Limit shape phase transitions

A merger of Arctic circles

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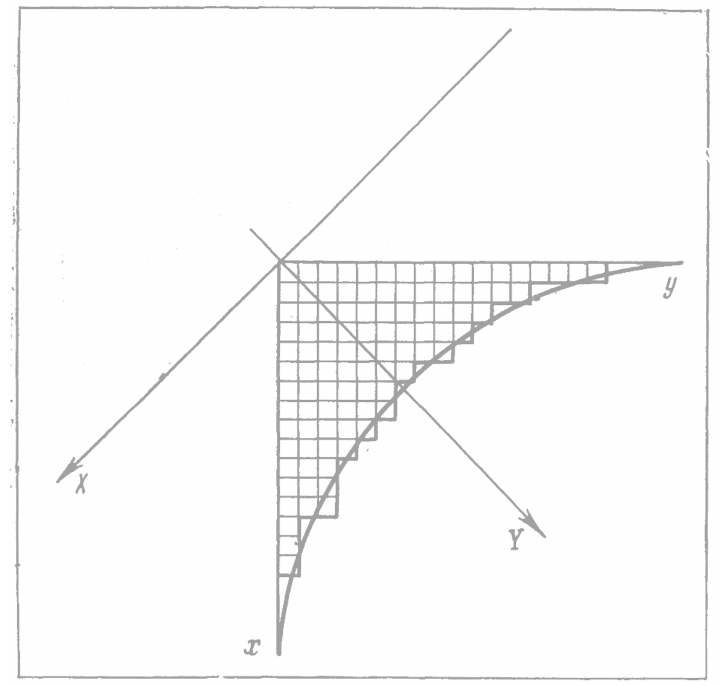
with: Dmitri Gangardt and James Pallister (Birmingham)

based on: https://arxiv.org/abs/2203.05269



Limit shape phenomenon

The limit shape phenomenon - the formation of a non-random shape in thermodynamic limit of random/statistical systems.

Vershik, Kerov, ’77; Logan, Schepp, ’77; Pokrovsky, Talapov, ’78, ’79;
Elkies, Kuperberg, Larsen, Propp, ’92; Jockusch, Propp, ’98;
Prahofer, Spohn; Borodin, Gorin; Nienhuis, Hilhorst, Blote; Cohn, Kenyon, Propp; Kenyon, Okounkov; AGA; Kenyon, Okounkov, Sheffield; Reshetikhin; Allegra, Dubail, Stephan, Viti; Colomo, Pronko, Zinn-Justin, Sportiello; Adler, Johansson, van Moerbeke; Corwin, …

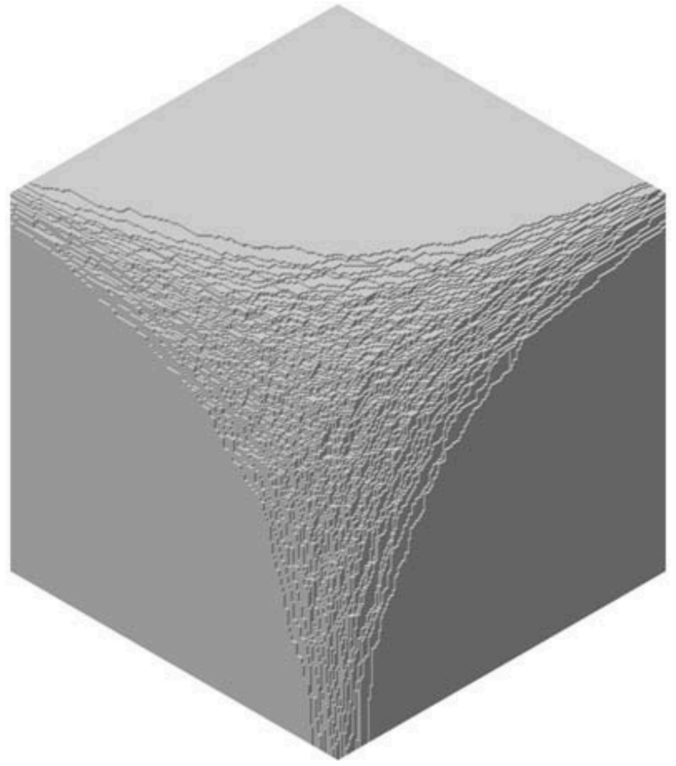
dimer models, polymer models, sorting networks, ASEP (asymmetric exclusion processes), sandpile models, bootstrap percolation models, polynuclear growth models, hydrodynamic flows, free fermions, 6-vertex models, Young diagrams, …

Outline

- Examples of limit shapes:
  - Emptiness formation
  - Arctic Circle
- From tilings to fluid dynamics
- Merging Arctic Circles - Gross-Witten-Wadia transition
- Limit shape phase transitions
- Universality of phase transitions

