Generalized Gibbs ensemble of the Ablowitz-Ladik lattice, and the Circular $\beta$-ensemble

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Overview

- Background, and motivations
- Ablowitz-Ladik lattice
- Generalized Gibbs Ensemble
- Circular $\beta$ ensemble
- Glimpses of the proofs


Consider a Poisson manifold \((M, \{, \})\), such that \{, \} is non-degenerate. Let \(x = (x_1, \ldots, x_{2N})\) be coordinates on \(M\). The evolution \(x(0) \rightarrow x(t)\) according to Hamilton equations with Hamiltonian \(H(x)\)

\[
\frac{dx_j}{dt} = \dot{x}_j = \{x_j, H\}, \quad j = 1, \ldots, 2N
\]

is integrable if there are \(H_1 = H, H_2, \ldots H_N\) independent conserved quantities \(\dot{H}_k = 0\) that Poisson commute: \(\{H_j, H_k\} = 0\). (Liouville)
Modern theory of integrable systems

Techniques to detect integrability:

1. Lax pair
2. Bi-Hamiltonian structure
Modern theory of integrable systems

Techniques to detect integrability:

1. Lax pair
2. Bi-Hamiltonian structure

The Hamilton equations

\[ \dot{x}_j = \{x_j, H\}, \quad j = 1, \ldots, 2N \]

admits a Lax pair formulation if there exist two matrices \( L = L(x) \) and \( A = A(x) \) such that

\[ \dot{L} = [A, L] := LA - AL \iff \dot{x}_j = \{x_j, H\}, \quad j = 1, \ldots, 2N \]

Then \( \text{Tr}L^k \), \( k \) integer, are constant of motions:

\[ \frac{d}{dt} \text{Tr}L^k = 0 \]
Gibbs measure

Consider the Gibbs measure

$$\mu = \frac{1}{Z(V)} e^{-\text{Tr}(V(L(x)))} \tilde{\mu}, \quad \tilde{\mu} = m(x)dx_1, \ldots dx_{2N},$$

here $V$ is a continuous function, and $\tilde{\mu}$ is invariant for the dynamics, thus also $\mu$ is invariant.

A classical example is the Gibbs measure for the harmonic oscillator chain:

$$\mu = \frac{\exp \left( -\frac{\beta}{2} \left( \sum_{j=1}^{N} p_j^2 + r_j^2 \right) \right) dr_1, \ldots dr_N dp_1 \ldots dp_N}{\int_{\mathbb{R}^{2N}} \exp \left( -\frac{\beta}{2} \left( \sum_{j=1}^{N} p_j^2 + r_j^2 \right) \right) dr_1, \ldots dr_N dp_1 \ldots dp_N},$$

here $r_j = q_{j+1} - q_j$. 
Consider the Gibbs measure

$$\mu = \frac{1}{Z(V)} e^{-\text{Tr}(V(L(x)))} \tilde{\mu}, \quad \tilde{\mu} = m(x) dx_1, \ldots dx_{2N},$$

$$\mu \rightarrow L$$

thus $L$ becomes a Random Matrix.
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thus \( L \) becomes a Random Matrix.

- Does \( L \) can be reduced to a known family of random matrices? Which is the spectrum of \( L \) when \( N \rightarrow \infty \) (density of states)?
- How do the correlation functions like
  \[ S(j, t) = \mathbb{E}(x_j(t)x_\ell(0)) - \mathbb{E}(x_j(t))\mathbb{E}(x_\ell(0)) \]
  behave when \( N \rightarrow \infty \) and \( t \rightarrow \infty \)?
Why:

Correlation functions $\rightarrow$ Transport properties
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Specific 1D phenomenon: conductivity **diverges** as the length of the chain grows (Anomalous transport).

Surprisingly, this is **measured** experimentally:

(Nature Nanotechnology 2021)
Why:

Correlation functions $\rightarrow$ Transport properties

For a general dynamical system, the computation of a general correlation function $S(j, t)$ as $t, N \rightarrow \infty$ is “utterly out of reach” (Spohn). Rigorous mathematical results in dimension bigger or equal to 3 (Lukkarinen-Spohn 2011).
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Numerical simulations show that:

$$S(j, t) \simeq \frac{1}{\lambda t^\gamma} f \left( \frac{j - vt}{\lambda t^\delta} \right).$$

