quad \partial_h \mathcal{F}_C(V+hz^k,\beta)\big|_{h=0} = \int_{\mathbb{T}} e^{ik\theta}\mu_\beta^V(\theta)\mathrm{d}\theta\,.$$

These equalities imply that

$$\int_{\mathbb{T}} e^{ik\theta}\nu_\beta^V(\theta)\mathrm{d}\theta = \partial_\beta\left(\beta\int_{\mathbb{T}} e^{ik\theta}\mu_\beta^V(\theta)\mathrm{d}\theta\right)$$

$$\mathcal{F}_{AL}(V,\beta) = \partial_\beta(\beta\mathcal{F}_C(V,\beta))\,.$$

Moreover, it holds true that

$$\partial_h \mathcal{F}_{AL}(V+hz^k,\beta)\big|_{h=0} = \int_{\mathbb{T}} e^{ik\theta}\nu_\beta^V(\theta)\mathrm{d}\theta\,, \quad \partial_h \mathcal{F}_C(V+hz^k,\beta)\big|_{h=0} = \int_{\mathbb{T}} e^{ik\theta}\mu_\beta^V(\theta)\mathrm{d}\theta\,.$$

These equalities imply that

$$\int_{\mathbb{T}} e^{ik\theta}\nu_\beta^V(\theta)\mathrm{d}\theta = \partial_\beta\left(\beta\int_{\mathbb{T}} e^{ik\theta}\mu_\beta^V(\theta)\mathrm{d}\theta\right)$$

and so

$$\nu_\beta^V = \partial_\beta(\beta\mu_\beta^V)$$

Ideas of the proof M.-Memin

We proved a large deviation principle for the family of empirical measures $\mu_N(\mathcal{E}) = \frac{1}{N}\sum_{j=1}^{N} \delta_{e^{i\theta_j}}$, implying that

$$\mu_N(\mathcal{E}) \xrightarrow[N\to\infty]{} \nu_\beta^V \,,$$

and ν_β^V is the unique minimizer of the functional

$$J^{(V,\beta)} \,:\, \mathcal{P}(\mathbb{T}) \to [0;\infty)\,.$$

Ideas of the proof M.-Memin

We proved a large deviation principle for the family of empirical measures $\mu_N(\mathcal{E}) = \frac{1}{N}\sum_{j=1}^N \delta_{e^{i\theta_j}}$, implying that

$$\mu_N(\mathcal{E}) \xrightarrow[N\to\infty]{} \nu_\beta^V \,,$$

and ν_β^V is the unique minimizer of the functional

$$J^{(V,\beta)} \;:\; \mathcal{P}(\mathbb{T}) \to [0;\infty)\,.$$

Moreover, we proved that we can rewrite the functional of Lambert, and Hardy $f^{(V,\beta)}$ (the one that is minimized by μ_β^V) as

$$f^{(V,\beta)}(\mu) = \lim_{\delta\to 0}\liminf_{q\to\infty} \inf_{\substack{\nu_{\beta/q},\dots,\nu_\beta \\ \frac{1}{q}\sum_i \nu_{i\beta/q}\in B_\mu(\delta)}} \left\{ \frac{1}{q}\sum_{i=1}^q J^{(V,i\beta/q)}(\nu_{i\beta/q}) \right\} ,$$

Ideas of the proof M.-Memin

We proved a large deviation principle for the family of empirical measures $\mu_N(\mathcal{E}) = \frac{1}{N}\sum_{j=1}^N \delta_{e^{i\theta_j}}$, implying that

$$\mu_N(\mathcal{E}) \xrightarrow[N\to\infty]{} \nu_\beta^V ,$$

and ν_β^V is the unique minimizer of the functional

$$J^{(V,\beta)} \;:\; \mathcal{P}(\mathbb{T}) \to [0;\infty) .$$

Moreover, we proved that we can rewrite the functional of Lambert, and Hardy $f^{(V,\beta)}$ (the one that is minimized by μ_β^V) as

$$f^{(V,\beta)}(\mu) = \lim_{\delta\to 0}\liminf_{q\to\infty} \inf_{\substack{\nu_{\beta/q},\dots,\nu_\beta \\ \frac{1}{q}\sum_i \nu_{i\beta/q}\in B_\mu(\delta)}} \left\{ \frac{1}{q}\sum_{i=1}^q J^{(V,i\beta/q)}(\nu_{i\beta/q}) \right\} ,$$

which implies that

$$\int_0^1 \nu_{t\beta}^V dt = \mu_\beta^V \implies \nu_\beta^V = \partial_\beta(\beta\mu_\beta^V)$$

Explicit Solutions

For the case $V = 0$, Lambert, and Hardy proved that

$$\mu_\beta^0 = \frac{1}{2\pi} \xrightarrow{G-M;M-M} \nu_\beta^0 = \frac{1}{2\pi}$$

Explicit Solutions

For the case $V = 0$, Lambert, and Hardy proved that

$$\mu_\beta^0 = \frac{1}{2\pi} \xrightarrow{G-M;M-M} \nu_\beta^0 = \frac{1}{2\pi}$$

For the classical Gibbs ensemble $V = \eta\Re(z)$, in Grava-M. we proved

$$\mu_\beta^V(\theta) = \frac{1}{2\pi} + \frac{1}{\pi\beta}\Re\left(\frac{zv'(z)}{v(z)}\right)\Big|_{z=e^{i\theta}}, \quad \nu_\beta^V = \frac{1}{2\pi} + \partial_\beta\left(\frac{1}{\pi}\Re\left(\frac{zv'(z)}{v(z)}\right)\Big|_{z=e^{i\theta}}\right).$$

where $v(z)$ is the unique analytic solution at 0 of the double confluent Heun equation

$$z^2 v''(z) + \left(-\eta + z(\beta+1) + \eta z^2\right) v'(z) + \eta\beta(z+\lambda)v(z) = 0\,,$$

and λ is determined as the unique solution of a transcendental equation.

Open problems

Explicitly compute the correlations functions. Despite having explicit solutions via finite-gap integration, and several insights for the GHD theory (Dojon, Spohn, El), the computation remains out of reach.

Open problems

Explicitly compute the correlations functions. Despite having explicit solutions via finite-gap integration, and several insights for the GHD theory (Dojon, Spohn, El), the computation remains out of reach.

α-ensemble	**Integrable System**
Gaussian	Toda lattice (Spohn; Guionnet-Memin)
Circular	Defocusing Ablowitz-Ladik lattice (Grava-M.; Memin-M.)
Laguerre	Exponential Toda lattice (Gisonni-Grava-Gubbiotti-M.)
Jacobi	Defocusing Schur flow (Spohn; Memin-M.)
Antisymmetric Gaussian	Volterra lattice (Gisonni-Grava-Gubbiotti-M.)

Open problems

Explicitly compute the correlations functions. Despite having explicit solutions via finite-gap integration, and several insights for the GHD theory (Dojon, Spohn, El), the computation remains out of reach.

α-**ensemble**	**Integrable System**
Gaussian	Toda lattice (Spohn; Guionnet-Memin)
Circular	Defocusing Ablowitz-Ladik lattice (Grava-M.; Memin-M.)
Laguerre	Exponential Toda lattice (Gisonni-Grava-Gubbiotti-M.)
Jacobi	Defocusing Schur flow (Spohn; Memin-M.)
Antisymmetric Gaussian	Volterra lattice (Gisonni-Grava-Gubbiotti-M.)
(??)2D β ensemble at high temperature	Focusing Ablowitz-Ladik and focusing mKdV

Thank you for the attention!