Free fermions and parafermions

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Free fermions

- The fundamental system in theoretical physics
- Many properties can be computed exactly
- Keeps on keeping on
  e.g. topological classification, entanglement, quenches…
- Appear even in some non-obvious guises
- Can solve algebraically

What is a free fermion?

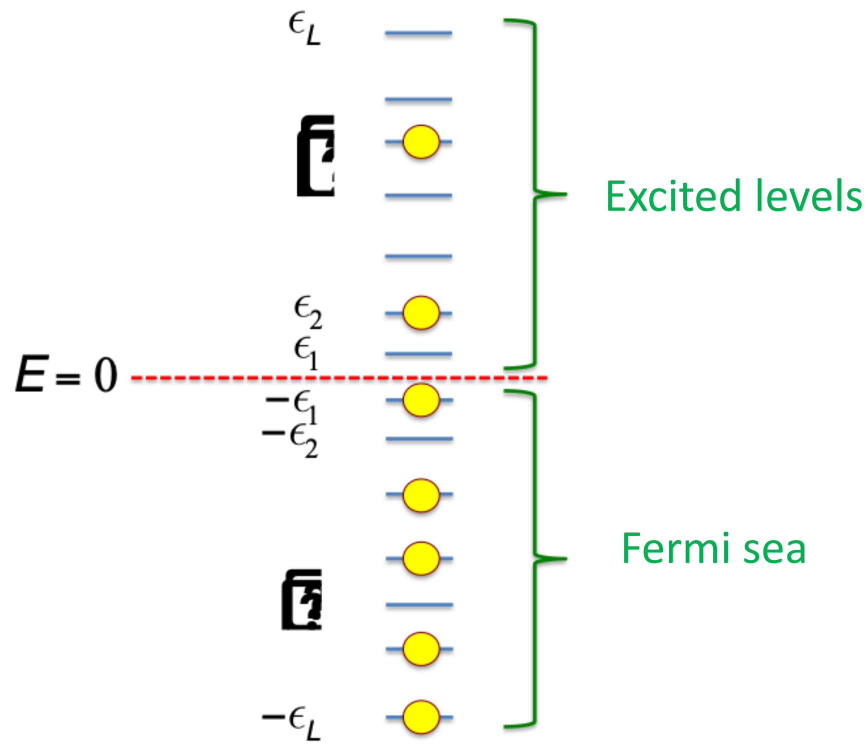
Forget statistics, forget operators, forget fields…

The basic property of free fermions is that their spectrum is of the form

$$E = \pm\epsilon_1 \pm \epsilon_2 \pm \ \dots \ \pm \epsilon_L$$

The choices of a given $\pm$ are independent, and do not affect the values of ϵ_l .

Levels are either filled or empty.



The usual (lattice) story

Automatically find such a spectrum when Hamiltonian is a bilinear in fermions:

$$H = \sum_{a,b} \mathcal{H}_{ab} \psi_a \psi_b$$

where $\mathcal{H}_{ab}$ is an antisymmetric matrix, and the (Majorana) fermions obey

$$\{\psi_a,\, \psi_b\} = 2\delta_{ab}$$

Examples of non-obvious free-fermionic systems:

1d quantum transverse-field/2d classical Ising

Kauffman, Onsager; its fermionic version now known as the ``Kitaev chain''

1d quantum XY

Jordan-Wigner; Lieb-Schultz-Mattis

2d Kitaev honeycomb model

In field theory: sine-Gordon at special point

Coleman; Luther-Emery

Canonical example: the Ising/Kitaev chain

transverse field
favouring disorder

interaction favouring order

$$H = -h \sum_{j=1}^{L} \sigma_j^x - J \sum_{j=1}^{L-1} \sigma_j^z \sigma_{j+1}^z$$

The σ_j^a are Pauli matrices acting on site *j* of the *L*-site chain.

The non-local Jordan-Wigner transformation maps it to

$$H = -ih \sum_{j=1}^{L} \psi_{2j-1}\psi_{2j} - iJ \sum_{j=1}^{L-1} \psi_{2j}\psi_{2j+1}$$

$$H = \sum_{a,b} \mathcal{H}_{ab}\psi_a\psi_b \qquad \mathcal{H} = i\begin{pmatrix} 0 & h & 0 & 0 & 0 & \cdots \\ -h & 0 & J & 0 & 0 & \cdots \\ 0 & -J & 0 & h & 0 & \cdots \\ 0 & 0 & -h & 0 & J & \cdots \\ & & & \vdots & & \end{pmatrix}$$