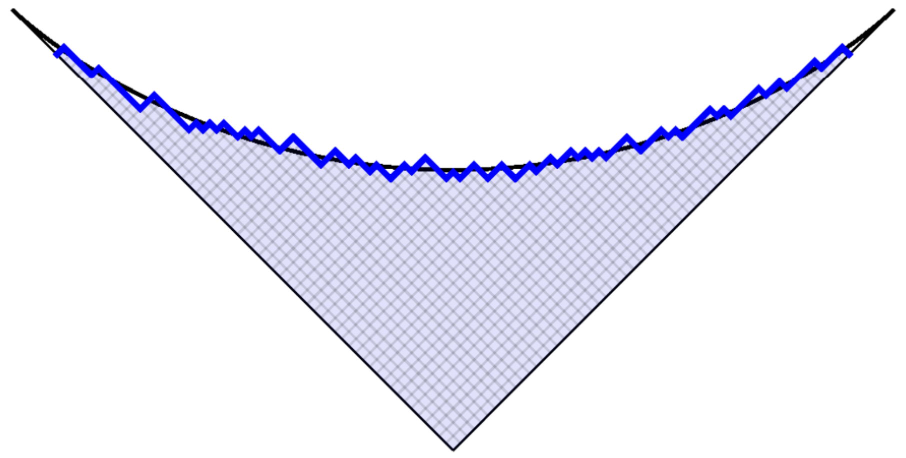
Examples of limit shapes

- Equilibrium crystal shape



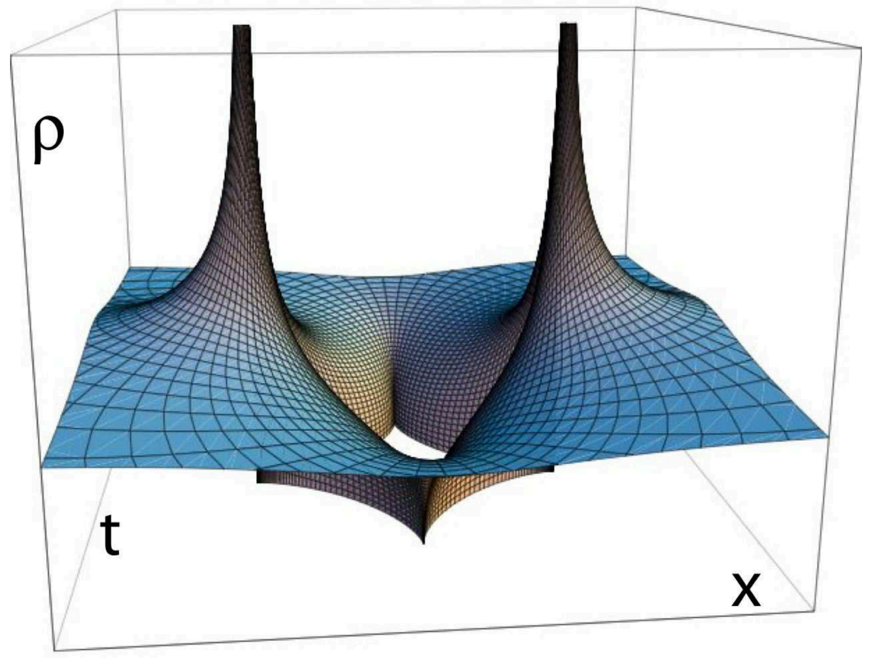
Ferrari, Spohn, 2003

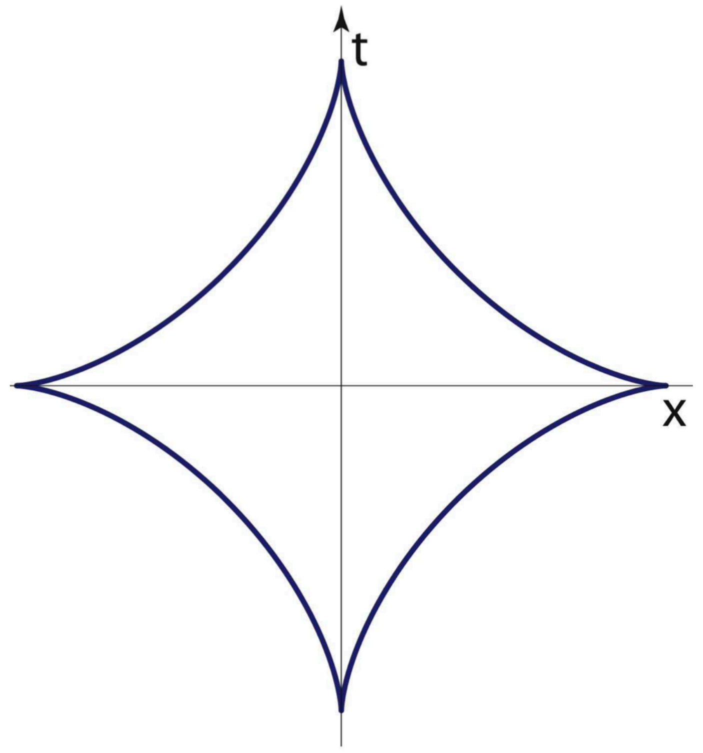
- Random Young tableau



Vershik, Kerov, 1977
Logan, Shepp, 1977

- Emptiness formation in 1d quantum fermions





AGA, 2005

Emptiness Formation

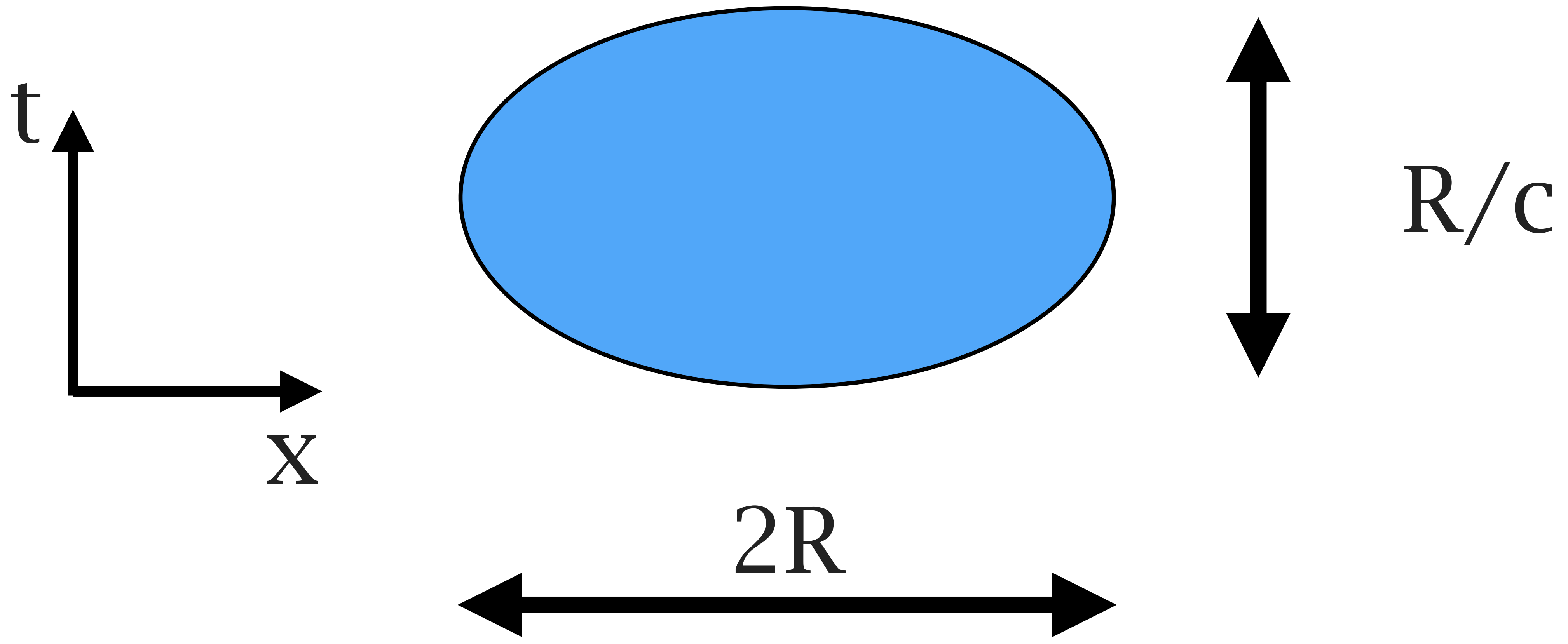
Imagine a quantum Fermi gas on a line

What is the probability that at time t=0, the gas parts away to form an emptiness of the size 2R at large R?

- For random points in one dimension the probability of an empty spot is $P \sim e^{-2\rho_0 R}$
- Wrong for a quantum system in the ground state
- For large R - collective dynamics

Estimate of the instanton's action

- For large R, the probability is determined by the action of an instanton
- What motion of the gas (in imaginary time) is optimal for the emptiness formation?



$$P \sim e^{-\# R^2}$$

$$P \sim e^{-\frac{1}{2}(k_F R)^2}$$

Hydrodynamics of free fermions

$$H = \int dx\, \rho \left(\frac{v^2}{2} + \frac{\pi^2 \hbar^2 \rho^2}{2} \right)$$

fluid's kinetic energy

Pauli's internal energy

$$\partial_\tau \rho + \partial_x (\rho v) = 0$$

$$\partial_\tau v + v \partial_x v = \partial_x \frac{\pi^2 \rho^2}{2}$$

$$i k_\tau + k k_x = 0$$

$$k = \pi \rho \pm i v$$

$$x + i k \tau = F(k)$$ - analytic

?

Find F(k) from boundary conditions and compute the action on the solution

$$P \sim e^{-S}$$

Emptiness solution

AGA, 2005

General solution:

$$ik_\tau + kk_x = 0$$

$$k = \pi\rho + iv$$

$$z \equiv \boxed{x + ik\tau = F(k)}$$

$\rightarrow$ $k, \bar{k}, \rho, v(x,\tau)$ $\longrightarrow$ $P \sim e^{-S}$

$F(k)$ - any analytic function

Emptiness: $F(k)$ - to be found from boundary conditions

$$\boxed{F(k) = R\frac{k}{\sqrt{k^2 - k_F^2}}}$$

$$k = k_F \frac{z}{\sqrt{z^2 - R^2}}$$

$\tau = 0,\ z = x$

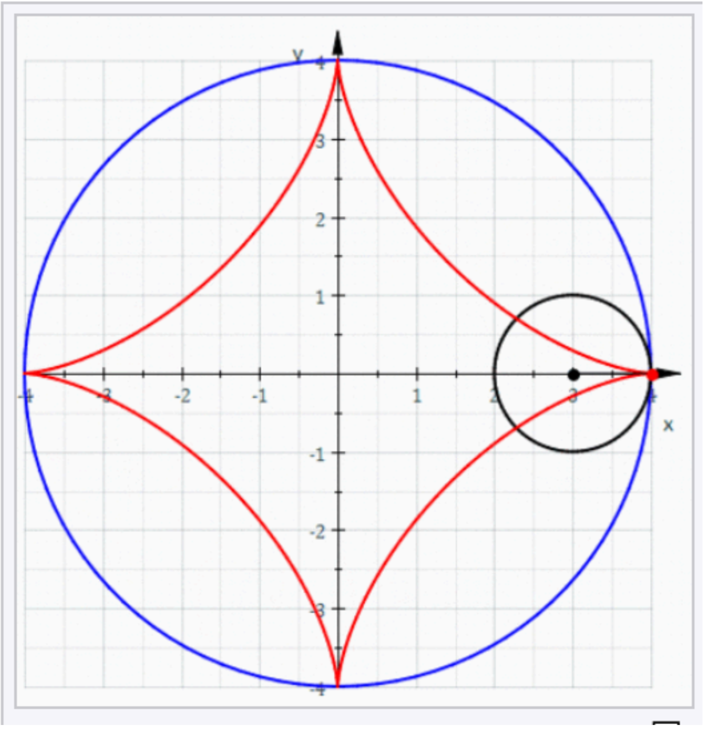
$\rightarrow \rho = 0$ for $x^2 - R^2 < 0$