- **Non integrable systems**, such as DNLS, FPUT, etc, $\gamma = \delta = \frac{2}{3}$ and $f = F_{TW}$.
- **Non linear integrable systems**, such as Toda, AL, $\gamma = \delta = 1$ and $f = e^{-x^2}$. 
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- **Short range harmonic chain**, we can perfectly describe the behaviour of the correlation functions (Mazur; . . . , M-Grava-McLaughlin-Kriecherbauer). The behaviour can be “wild”, for different position-time scales the behaviour is described by Airy, Pearcy integral, . . .
Recent Breakthrough

- H. Spohn was able to characterize the density of states for the GGE of the Toda lattice with polynomial potential in terms of the equilibrium measure of the Gaussian $\beta$ ensemble at high temperature.

- Applying the theory of Generalized Hydrodynamic, he argued that the decay of correlation functions is ballistic ($\delta = \gamma = 1$).

- A. Guionnet, and R. Memin generalized Spohn results, obtaining a Large deviation principle for the empirical measures with continuous potential.
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The $\alpha$-ensembles are related to the classical $\beta$ ones in the high temperature regime.

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The \( \alpha \)-ensembles are related to the classical \( \beta \) ones in the high temperature regime.
The Ablowitz-Ladik lattice

\[ i \dot{\alpha}_j = (2\alpha_j - \alpha_{j-1} - \alpha_{j+1}) + |\alpha_j|^2(\alpha_{j-1} + \alpha_{j+1}), \quad j = 1, \ldots, N \]

where \( \alpha_j \in \mathbb{C} \), and we consider periodic boundary condition, thus \( \alpha_{j+N} = \alpha_j \).
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\[ i\partial_t \psi(x, t) = -\frac{1}{2} \partial_x^2 \psi(x, t) + |\psi(x, t)|^2 \psi(x, t). \]

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- The DNLS is another discretization, but it is not integrable.
Hamiltonian Structure

There are two conserved quantities:

\[ K^{(0)} = \prod_{j=1}^{N} \left(1 - |\alpha_j|^2\right), \quad K^{(1)} := -\sum_{j=1}^{N} \alpha_j \overline{\alpha}_{j+1}. \]

Since \( K^{(0)} \) is conserved, this implies that if \( |\alpha_j(0)| < 1 \forall j \), then \( |\alpha_j(t)| < 1 \forall t \). Thus we can consider \( \mathbb{D}^N \) as phase space, \( \mathbb{D} = \{z \in \mathbb{C} \mid |z| < 1\} \).
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\[
\{ f, g \} = i \sum_{j=1}^{N} (1 - |\alpha_j|^2) \left( \frac{\partial f}{\partial \alpha_j} \frac{\partial g}{\partial \bar{\alpha}_j} - \frac{\partial g}{\partial \bar{\alpha}_j} \frac{\partial f}{\partial \alpha_j} \right)
\]

(Ercolani, Lozano)

\[
\dot{\alpha}_j = \begin{cases} 
\alpha_j, & \text{if } -2 \log \left( K^{(0)} \right) + 2 \Re(K^{(1)}) \geq 0 \\
-2 \log \left( K^{(0)} \right) + 2 \Re(K^{(1)}) & \text{else} \\
\end{cases}
:= H_{AL}
\]
Integrability (N even)

Nenciu, and Simon proved that the AL equations of motion are equivalent to the Lax pair:

\[ \dot{\mathcal{E}} = i [\mathcal{E}, A(\mathcal{E})] \]

where \( \mathcal{E} = \mathcal{L}\mathcal{M} \), such that

\[
\mathcal{M} = \begin{pmatrix}
-\alpha_1 & \Xi_3 & & \\
& \ddots & \ddots & \\
& & -\alpha_1 & \rho_1 \\
\rho_1 & & & \\
\end{pmatrix}, \quad \mathcal{L} = \begin{pmatrix}
\Xi_2 & & \\
& \ddots & \\
& & \Xi_N \\
\end{pmatrix},
\]

here \( \Xi_j = \begin{pmatrix} \bar{\alpha}_j & \rho_j \\ \rho_j & -\alpha_j \end{pmatrix} \) and \( \rho_j = \sqrt{1 - |\alpha_j|^2} \).
Structure of periodic CMV matrix

\[ \mathcal{E} = \begin{pmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ \vdots & \vdots & \vdots & \vdots \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{pmatrix} . \]