Really easy to find spectra of such Hamiltonians

Commuting a linear in fermions with a bilinear gives a linear back:

$$\left[H, \sum_{a=1}^{2L} r_a \psi_a\right] = \sum_{b=1}^{2L} s_b \psi_b \qquad \text{where} \quad \mathcal{H}_{ba} r_a = s_b$$

The raising and lowering operators $\Psi_{\pm k} = \sum_a v_a^{(\pm k)} \psi_a$

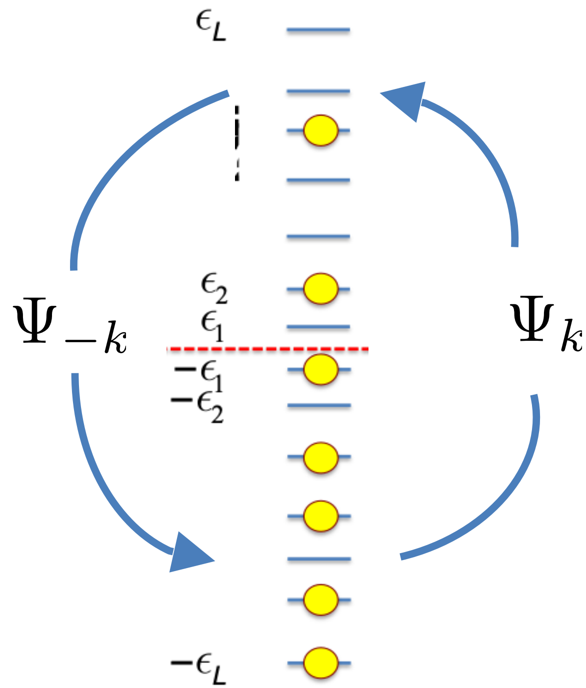
come from eigenvectors $\mathcal{H} v^{(\pm k)} = \pm 2\epsilon_k \, v^{(\pm k)}$

$$\left[H,\ \Psi_{\pm k}\right] = \pm 2\epsilon_k \Psi_{\pm k}$$

The eigenvalues ϵ_k are the zeroes of a polynomial whose coefficients depend on the couplings.

One more step: To find free-fermion spectrum, need to show that raising/lowering operators have algebra

$$\left\{\Psi_l,\ \Psi_{l'}\right\} = \delta_{l,-l'} \quad \text{in addition to} \quad \left[H,\ \Psi_{\pm k}\right] = \pm 2\epsilon_k \Psi_{\pm k}$$



$$E = \pm\epsilon_1 \pm \epsilon_2 \pm \ \ldots \ \pm \epsilon_L$$

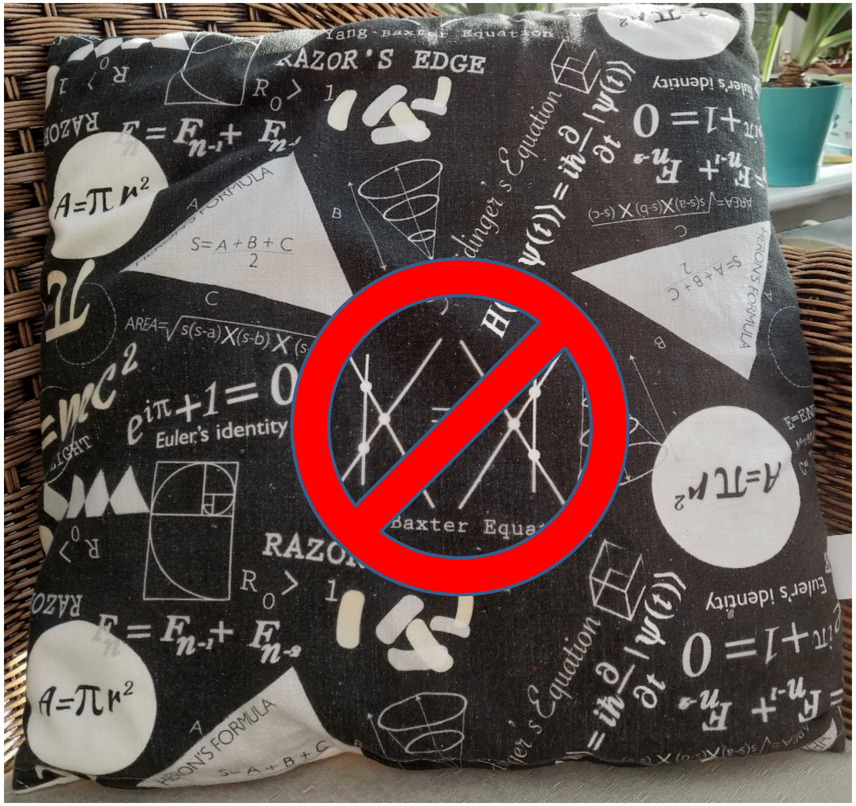
By now many many models have been solved by J-W transformations. Do one on your fave chain, and if the Hamiltonian is quadratic in fermions, you win!