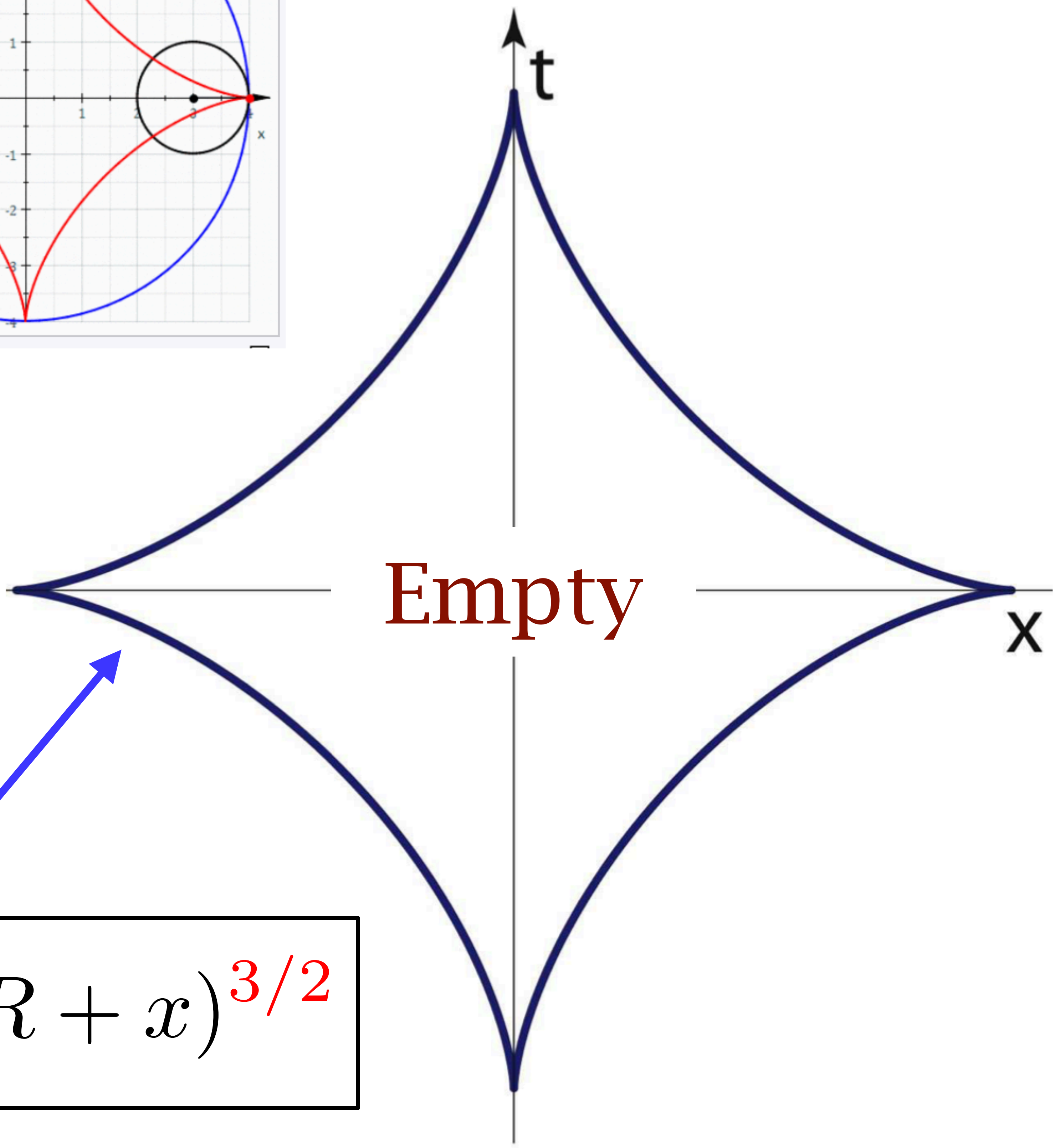
$z \rightarrow \infty$

$\rightarrow k \rightarrow k_F$

Emptiness and Astroid

AGA, 2005

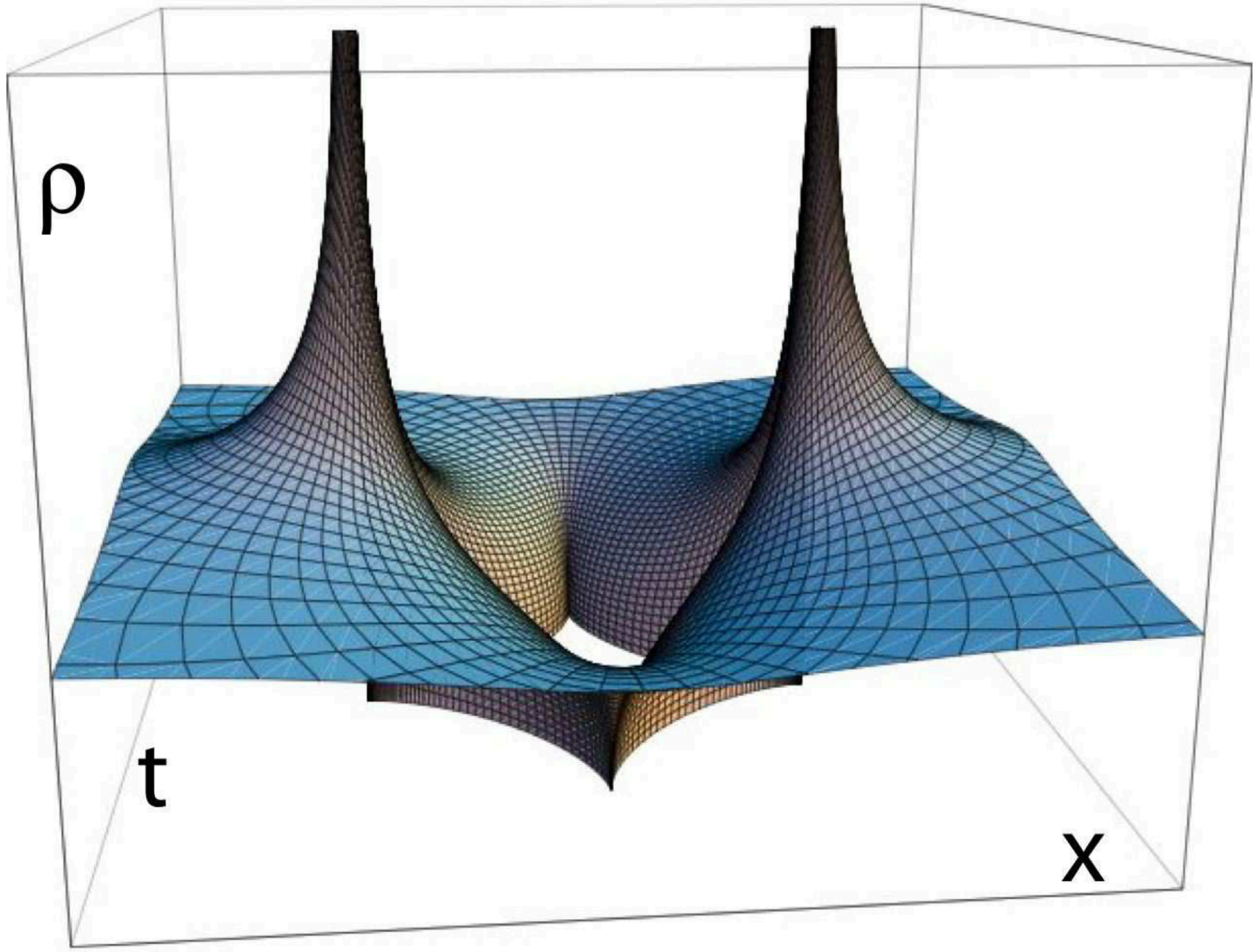




$$t \sim -(R+x)^{3/2}$$

The shape of the empty space - Astroid!

$$x^{2/3} + t^{2/3} = R^{2/3}$$



The density of gas around the emptiness

Remarks

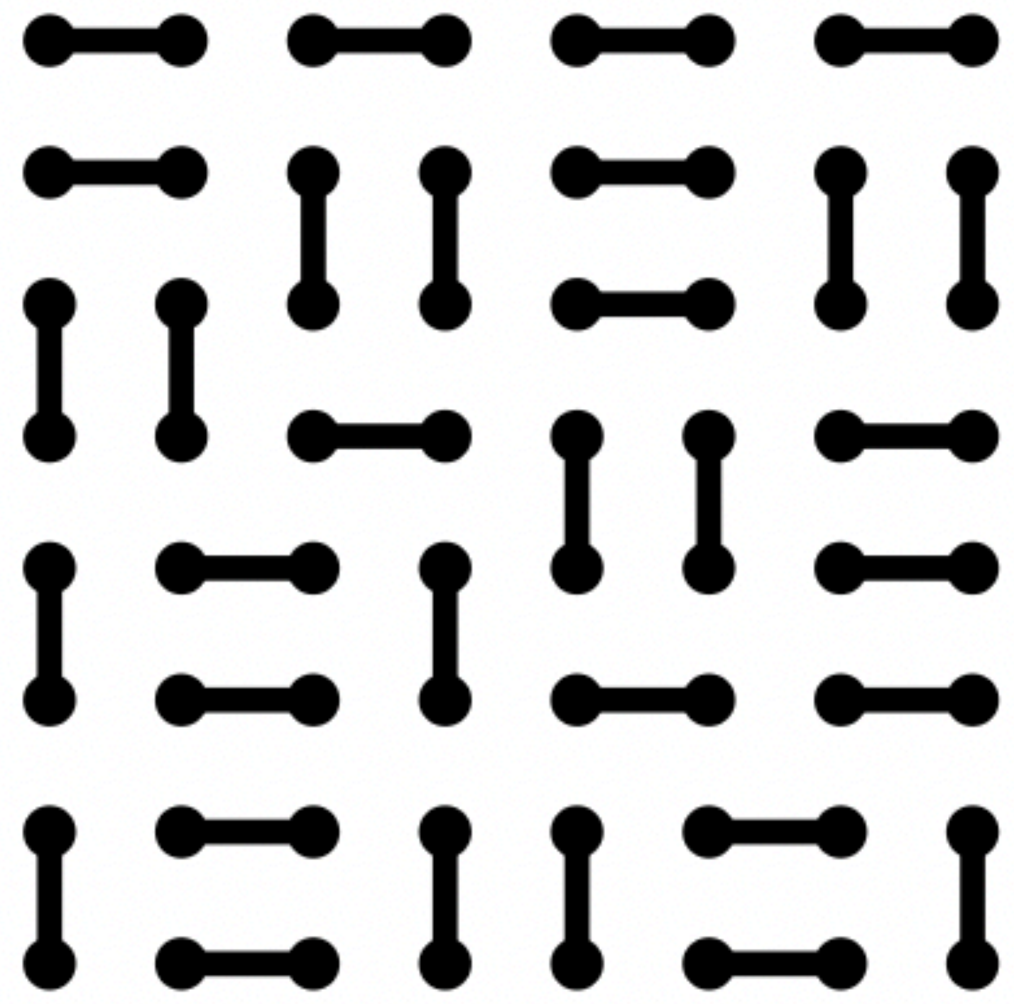
- The probability of large fluctuation is determined by the action of an instanton
- Instanton - solution of classical hydrodynamic equations in imaginary time
- Frozen regions minimize action and are allowed as parts of an instanton solution
- Globally: combination of the solution of hydro equations and frozen regions
- Free fermions: find F(k) from the boundary conditions imposed in spacetime
- Technical: the action computed on equations of motion - integral over the boundary of the frozen region
- Emptiness formation probability can also be computed from the ground state wave function without fluid dynamics

Domino Tilings and Arctic Circle Theorem

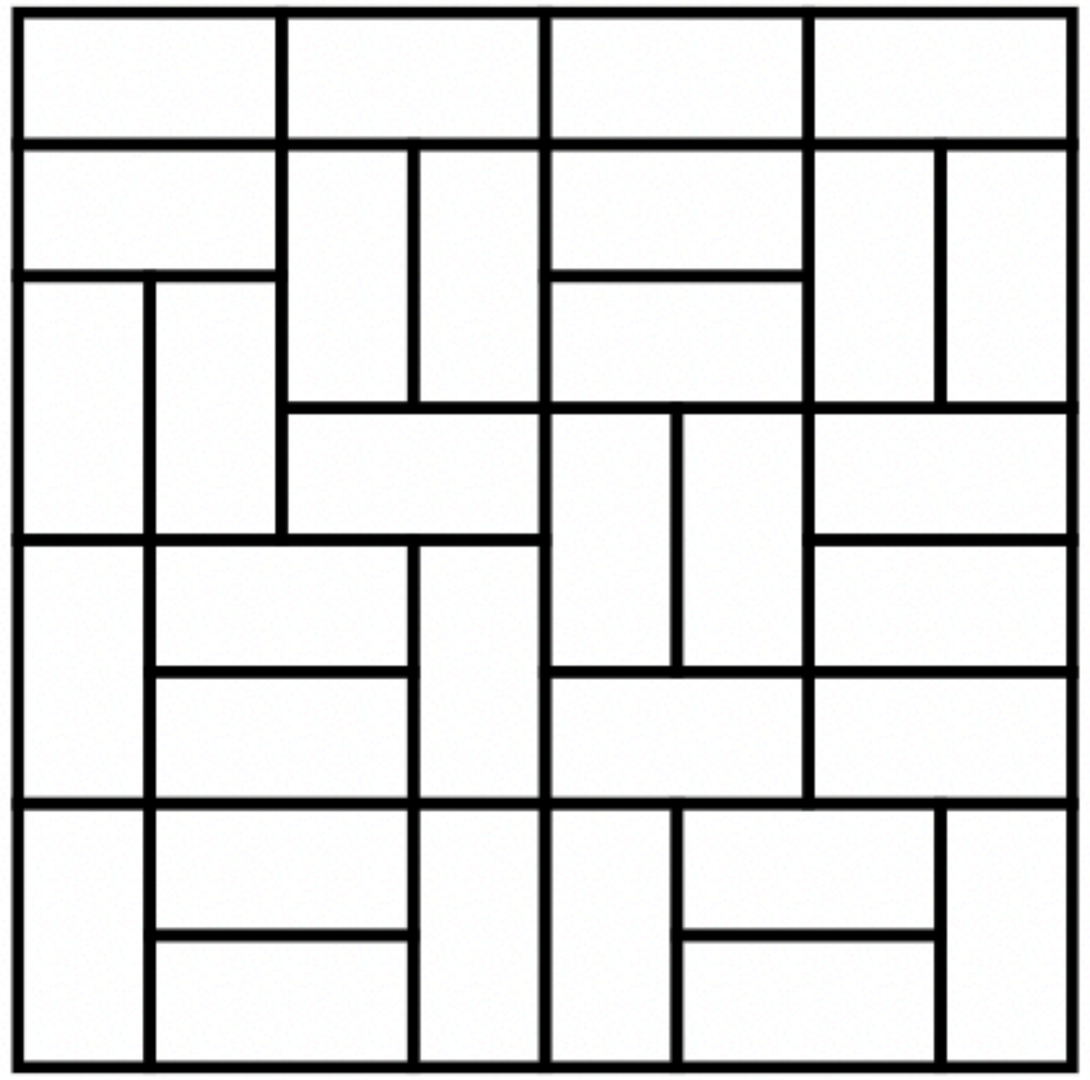
Domino tiling

Problem: In how many ways one could tile the 8 x 8 chessboard by dominos of the size 2 x 1?