- Periodic CMV (Cantero Moral Velazquez) Matrix:
  - unitary \( \lambda_j = e^{i\theta_j}, \theta_j \in \mathbb{T} \)
In view of the Lax pair:
\[ \dot{\mathcal{E}} = i [\mathcal{E}, A(\mathcal{E})] , \]
then
\[ K^{(\ell)} = \text{Tr} \left( \mathcal{E}^{\ell} \right) , \quad \ell = 1, \ldots, N - 1 \]
are conserved.
Generalized Gibbs Ensemble

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are conserved.
So we can define the Generalized Gibbs Ensemble as

\[ \mu_{AL} = \frac{1}{Z^N_{AL}(V, \beta)} \prod_{j=1}^{N} (1 - |\alpha_j|^2)^{\beta-1} \exp(-\text{Tr}(V(\mathcal{E})))d^2\alpha , \quad \alpha_j \in \mathbb{D} \]

here \( V(z) \) is a continuous function, \( V(z) : \mathbb{D} \to \mathbb{R} \).
The \(-1\) comes from the Poisson bracket (volume form)
Integrability and Random matrix

$$\mu_{AL} \longrightarrow \mathcal{E}$$

thus $\mathcal{E}$ becomes a Random Matrix.
Integrability and Random matrix

\[ \mu_{AL} \rightarrow \mathcal{E} \]

thus \( \mathcal{E} \) becomes a Random Matrix.

Define the empirical measure as

\[ \mu_N(\mathcal{E}) = \frac{1}{N} \sum_{j=1}^{N} \delta_{e^{i\theta_j}}, \]

where \( e^{i\theta_j} \)'s are the eigenvalues of \( \mathcal{E} \).

Study the weak limit of \( \mu_N(\mathcal{E}) \), or density of states

\[ \mu_N(\mathcal{E}) \rightharpoonup \nu^V_\beta \]

The eigenvalues are the fundamental ingredient of the finite-gap integration.
Circular $\beta$ Ensemble

$$d\mathbb{P}_C(\theta_1, \ldots, \theta_N) = (Z^E_N(V, \bar{\beta}))^{-1}|\Delta(e^{i\theta})|^{\bar{\beta}} \exp \left( - \sum_{j=1}^{N} V(e^{i\theta_j}) \right) d\theta_1 \ldots d\theta_N,$$

where $\Delta(e^{i\theta}) = \prod_{\ell \neq j}(e^{i\theta_j} - e^{i\theta_\ell})$, $\theta_j \in [-\pi, \pi)$, and $Z^E_N(V, \bar{\beta})$ is the partition function.
Circular $\beta$ Ensemble

$$
\text{d}\mathbb{P}_C(\theta_1, \ldots, \theta_N) = (Z^E_N(V, \tilde{\beta}))^{-1} \left| \Delta(e^{i\theta}) \right|^{\beta} \exp \left( - \sum_{j=1}^{N} V(e^{i\theta_j}) \right) \text{d}\theta_1 \ldots \text{d}\theta_N,
$$

where $\Delta(e^{i\theta}) = \prod_{\ell \neq j} (e^{i\theta_j} - e^{i\theta_\ell})$, $\theta_j \in [-\pi, \pi)$, and $Z^E_N(V, \tilde{\beta})$ is the partition function.

**Physical Interpretation:** charged particles constrained on the unit circle, subjected to an external potential $V(z)$ at temperature $\tilde{\beta}^{-1}$
Definition

We said that a complex random variable $X$ with values on the unit disk $D$ is $\Theta_\nu$-distributed ($\nu > 1$) if:

$$E[f(X)] = \nu - \frac{1}{2\pi} \int_D f(z) (1 - |z|^2)^{\nu - 3/2} \, d^2z.$$ 

If $\nu = 1$ let $\Theta_1$ denote the uniform distribution on the unit circle.