Recently, a general result for when J-W transformation to free fermions is possible. Describe terms in Hamiltonian by a ``frustration” graph, and for it to be quadratic in fermions, graph must have certain properties.

Chapman-Flammia, 2003.05465

Is that all there is?

I'll describe how to construct raising and lowering operators in some interacting models using elementary algebra.



Strangely, technique only works for open boundary conditions, model remains integrable for periodic but not free. But first, the models….

Rewriting the models in a more algebraic form

Generalise Ising/Kitaev Hamiltonian to allow spatial dependence

constants

$$H = \sum_{m=1}^{2L-1} h_m \qquad h_{2j-1} = t_{2j-1}\,\sigma_j^x \qquad h_{2j} = t_{2j}\,\sigma_j^z \sigma_{j+1}^z$$

These operators obey a very simple algebra

$$h_m^2 = t_m^2\,, \qquad h_m h_{m+1} = -h_{m+1} h_m\,, \qquad h_m h_{m'} = h_{m'} h_m \quad \text{for } |m-m'| > 1$$

Frustration graph:

Using this algebra, construct operators that obey

$$\left\{\Psi_l,\ \Psi_{l'}\right\} = \delta_{l,-l'} \qquad \left[H,\ \Psi_{\pm k}\right] = \pm 2\epsilon_k \Psi_{\pm k}$$

Spectrum then follows independent of representation.

Free fermions in disguise

Fendley 2019

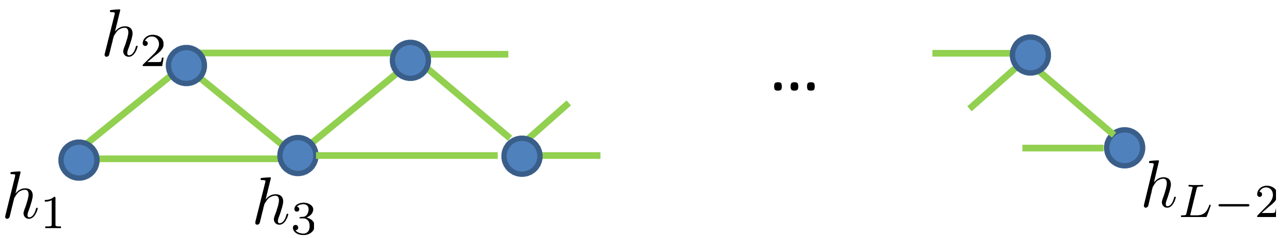
$$H = \sum_{m=1}^{L-2} h_m \ , \qquad\qquad h_m = t_m \, \sigma^x_m \sigma^z_{m+1} \sigma^z_{m+2}$$

constants

The generators anticommute two sites apart as well:

$$h_m^2 = t_m^2 \, , \quad h_m h_{m+1} = -h_{m+1} h_m \, , \quad h_m h_{m+2} = -h_{m+2} h_m \, ,$$

$$h_m h_{m'} = h_{m'} h_m \quad \text{for } |m - m'| > 2$$



The J-W transformation gives a purely four-fermi Hamiltonian:

$$H = \sum_{m=1}^{L-2} t_m \, \psi_{2m-1} \psi_{2m} \psi_{2m+2} \psi_{2m+3}$$

This model has an $N=2$ supersymmetry, with generators made of fermion trilinears.

Baxter's $\mathbb{Z}_n$ Hamiltonian

Analogs of Pauli matrices for an *n*-state spin are

$$\tau = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & & & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{pmatrix} \qquad \sigma = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & \omega & 0 & \cdots & 0 \\ 0 & 0 & \omega^2 & & 0 \\ \vdots & \vdots & & & \vdots \\ 0 & 0 & 0 & \cdots & \omega^{n-1} \end{pmatrix}$$

``shift'' $\qquad \omega = e^{2\pi i/n} \qquad$ ``clock''

Like with Ising, set $\quad H = \sum_{m=1}^{2L-1} h_m$

$$h_{2j-1} = t_{2j-1}\tau_j$$
$$h_{2j} = t_{2j}\sigma_j^\dagger \sigma_{j+1}$$

Not Hermitian! No + h.c.

Note!