Kasteleyn, 1963



Dimer covering of the lattice 8 x 8

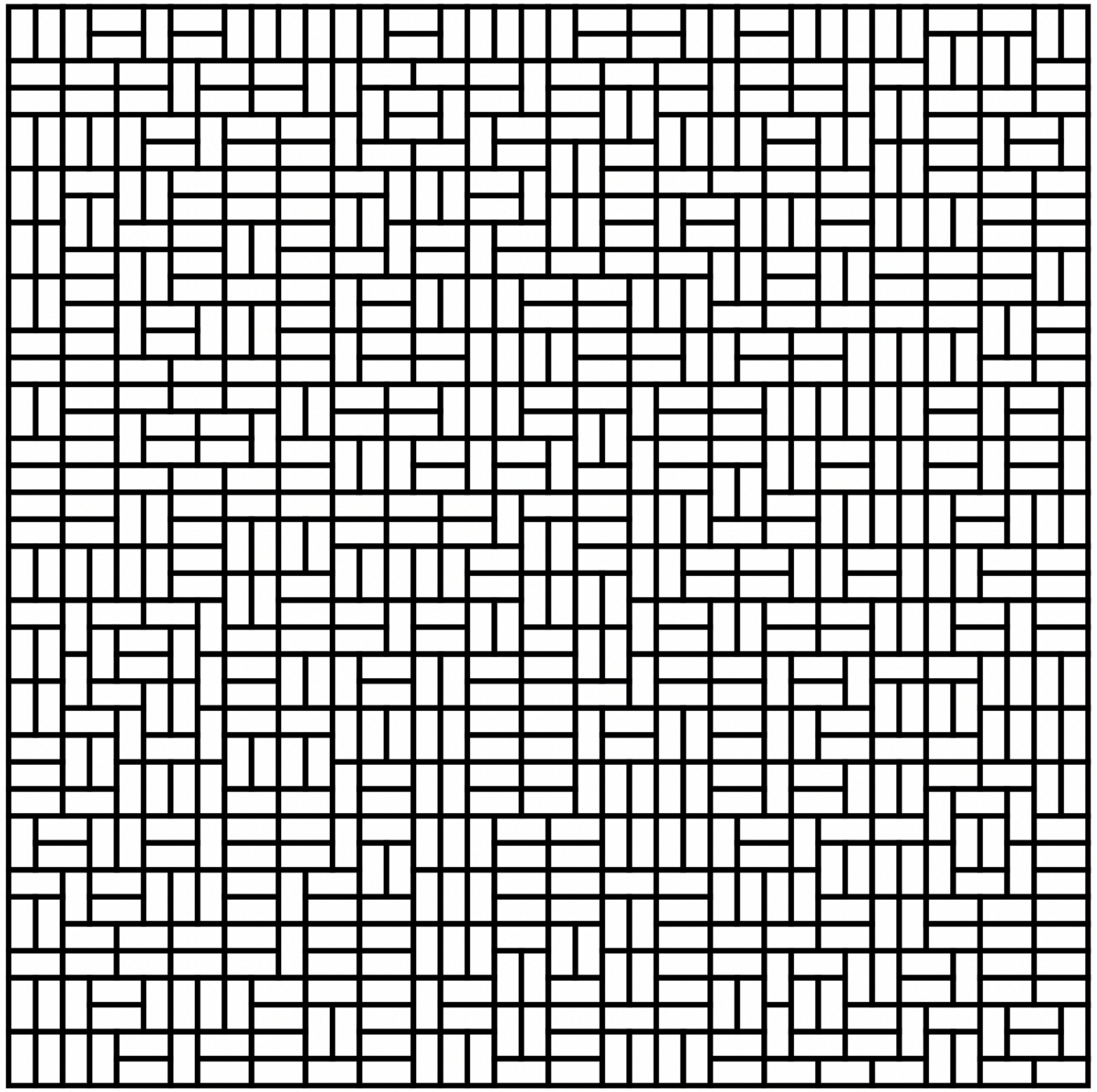


Example of a domino tiling 8 x 8

Kenyon, 2009

Motivation: random dimer coverings in problems of absorption of diatomic molecules by a crystal surface.

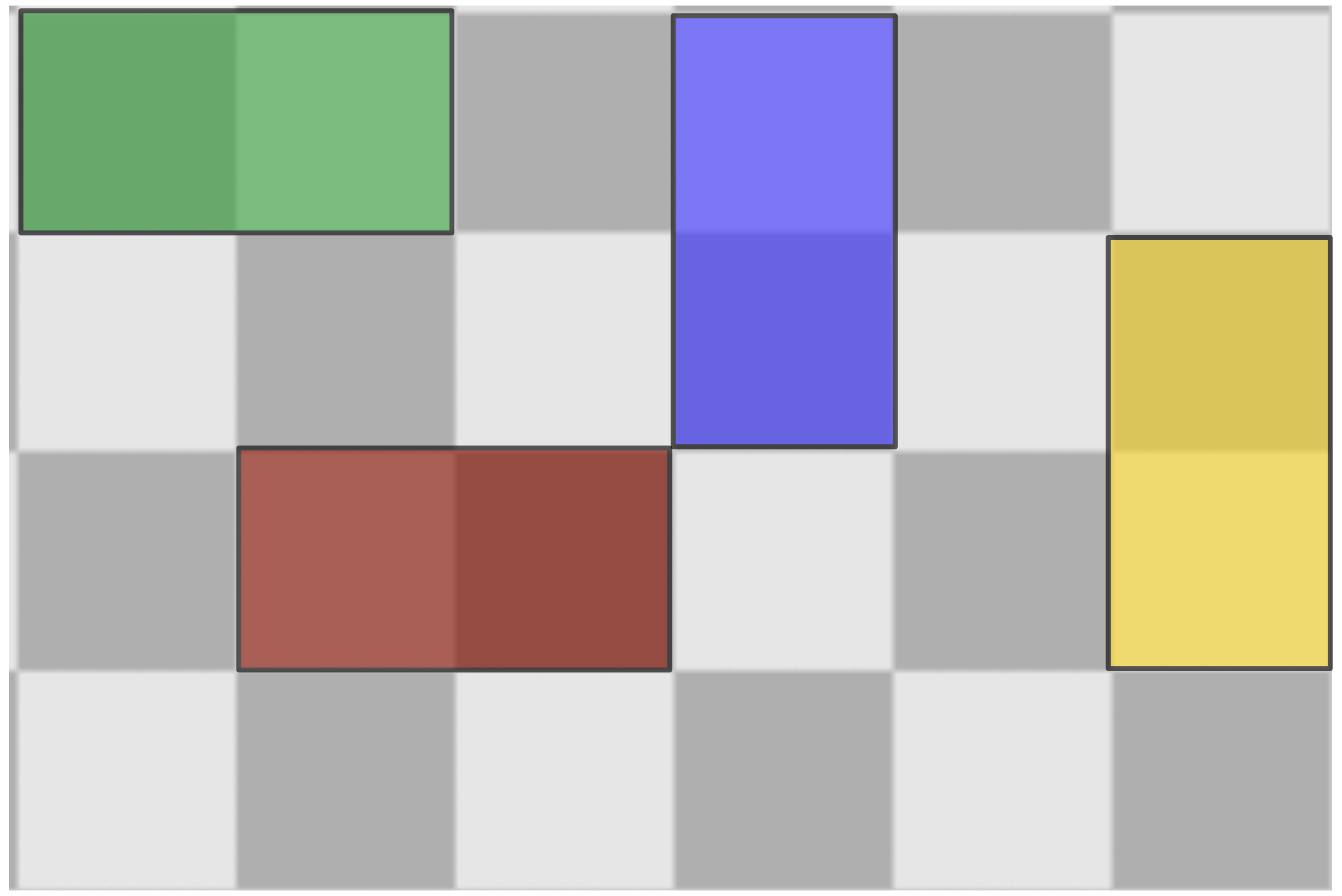
Colored dominos



Random tiling of the board 40 x 40
(totally about 10^{197} tilings)

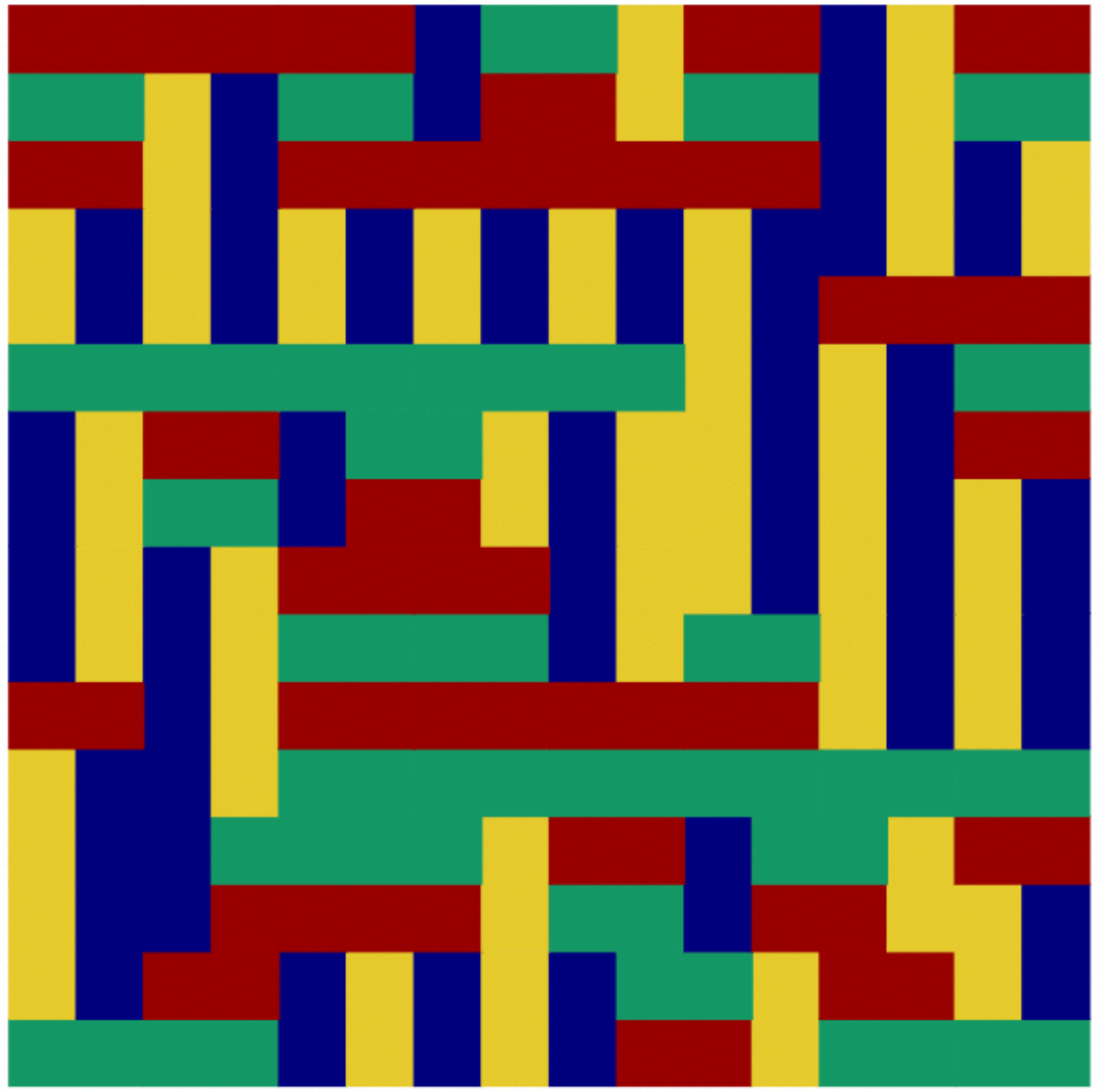
Dominos can be horizontal, vertical, …

Let's color them!