Remark: let $\nu \in \mathbb{N}$, if $u$ is chosen at random according to the surface measure on the unit sphere $S_\nu$ in $\mathbb{R}^{\nu+1}$, then $u_1 + iu_2$ is $\Theta_\nu$-distributed.
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Theorem (Killip, Nenciu)

Let \( \alpha_j \sim \Theta_{\tilde{\beta}(N-j)+1} \), \( \rho_j = \sqrt{1 - |\alpha_j|^2} \), and define \( \Xi_j \) as

\[
\Xi_j = \begin{pmatrix} \overline{\alpha}_j & \rho_j \\ \rho_j & -\alpha_j \end{pmatrix}.
\]

for \( 1 \leq j \leq N - 1 \) while \( \Xi_0 = (1) \) and \( \Xi_N = (\overline{\alpha}_N) \) are \( 1 \times 1 \) matrices. From these define the \( N \times N \) block diagonal matrices as:

\[
L = \text{diag} (\Xi_1, \Xi_3, \Xi_5, \ldots) \quad \text{and} \quad M = \text{diag} (\Xi_0, \Xi_2, \Xi_4, \ldots).
\]

The eigenvalues of the two CMV matrices \( E = LM \) and \( \tilde{E} = ML \) are distributed according to the Circular Beta Ensemble:

\[
dP_C (\theta_1, \ldots, \theta_N) = (z_N^E(0, \tilde{\beta}))^{-1} |\Delta(e^{i\theta})|^{\tilde{\beta}} \ d\theta_1 \ldots d\theta_N, \quad \theta_j \in [-\pi, \pi).
\]
Structure of CMV matrix

\[ E = \begin{bmatrix}
* & * & * & & \\
* & * & * & & \\
* & * & * & * & \\
 & & & & \\
* & * & * & & \\
* & * & * & & \\
\end{bmatrix}. \]

- Pentadiagonal
- Unitary
- finite rank perturbation of \( E \)
$$d\mathbb{P}_C(\theta_1, \ldots, \theta_N) = (Z_N^E(0, \tilde{\beta}))^{-1} |\Delta(e^{i\theta})|^{\tilde{\beta}} d\theta_1 \ldots d\theta_N, \quad \theta_j \in [-\pi, \pi),$$

$$d\mathbb{P}_\alpha(\alpha_1, \ldots, \alpha_N) = (Z_N^E(0, \tilde{\beta}))^{-1} \prod_{j=1}^{N-1} (1 - |\alpha_j|^2)^{\tilde{\beta}(N-j)/2-1} d\alpha_j d\alpha_N .$$
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\]

\[
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\]
\[ d\mathbb{P}_C(\theta_1, \ldots, \theta_N) = (Z_N^E(0, \tilde{\beta}))^{-1} |\Delta(e^{i\theta})|^{\tilde{\beta}} \, d\theta_1 \ldots d\theta_N, \quad \theta_j \in [-\pi, \pi), \]

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The last one looks similar to

\[ \mu_{AL} = Z_N^{AL}(V, \beta)^{-1} \prod_{j=1}^{N} (1 - |\alpha_j|^2)^{\beta-1} \exp(-\text{Tr}(V(E))) \, d^2\alpha, \quad \alpha_j \in \mathbb{D}, \]
High temperature regime - $\tilde{\beta} = \frac{2\beta}{N}$

$$d\mathbb{P}_\alpha(\alpha_1, \ldots, \alpha_N) = \frac{\prod_{j=1}^{N-1} (1 - |\alpha_j|^2)^{\beta(1-j/N)-1} \exp(-\text{Tr}(V(E))) d\alpha_j d\alpha_N}{Z^E_N \left(V, \frac{2\beta}{N}\right)}.$$
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\]

**Theorem (Hardy, and Lambert)**

Let \( \beta > 0 \), and \( V : \mathbb{T} \rightarrow \mathbb{R} \) continuous. Then

- the sequence \( \mu_N(E) = \frac{1}{N} \sum_{j=1}^{N} \delta_{e^{i\theta_j}} \) satisfies a large deviation principle, and in particular
  \[
  \mu_N(E) \xrightarrow{a.s.} \mu^V_{\beta},
  \]

- \( \mu^V_{\beta} \in \mathcal{P}(\mathbb{T}) \), and it is the unique minimizer of the functional

\[
f(V,\beta)(\rho) = \int_{\mathbb{T}} V(\theta)\rho(\theta)d\theta - \beta \int_{\mathbb{T} \times \mathbb{T}} \log \sin \left( |e^{i\theta} - e^{i\phi}| \right) \rho(\theta)\rho(\phi)d\theta d\phi
+ \int_{\mathbb{T}} \log (\rho(\theta)) \rho(\theta)d\theta + \log(2\pi).
\]
Recap

\[ \mu_{AL} = (Z_{N}^{AL}(V, \beta))^{-1} \prod_{j=1}^{N} (1 - |\alpha_j|^2)^{\beta-1} \exp(-\text{Tr}(V(\mathcal{E}))) d^2\alpha. \]