Algebra:
$$h_m^n = t_m^n\,, \qquad h_m h_{m+1} = \omega h_{m+1} h_m\,,$$
$$h_m h_{m'} = h_{m'} h_m \quad \text{for } |m-m'|>1$$

The KBBS Hamiltonian

aka τ_2 model, aka BS model

Korepanov 87; Baxter 89; Bazhanov and Stroganov 89
Baxter 2004; Baxter 2013; Au-Yang+Perk 2014

The same shift-clock $\mathbb{Z}_n$ algebra as before:

$$h_m^n = t_m^n\,, \qquad h_m h_{m+1} = \omega h_{m+1} h_m\,,$$

$$h_m h_{m'} = h_{m'} h_m \quad \text{for } |m-m'|>1$$

but Hamiltonian is long-ranged:

$$H = \sum_{m \le m' \le 2L-1} u_m\, h_m h_{m+1} \cdots h_{m'}$$

constants

In terms of parafermions, each term is bilinear $\psi^\dagger_{2m-1}\psi_{2m'}$

How to show these models are ``free'':

1. Find a transfer matrix $T(u)$ obeying $[H,\ T(u)]=0$ for all u.

   In these examples, easy to do using the algebra of the h_m

2. Show that $\left[T(u),\, T(u')\right] = 0$

   Usually use Yang-Baxter, but here straightforward to do directly

3. Find ``inverse'' such that $T(u)T^{-}(u) = P(u^n)$ polynomial

4. Raising and lowering (more generally shift) operators are constructed for each zero u_k obeying $P((u_k)^n) = 0$

$$\Psi_k = T(u_k)\,\chi\, T^{-}(u_k)$$

where the "simplicial" mode χ is an extra Hamiltonian generator at an edge.

5. They obey $$\left[H,\, \Psi_k\right] = (\omega - 1)\epsilon_k\, \Psi_k \;\; , \quad \epsilon_k = \frac{1}{u_k}$$

Rather miraculous.

The transfer matrix

Ising: adjacent h_m anticommute, others commute.

Non-local conserved charges $Q^{(r)}$ involve h_m at least 2 sites apart:

$$\Big[H,\, Q^{(r)}\Big] = 0$$

$$Q^{(1)} = H = \sum_m h_m$$

$$Q^{(2)} = \sum_{m_1+1<m_2} h_{m_1} h_{m_2}$$

$$Q^{(3)} = \sum_{m_1+1<m_2<m_3-1} h_{m_1} h_{m_2} h_{m_3}$$

$$T(u) = \sum_r (-u)^r Q^{(r)}$$ yields the desired $$\Big[H,\, T(u)\Big] = 0$$

Local conserved charges follow from logarithmic derivative

And for the four-fermi model

Also include in algebra $$h_m h_{m+2} = -h_{m+2} h_m$$

Non-local conserved charges $Q^{(r)}$ involve h_m at least 3 sites apart:

$$\left[H,\, Q^{(r)}\right] = 0$$

$$Q^{(1)} = H = \sum_m h_m$$

$$Q^{(2)} = \sum_{m_1+2<m_2} h_{m_1} h_{m_2}$$

$$Q^{(3)} = \sum_{m_1+2<m_2<m_3-2} h_{m_1} h_{m_2} h_{m_3}$$

$T(u) = \sum_r (-u)^r Q^{(r)}$ yields the desired $\left[H,\, T(u)\right] = 0$

Local conserved charges follow from logarithmic derivative

Same form for parafermion case!

even though here $$h_m h_{m+1} = \omega\, h_{m+1} h_m$$

Non-local conserved charges $Q^{(r)}$ involve h_m at least 2 sites apart:

$$\left[H,\ Q^{(r)}\right] = 0$$

$$Q^{(1)} = H = \sum_m h_m$$

$$Q^{(2)} = \sum_{m_1+1<m_2} h_{m_1}\, h_{m_2}$$

$$Q^{(3)} = \sum_{m_1+1<m_2<m_3-1} h_{m_1}\, h_{m_2}\, h_{m_3}$$

$T(u) = \sum_r (-u)^r Q^{(r)}$ obeys ``inversion'' relation

$$T(u)\, T(\omega u)\, T(\omega^2 u)\, \cdots\, T(\omega^{n-1} u) = P(u^n)$$

polynomial

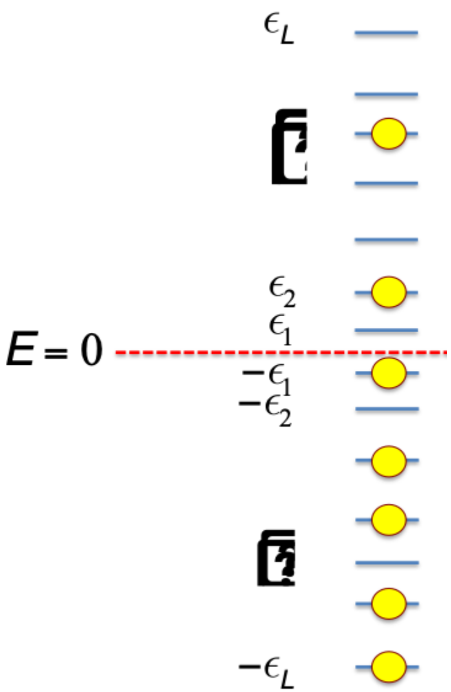
The KBBS model works out similarly.

Free fermions in disguise

Even though the starting algebras are different, construct raising/lowering operators in both Ising and 4-fermi models obeying the same algebra

$$\{\Psi_l,\ \Psi_{l'}\} = \delta_{l,-l'} \qquad \left[H,\ \Psi_{\pm k}\right] = \pm 2\epsilon_k \Psi_{\pm k}$$

$$P\left((\epsilon_k)^{-2}\right) = 0$$



$$E = \pm\epsilon_1 \pm \epsilon_2 \pm \cdots \pm \epsilon_M$$

Only distinction: $M = L/2$ for Ising

$M = L/3$ for 4-fermi

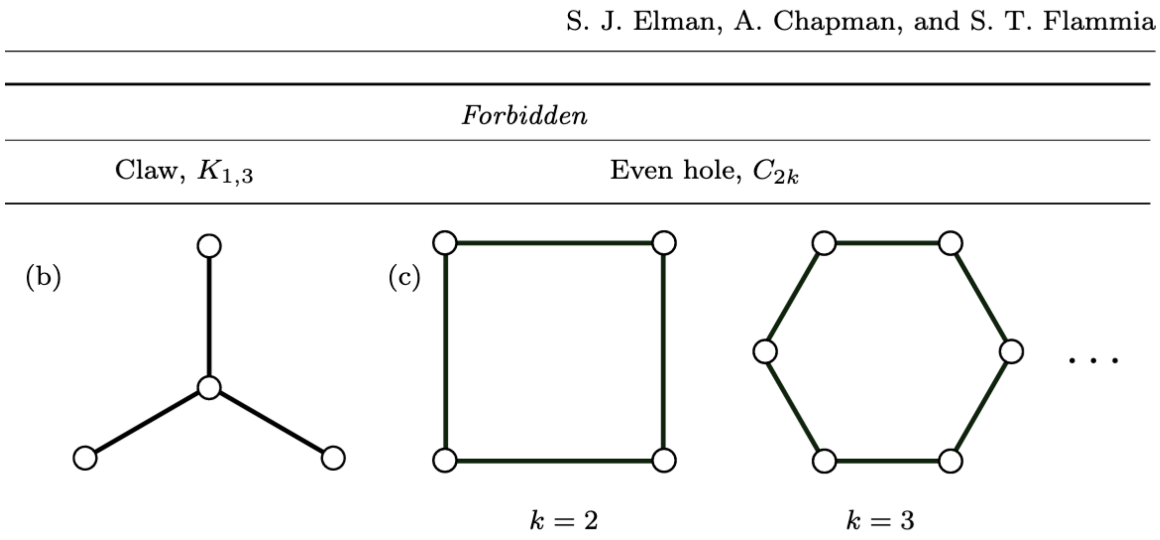
Thus exponentially large degeneracies for each energy in 4-fermi model!

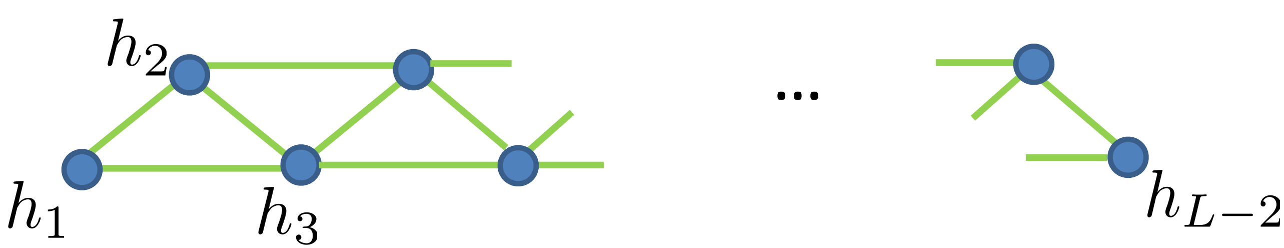
Connection to graph theory

Remarkably, can find a general set of criteria that explains when free fermions are in disguise.