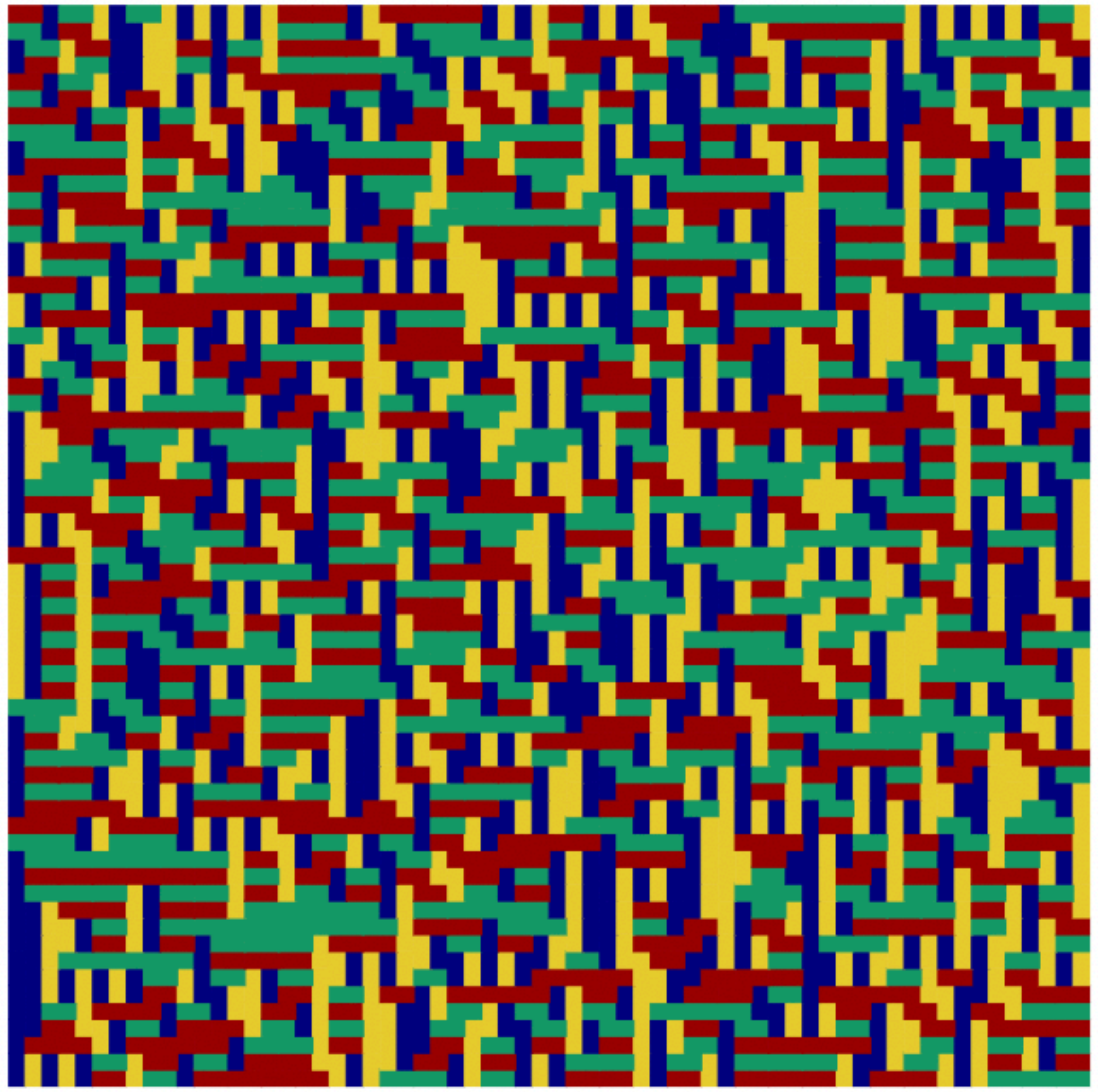


Colored tilings L x L

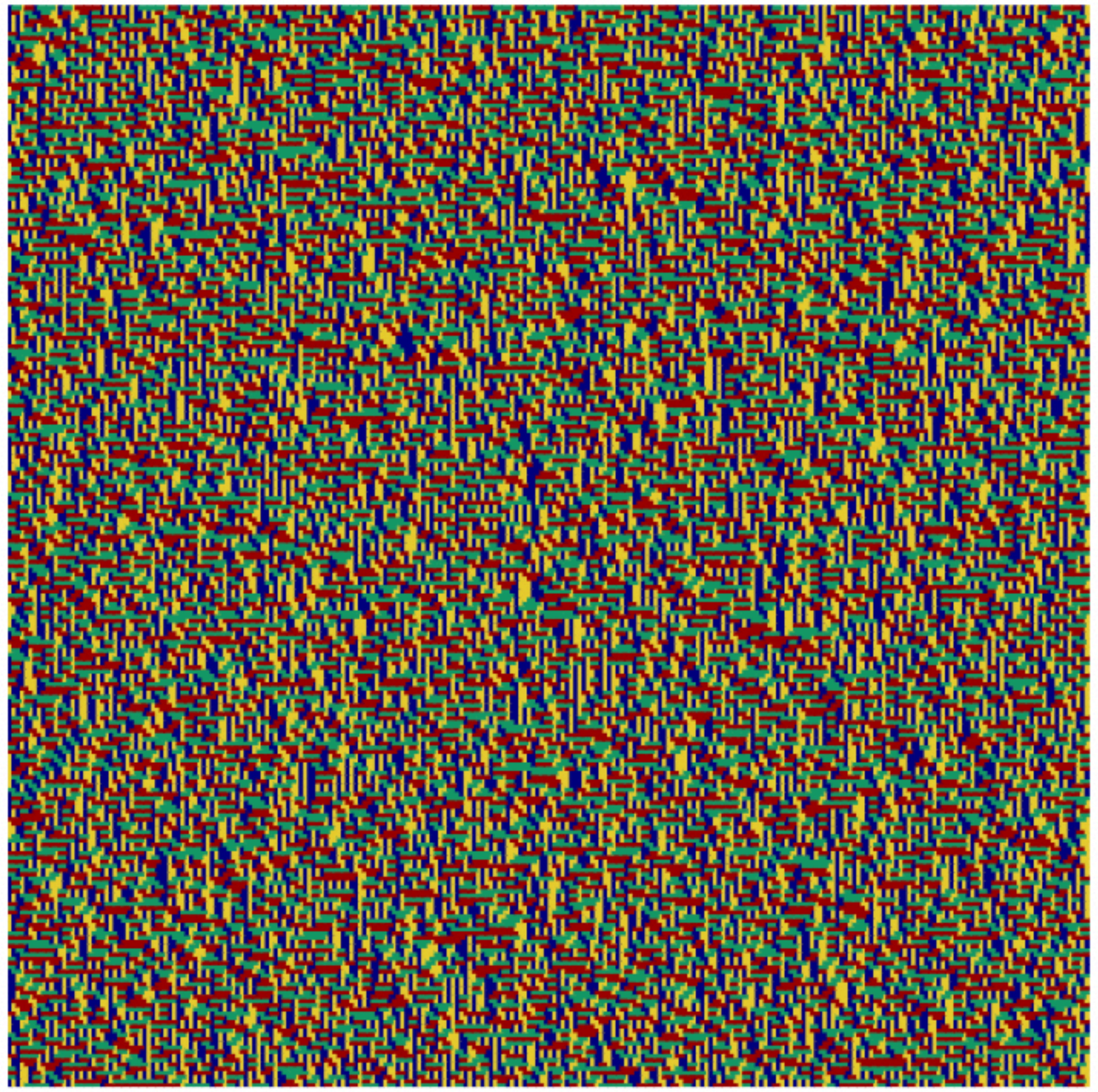
Stéphan, 2020



L=16



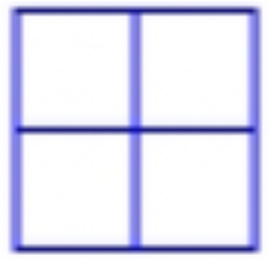
L=64



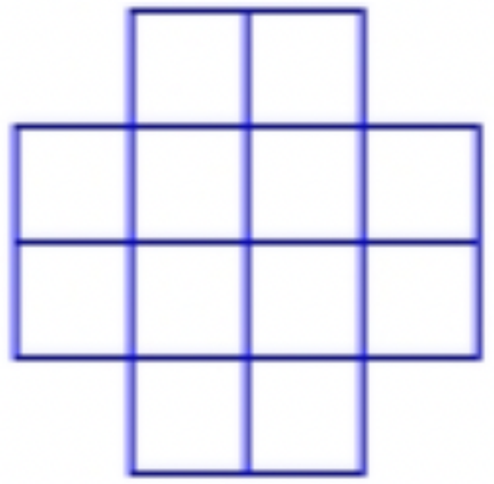
L=256

$L \to \infty$ - Thermodynamic limit

Aztec Diamond



1



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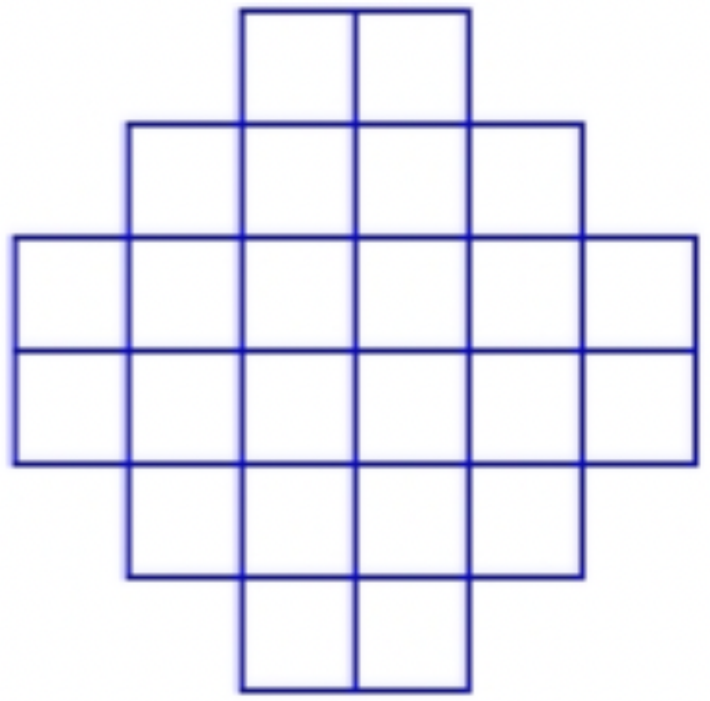

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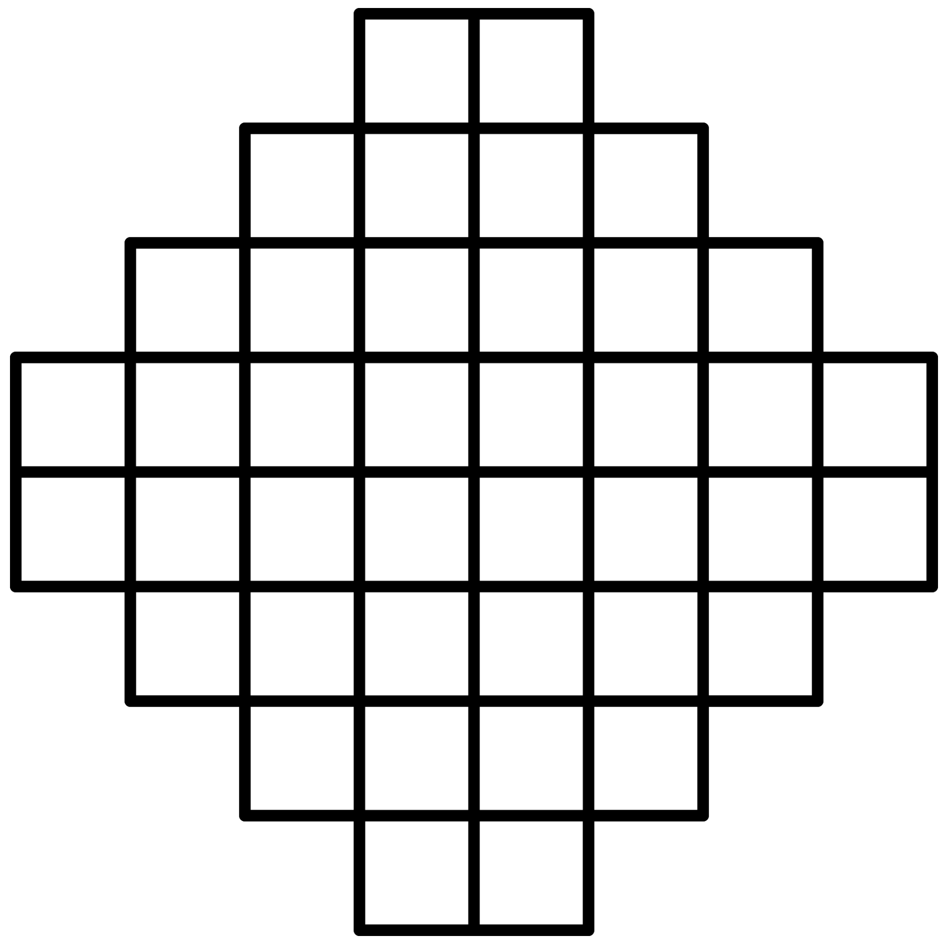