\[ d\mathbb{P}_\alpha = \left( Z_{N}^{E} \left( V, \frac{2\beta}{N} \right) \right)^{-1} \prod_{j=1}^{N-1} (1 - |\alpha_j|^2)^{\beta(1-j/N)-1} \exp(-\text{Tr}(V(E))) d\alpha_j d\alpha_N, \]

\[ \mu_{N}(E) \overset{a.s.}{\to} \mu_{\beta}^{V} \]

\[ \mu_{\beta}^{V} \in \mathcal{P}(\mathbb{T}), \text{ and it is the unique minimizer of } f^{(V, \beta)}(\rho). \]

The structure of \( E, \mathcal{E} \) is similar.
Recap

\[ \mu_{AL} = (Z_{AL}^N(V, \beta))^{-1} \prod_{j=1}^{N} (1 - |\alpha_j|^2)^{\beta-1} \exp(-\text{Tr}(V(E))) d^2\alpha . \]

\[ d\mathbb{P}_\alpha = \left( Z_N^E \left( V, \frac{2\beta}{N} \right) \right)^{-1} \prod_{j=1}^{N-1} (1 - |\alpha_j|^2)^{\beta(1-j/N)-1} \exp(-\text{Tr}(V(E))) d\alpha_j d\alpha_N , \]

\[ \mu_{N}(E) \xrightarrow{a.s.} \mu_{\beta}^V \]

\[ \mu_{\beta}^V \in \mathcal{P}(\mathbb{T}) , \text{ and it is the unique minimizer of } f(V, \beta)(\rho) . \]

The structure of \( E, \mathcal{E} \) is similar.

**Question**

Can we recover, or at least characterize, the density of states \( \nu_{\beta}^V \), in terms of \( \mu_{\beta}^V \)?
Theorem G.M., and T. Grava

Let $\beta > 0$, $V : \mathbb{T} \to \mathbb{R}$ a Laurent polynomial. Then the mean density of states of the Ablowitz-Ladik lattice $\nu_\beta^V$ can be computed explicitly as

$$\nu_\beta^V = \partial_\beta (\beta \mu_\beta^V),$$

where $\mu_\beta^V$ is the unique minimizer of the functional

$$f^{(V,\beta)}(\rho) = \int_\mathbb{T} V(\theta)\rho(\theta) d\theta - \beta \int \int_{\mathbb{T} \times \mathbb{T}} \log \sin \left( |e^{i\theta} - e^{i\phi}| \right) \rho(\theta)\rho(\phi) d\theta d\phi$$

$$+ \int_\mathbb{T} \log (\rho(\theta)) \rho(\theta) d\theta + \log(2\pi).$$

Indepedently, Spohn obtained the same result.
Generalization

Theorem G.M., and R. Memin

Let $\beta > 0$, $V : \mathbb{T} \to \mathbb{R}$ a continuous and bounded function. Then the mean density of states of the Ablowitz-Ladik lattice $\nu^V_\beta$ can be computed explicitly as

$$\nu^V_\beta = \partial_\beta (\beta \mu^V_\beta),$$

where $\mu^V_\beta$ is the unique minimizer of the functional

$$f^{(V, \beta)}(\rho) = \int_\mathbb{T} V(\theta)\rho(\theta)d\theta - \beta \int \int_{\mathbb{T} \times \mathbb{T}} \log \sin \left( |e^{i\theta} - e^{i\phi}| \right) \rho(\theta)\rho(\phi)d\theta d\phi$$

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<table>
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<th>M-Memin</th>
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<tbody>
<tr>
<td>Transfer operator technique</td>
<td>Large deviations principles</td>
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<tr>
<td>Moment method</td>
<td>makes use of some ideas of M-Grava</td>
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<td>(It is not the only result of the paper)</td>
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Ideas of the proof M.-Grava

Define the free energies as

\[ \mathcal{F}_{AL}(V, \beta) = -\lim_{N \to \infty} \frac{1}{N} \ln(Z_{AL}^N(V, \beta)), \quad \mathcal{F}_C(V, \beta) = -\lim_{N \to \infty} \frac{1}{N} \ln \left( Z_{E}^N \left( V, \frac{2\beta}{N} \right) \right), \]

where

\[ Z_{AL}^N(V, \beta) = \int_{\mathbb{D}^N} \prod_{j=1}^{N} (1 - |\alpha_j|^2)^{\beta-1} \exp(-\text{Tr}(V(E))) d^2\alpha \]

\[ Z_{E}^N \left( V, \frac{2\beta}{N} \right) = \int_{\mathbb{D}^{N-1} \times T} \prod_{j=1}^{N-1} (1 - |\alpha_j|^2)^{\beta(1-j/N)-1} \exp(-\text{Tr}(V(E))) d^2\alpha_j d\alpha_N \]
Ideas of the proof M.-Grava