Elman, Chapman and Flammia

Works when the frustration graph is even-hole-free, claw-free.





Construction works for free parafermions as well

Anticommutation relations generalise to

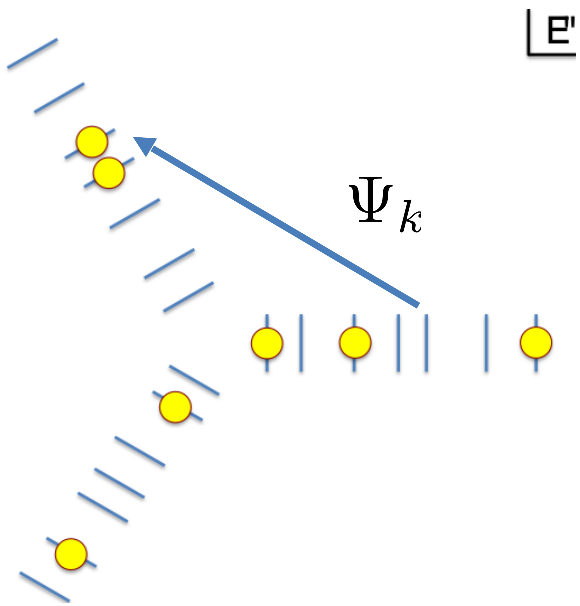
$$(\epsilon_k - \omega\epsilon_l)\Psi_k\Psi_l = (\epsilon_l - \omega\epsilon_k)\Psi_l\Psi_k$$

Baxter's $\mathbb{Z}_n$ chains have spectrum

$$E = \omega^{m_1}\epsilon_1 \;+\; \omega^{m_2}\epsilon_2 \;+\; \cdots \;+\; \omega^{m_L}\epsilon_L$$

where $\omega = e^{2\pi i/n}$

and each $m_j = 0, 1, \ldots n-1$



Baxter 1989
2014

Fendley 2013
2022

Au-Yang/Perk
2014, 2016

The physics of the four-fermi chain

$$H = \sum_m t_m \psi_m \psi_{m+1} \psi_{m+3} \psi_{m+4}$$

$$= \sum_j \left(t_{2j} \sigma_j^z \sigma_{j+1}^z \sigma_{j+2}^x + t_{2j+1} \sigma_j^x \sigma_{j+1}^z \sigma_{j+2}^z \right)$$

The free-fermion raising/lowering operators are non-linear and non-local in the original fermions. Thus e.g. no Wick's theorem in real space.

Nevertheless, the spectrum is free-fermionic:

$$E = \pm\epsilon_1 \pm \epsilon_2 \pm \cdots \pm \epsilon_S$$

where the $(\epsilon_k)^2$ are the roots of an order *S*=2*L*/3 polynomial.

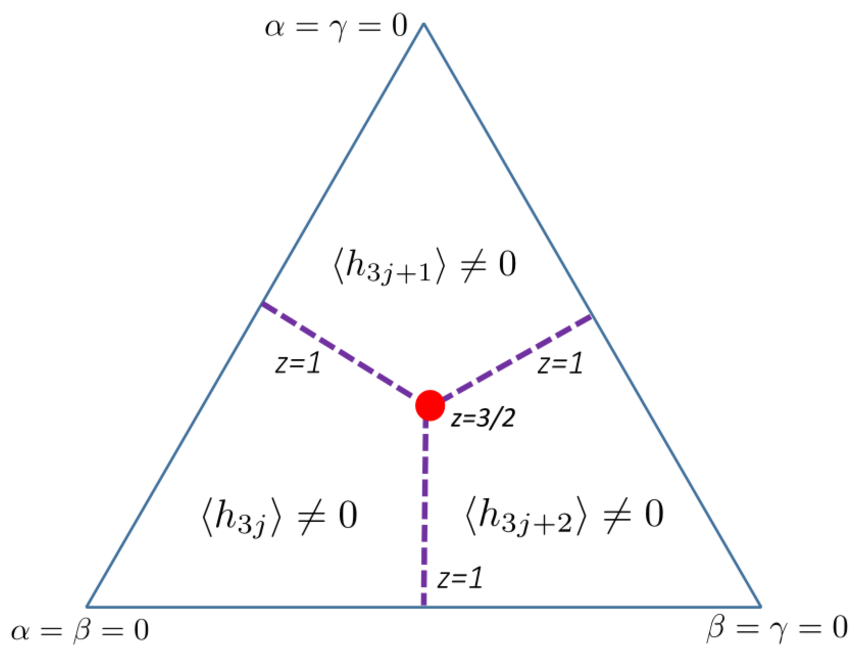
One unusual property: each energy is exponentially $(2^{L/3})$ degenerate. Degeneracies related to an extended supersymmetry algebra.