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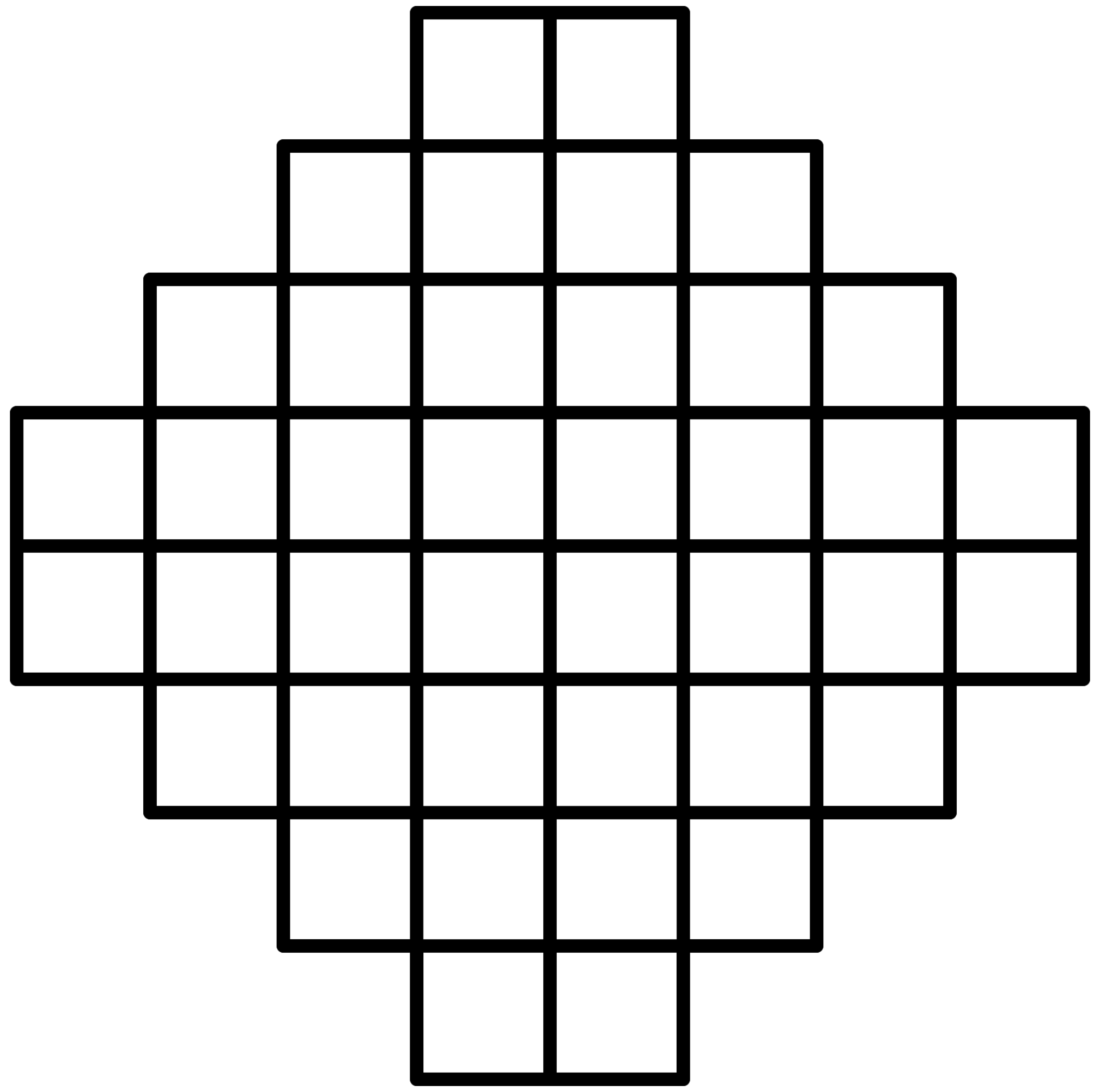
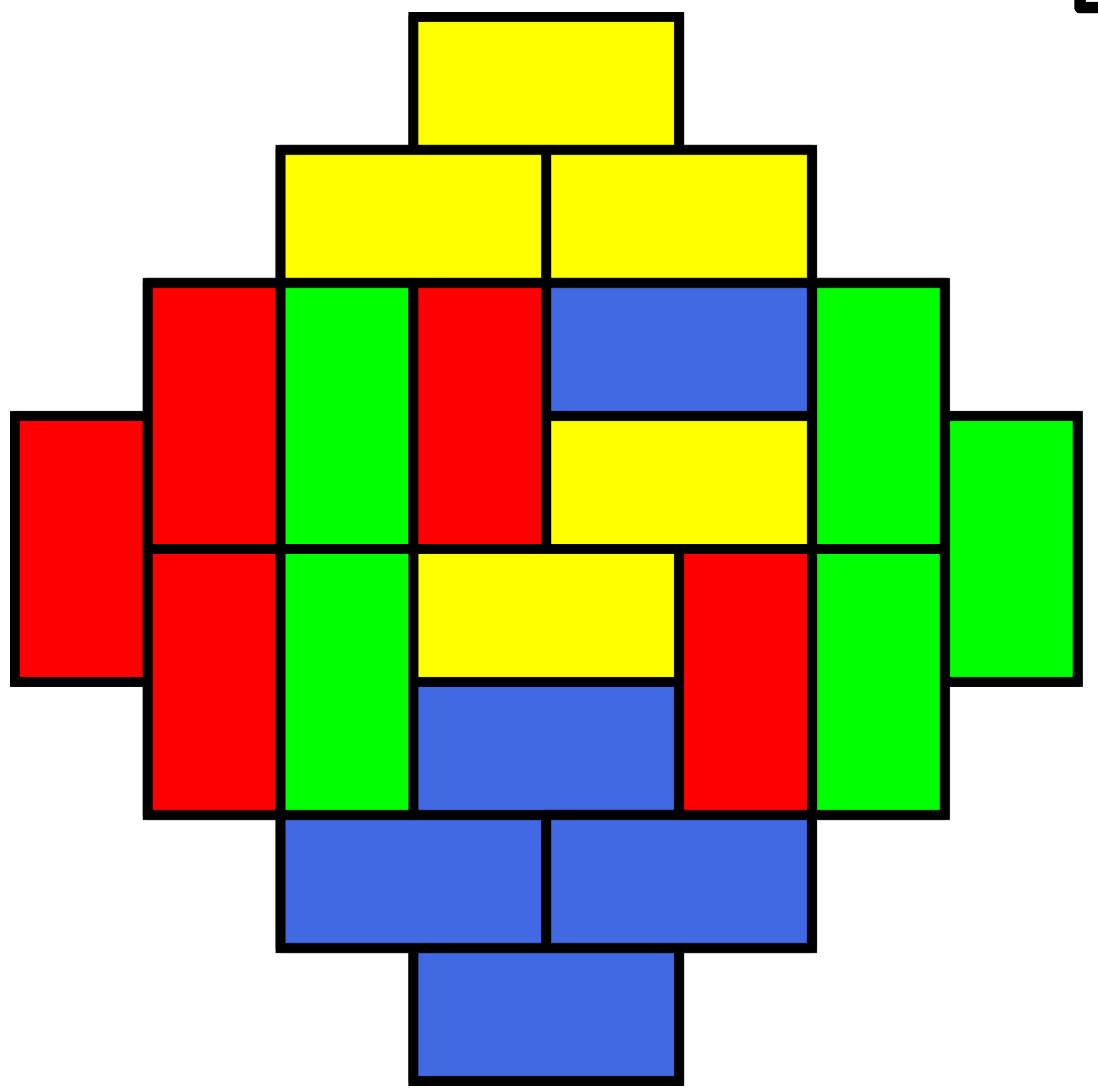