Define the free energies as

\[ \mathcal{F}_{AL}(V, \beta) = - \lim_{N \to \infty} \frac{1}{N} \ln(Z^{AL}_N(V, \beta)), \quad \mathcal{F}_C(V, \beta) = - \lim_{N \to \infty} \frac{1}{N} \ln \left( Z_{N}^{E} \left( V, \frac{2\beta}{N} \right) \right), \]

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\[ Z^{AL}_N(V, \beta) = \int_{\mathbb{D}^N} \prod_{j=1}^{N} (1 - |\alpha_j|^2)^{\beta-1} \exp(-\text{Tr}(V(E))) d^2 \alpha \]

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It is rather technical to prove that

\[ \mathcal{F}_{AL}(V, \beta) = \partial_\beta (\beta \mathcal{F}_C(V, \beta)) \]
\[ \mathcal{F}_{AL}(V, \beta) = \partial_\beta (\beta \mathcal{F}_C(V, \beta)) \]

Consider the case \( V = 0 \). Then, it is possible to compute explicitly

\[ Z^E_N \left(0, \frac{2\beta}{N} \right), Z^{AL}_N (0, \beta) : \]

\[
Z^E_N \left(0, \frac{2\beta}{N} \right) = 2 \frac{\pi^N}{\beta^{N-1}} \prod_{j=1}^{N-1} \frac{1}{1 - \frac{j}{N}}
\]

\[
Z^{AL}_N (0, \beta) = \frac{\pi^N}{\beta^N}
\]

This implies that

\[ F_C(0, \beta) = \int_0^\beta \ln \beta x \pi dx, \quad F_{AL}(0, \beta) = \ln \beta \pi \]

It is possible to generalize this result applying the transfer operator technique.
\[ \mathcal{F}_{AL}(V, \beta) = \partial_\beta (\beta \mathcal{F}_C(V, \beta)) \]

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\[
Z^AL_N(0, \beta) = \frac{\pi^N}{\beta^N}
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This implies that

\[
\mathcal{F}_C(0, \beta) = \int_0^1 \ln \left( \frac{\beta x}{\pi} \right) dx, \quad \mathcal{F}_{AL}(0, \beta) = \ln \left( \frac{\beta}{\pi} \right)
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\[ \mathcal{F}_{AL}(V, \beta) = \partial_\beta (\beta \mathcal{F}_C(V, \beta)) \]

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Z^E_N\left(0, \frac{2\beta}{N}\right) = 2 \frac{\pi^N}{\beta^{N-1}} \prod_{j=1}^{N-1} \frac{1}{1 - \frac{j}{N}} = \prod_{j=1}^{N-1} F\left(\beta \left(1 - \frac{j}{N}\right)\right)
\]

\[
Z^{AL}_N(0, \beta) = \frac{\pi^N}{\beta^N} = \prod_{j=1}^{N} F(\beta)
\]

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It is possible to generalize this result applying the transfer operator technique.
\[ \mathcal{F}_{AL}(V, \beta) = \partial_{\beta}(\beta \mathcal{F}_{C}(V, \beta)). \]

Moreover, it holds true that

\[ \partial_{h}\mathcal{F}_{AL}(V + hz^{k}, \beta) \bigg|_{h=0} = \int_{\mathbb{T}} e^{ik\theta} \nu_{\beta}(\theta) d\theta, \quad \partial_{h}\mathcal{F}_{C}(V + hz^{k}, \beta) \bigg|_{h=0} = \int_{\mathbb{T}} e^{ik\theta} \mu_{\beta}(\theta) d\theta. \]
\[ \mathcal{F}_{AL}(V, \beta) = \partial_{\beta}(\beta \mathcal{F}_{C}(V, \beta)). \]