Find for uniform couplings, theory is critical, but not described by a CFT.

It has dynamical critical exponent $z = 3/2$, i.e. excitations have energy

$$E \propto L^{-\frac{3}{2}}$$

Staggering on every third site gives phase diagram

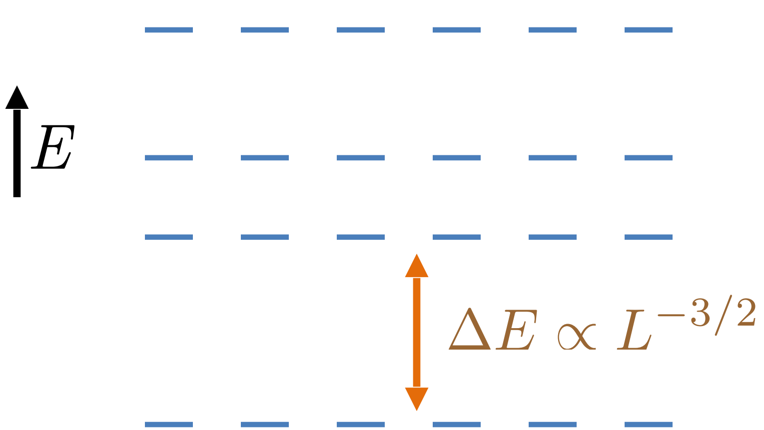


Triality!

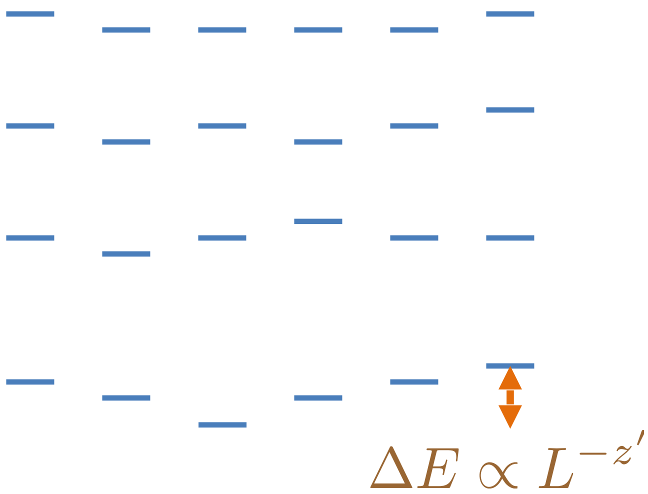
Not free-fermionic for periodic boundary condition

Also breaks degeneracies, giving a second dynamical critical exponent:

Open b.c.:



Periodic:



Integrability is preserved for uniform couplings.

Can z' be computed analytically?

Hello integrability experts? Hello numerical experts?

Combining Ising and four-fermi chains

$$H_{\rm Ising} - gH_{4-{\rm fermi}}$$

O'Brien and Fendley

- Combination is not only not free-fermion, it's not even integrable.