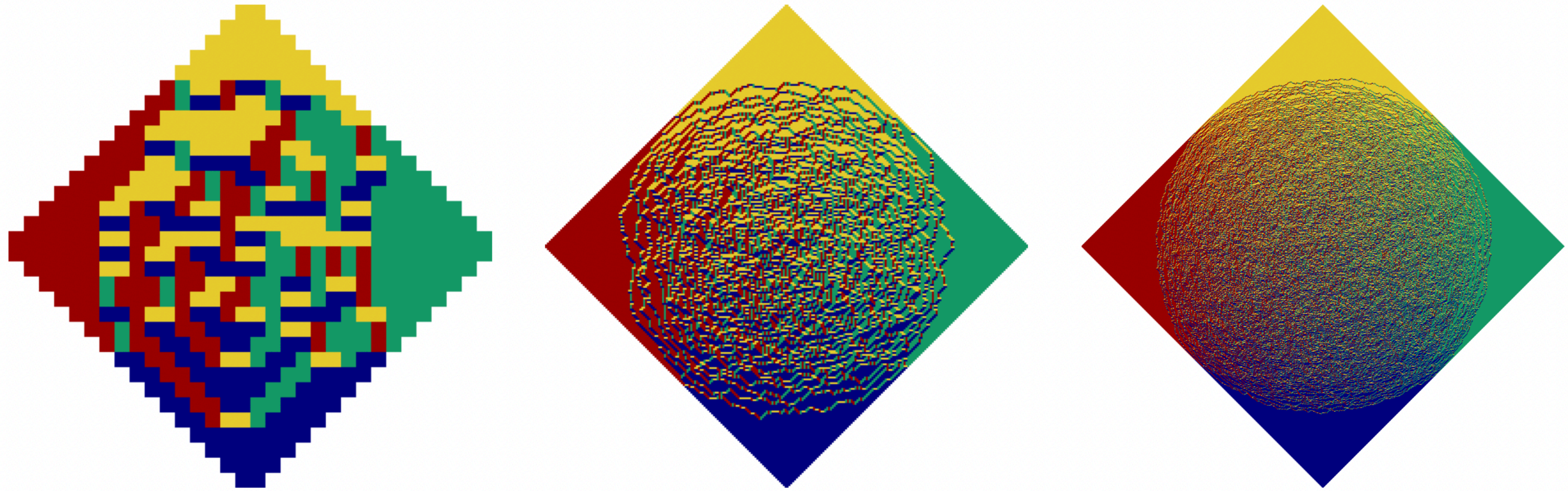
Aztec diamond of order 4



Random tiling of Aztec diamond

Thermodynamic limit

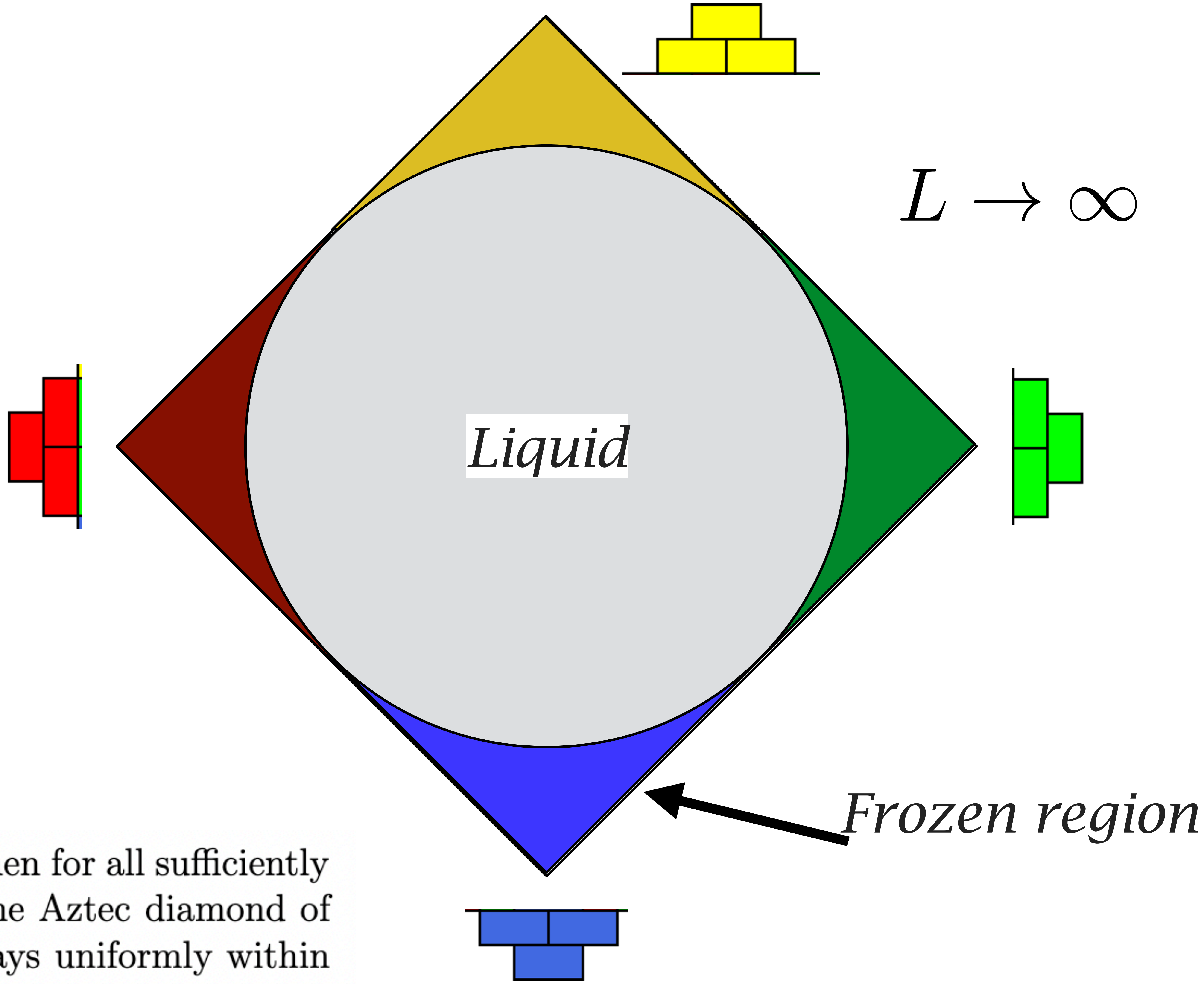
Stéphan, 2020



L=16 L=128 L=1024

Arctic circle theorem

In thermodynamic limit with probability going to 1 the random tiling of Aztec diamond looks like the one shown in picture.

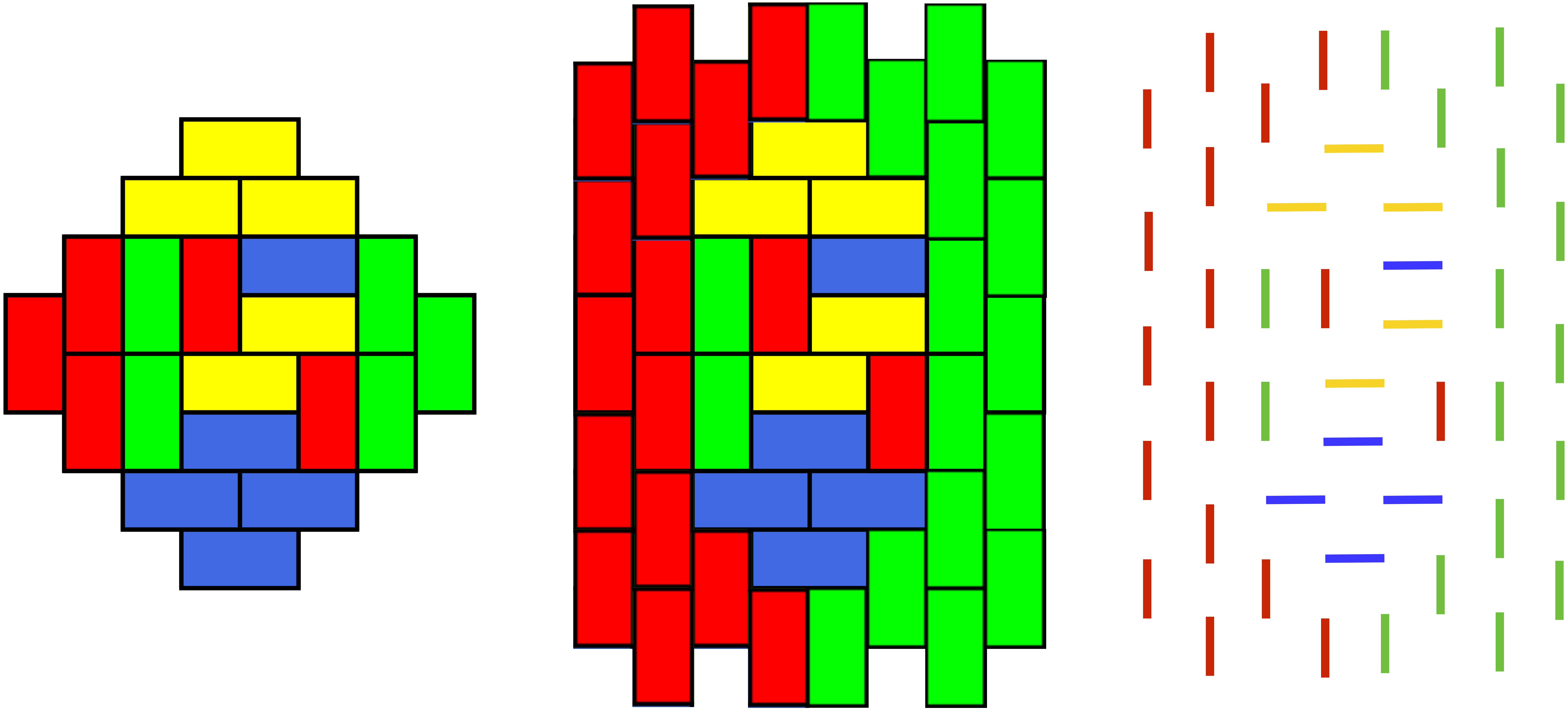


William Jockusch, James Propp, and Peter Shor. "Random domino tilings and the arctic circle theorem." *arXiv preprint math/9801068* (1998).

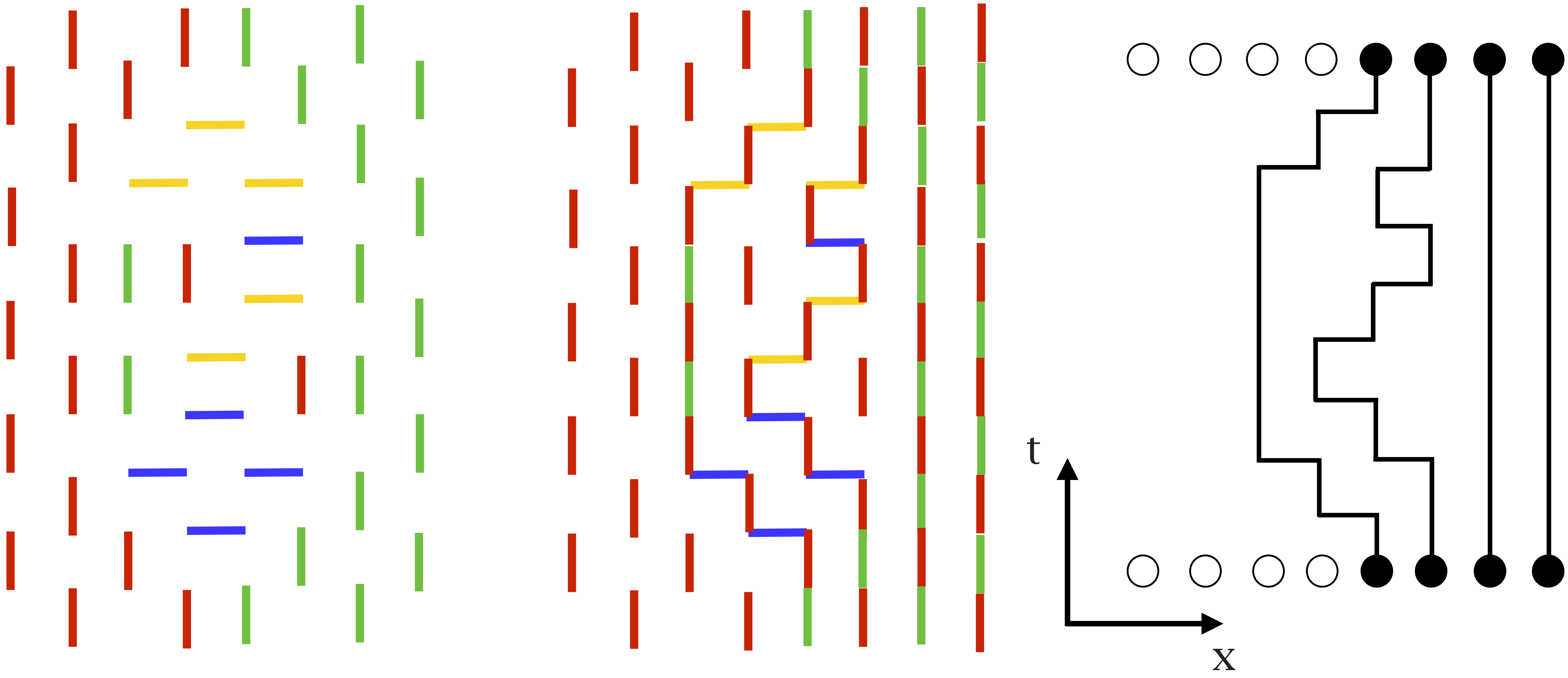
Theorem 1 (the Arctic Circle Theorem): Fix $\epsilon > 0$. Then for all sufficiently large n, all but an ϵ fraction of the domino tilings of the Aztec diamond of order n will have a temperate zone whose boundary stays uniformly within distance ϵn of the inscribed circle.