Moreover, it holds true that

\[ \partial_{h}\mathcal{F}_{AL}(V + hz^{k}, \beta)_{|_{h=0}} = \int_{\mathbb{T}} e^{ik\theta} \nu_{\beta}^{V}(\theta) d\theta, \quad \partial_{h}\mathcal{F}_{C}(V + hz^{k}, \beta)_{|_{h=0}} = \int_{\mathbb{T}} e^{ik\theta} \mu_{\beta}^{V}(\theta) d\theta. \]

These equalities imply that

\[ \int_{\mathbb{T}} e^{ik\theta} \nu_{\beta}^{V}(\theta) d\theta = \partial_{\beta} \left( \beta \int_{\mathbb{T}} e^{ik\theta} \mu_{\beta}^{V}(\theta) d\theta \right) \]
\[ F_{AL}(V, \beta) = \partial_\beta (\beta F_C(V, \beta)). \]

Moreover, it holds true that
\[
\partial_h F_{AL}(V + h z^k, \beta) \bigg|_{h=0} = \int_T e^{ik\theta} \nu_\beta^V(\theta) d\theta, \quad \partial_h F_C(V + h z^k, \beta) \bigg|_{h=0} = \int_T e^{ik\theta} \mu_\beta^V(\theta) d\theta.
\]

These equalities imply that
\[
\int_T e^{ik\theta} \nu_\beta^V(\theta) d\theta = \partial_\beta \left( \beta \int_T e^{ik\theta} \mu_\beta^V(\theta) d\theta \right)
\]

and so
\[
\nu_\beta^V = \partial_\beta (\beta \mu_\beta^V).
\]
Ideas of the proof M.-Memin

We proved a large deviation principle for the family of empirical measures
\[ \mu_N(E) = \frac{1}{N} \sum_{j=1}^{N} \delta_{e^{i\theta_j}}, \]
implying that
\[ \mu_N(E) \xrightarrow{N \to \infty} \nu^V, \]
and \( \nu^V \) is the unique minimizer of the functional
\[ J(V, \beta) : \mathcal{P}(\mathbb{T}) \to [0; \infty). \]
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\[ J(V,\beta) : \mathcal{P}(\mathbb{T}) \to [0; \infty). \]
Moreover, we proved that we can rewrite the functional of Lambert, and Hardy
\( f(V,\beta) \) (the one that is minimized by \( \mu^V_\beta \)) as
\[
f(V,\beta)(\mu) = \lim_{\delta \to 0} \lim_{q \to \infty} \inf_{\nu^V_\beta/q \cdots \nu^V_\beta} \left\{ \frac{1}{q} \sum_{i=1}^{q} J(V,i\beta/q)(\nu_{i\beta/q}) \right\},
\]
Ideas of the proof M.-Memin

We proved a large deviation principle for the family of empirical measures
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\]

which implies that

\[
\int_0^1 \nu^V t dt = \mu^V_\beta \implies \nu^V_\beta = \partial_\beta(\beta \mu^V_\beta)
\]
Explicit Solutions

For the case $V = 0$, Lambert, and Hardy proved that

$$\mu_0 = \frac{1}{2\pi} \rightarrow \nu_0 = \frac{1}{2\pi}$$
Explicit Solutions

For the case $V = 0$, Lambert, and Hardy proved that

$$\mu_0^V = \frac{1}{2\pi} \overset{G-M;M-M}{\longrightarrow} \nu_0^V = \frac{1}{2\pi}$$

For the classical Gibbs ensemble $V = \eta \Re(z)$, in Grava-M. we proved

$$\mu^V_{\beta}(\theta) = \frac{1}{2\pi} + \frac{1}{\pi \beta} \Re \left( \frac{zv'(z)}{v(z)} \right) \bigg|_{z=e^{i\theta}}, \quad \nu^V_{\beta} = \frac{1}{2\pi} + \partial_{\beta} \left( \frac{1}{\pi} \Re \left( \frac{zv'(z)}{v(z)} \right) \bigg|_{z=e^{i\theta}} \right).$$

where $v(z)$ is the unique analytic solution at 0 of the double confluent Heun equation

$$z^2 v''(z) + (-\eta + z(\beta + 1) + \eta z^2) v'(z) + \eta \beta (z + \lambda) v(z) = 0,$$

and $\lambda$ is determined as the unique solution of a transcendental equation.
Open problems

Explicitly compute the correlations functions. Despite having explicit solutions via finite-gap integration, and several insights for the GHD theory (Dojon, Spohn, EI), the computation remains out of reach.
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??
Thank you for the attention!