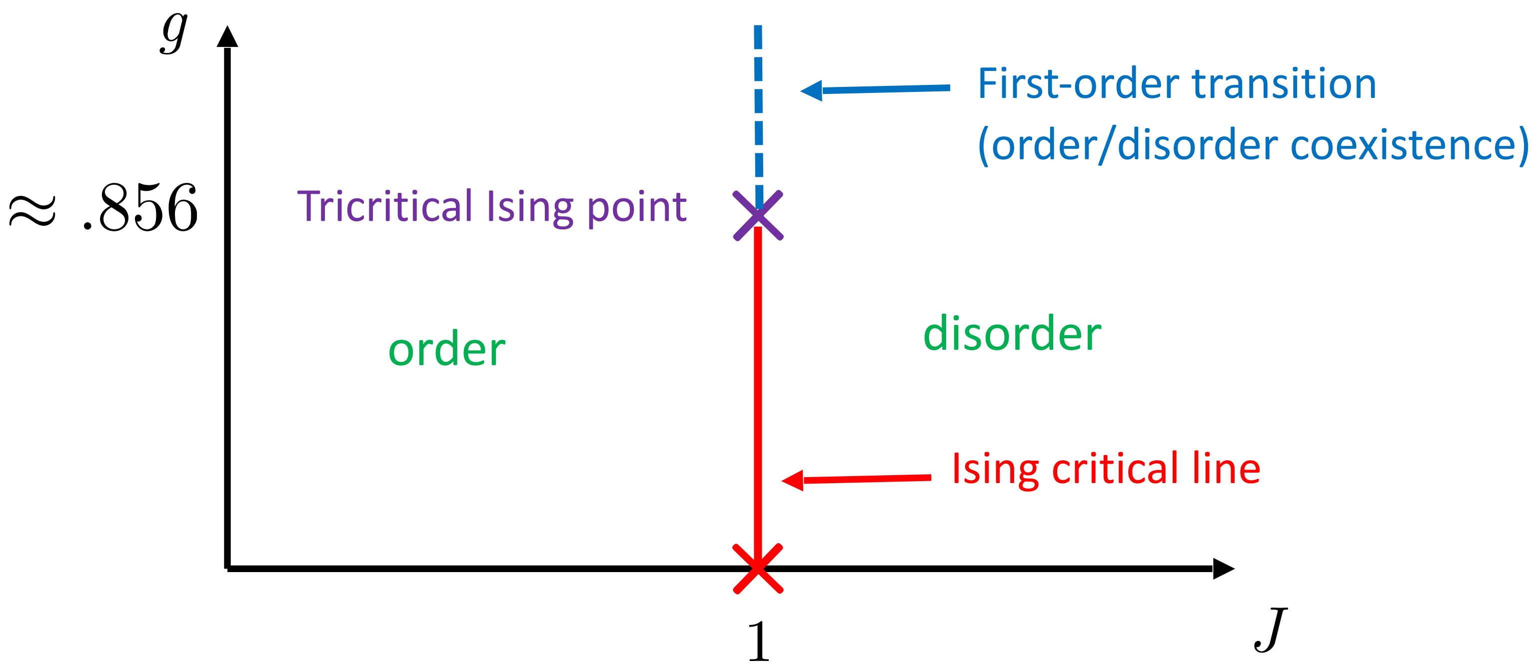
- Find a non-trivial critical point with only one-parameter tuning and without changing the Hilbert space. Thus ideally suited for testing numerical methods.

- Along self-dual line, interesting properties such as supersymmetry and order-disorder coexistence.

This lattice model gives direct insight into the long-distance physics.

$$H = -\sum_j \left(\sigma_j^x + J \sigma_j^z \sigma_{j+1}^z + g\left(\sigma_j^x \sigma_{j+1}^z \sigma_{j+2}^z + \sigma_j^z \sigma_{j+1}^z \sigma_{j+2}^x \right) \right)$$

In Temperley-Lieb language: $H = \sum_j \left(e_j + e_j e_{j+3} \right)$



Along self-dual line the supersymmetry is apparent:

$$H_{\mathrm{s-d}} = \sum_j \left(2i\psi_a\psi_{a+1} - g\,\psi_{a-1}\psi_a\psi_{a+2}\psi_{a+3}\right)$$

$$= (Q^+)^2 + (Q^-)^2 + E_0$$

$$Q^{\pm} = \frac{1}{2\sqrt{g}} \sum_{a=1}^{2L} (\pm 1)^a \left(\psi_a \pm ig\,\psi_{a-1}\psi_a\psi_{a+1}\right)$$

The supersymmetry generators include Majorana trilinears.

Except at the pure four-fermi point, the supersymmetry is not exact on the lattice, i.e. $\left[Q^{\pm},\, H\right] = 0$ is true only in the continuum.

$$H_{\mathrm{s-d}} = (Q^+)^2 + (Q^-)^2 + E_0$$

holds all along self-dual line, not just at TCI point.

Any state obeying $Q^+|g\rangle = Q^-|g\rangle = 0$ is a ground state.

At $g = 1$, there are three exact ground states!

$$\left|\uparrow\uparrow\uparrow\cdots\uparrow\right\rangle\ ,\ \left|\downarrow\downarrow\downarrow\cdots\downarrow\right\rangle \quad \text{and} \quad \sum_{\text{all states } s} |s\rangle$$

Given the first two ground states, the self-duality requires that the latter also is a ground state.

Lots more to do

- Graph theory for parafermions?

- Field theory?

- Connection to experiment both in Ising+4-fermion and in chiral Potts
  Aasen et al 2020 Rydberg blockade, Lukin et al

- The (superintegrable) chiral Potts transfer matrix is inverse of KBBS one. Lots of mysteries, including a close connection to integrable Bazhanov-Baxter models in 3d

- Connection to chiral CFTs?