Tilings as flows

From dominos to dimers

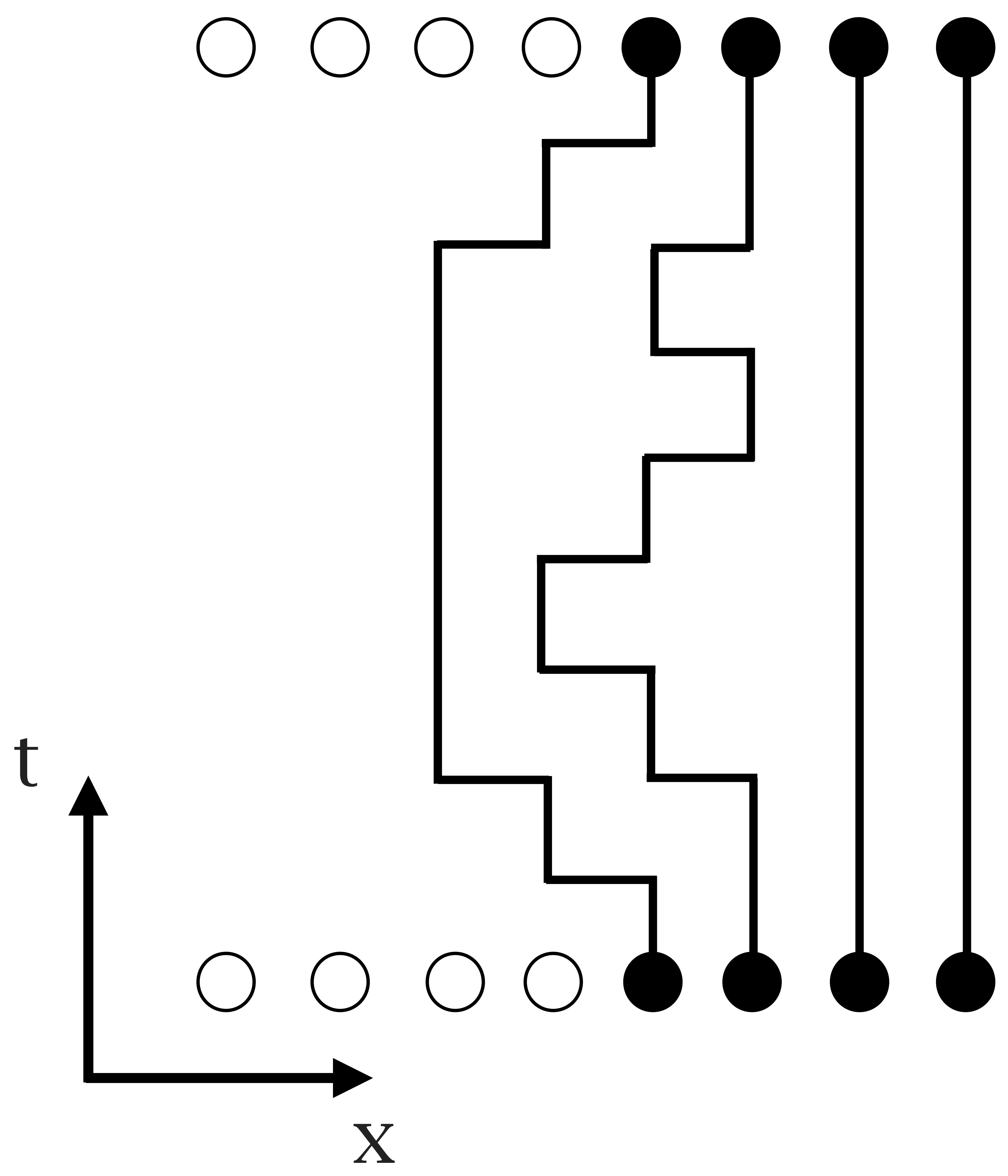


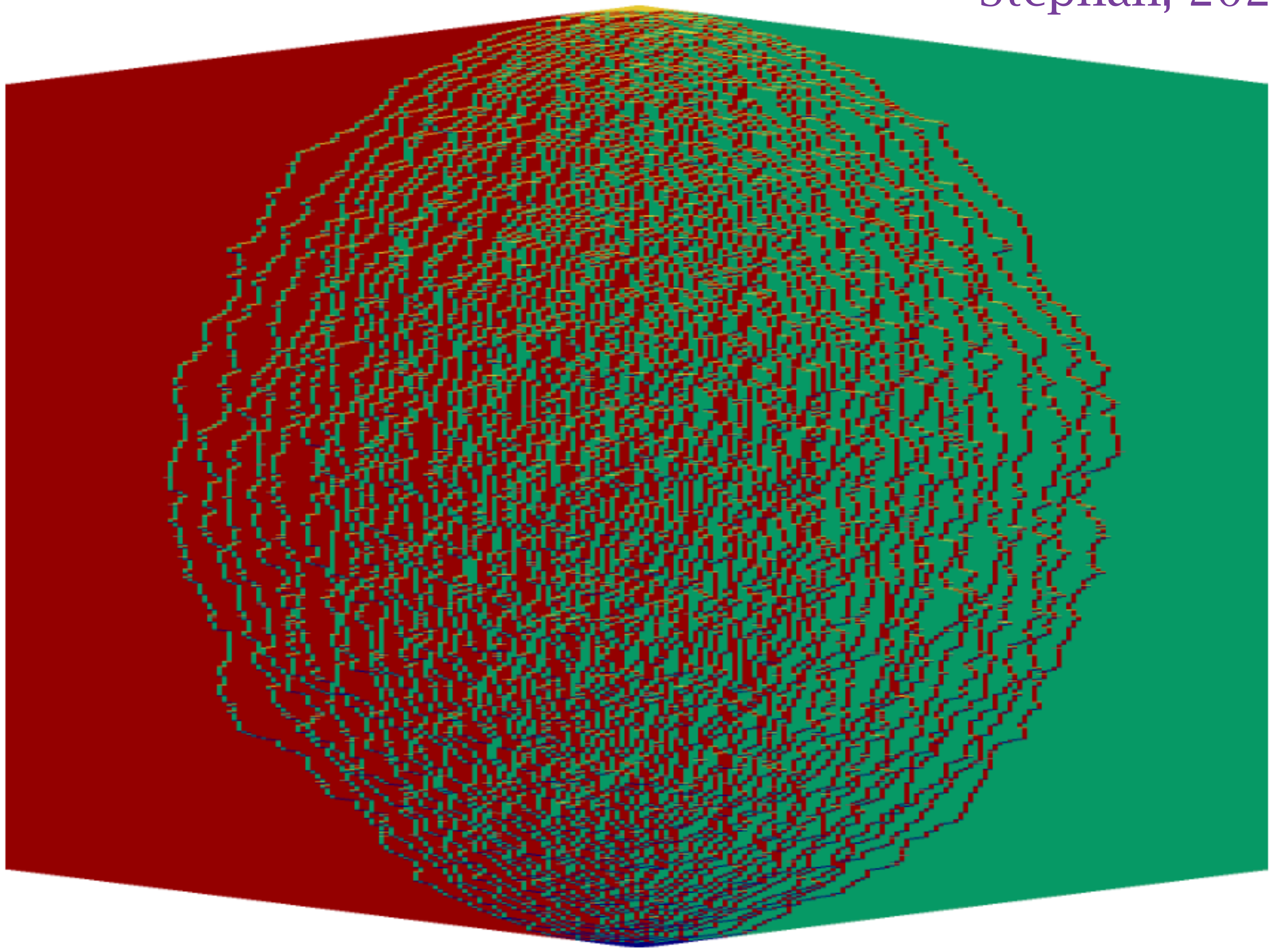
From dimers to worldlines



From lines to gas

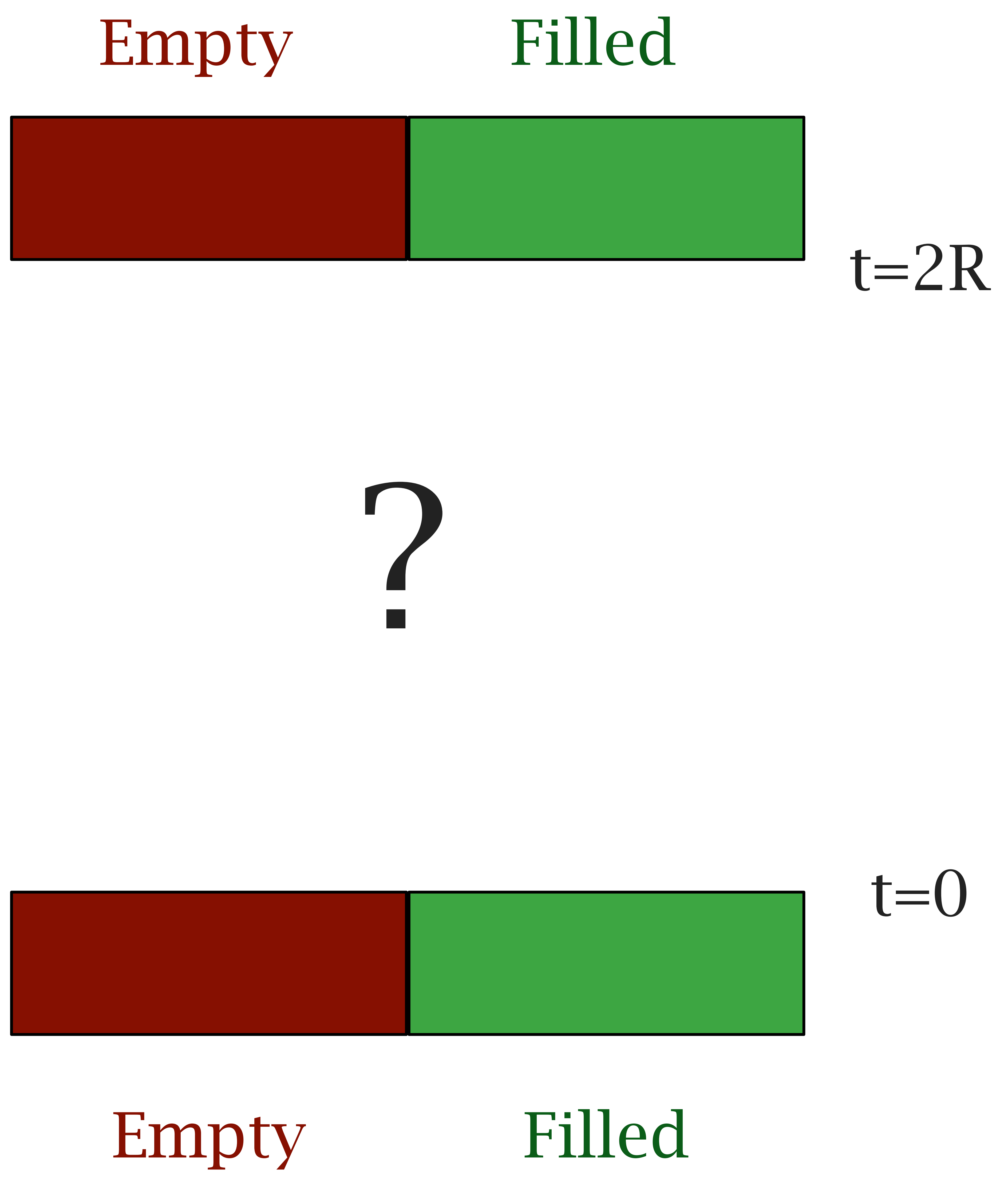
Stéphan, 2020





Green lines - particle trajectories

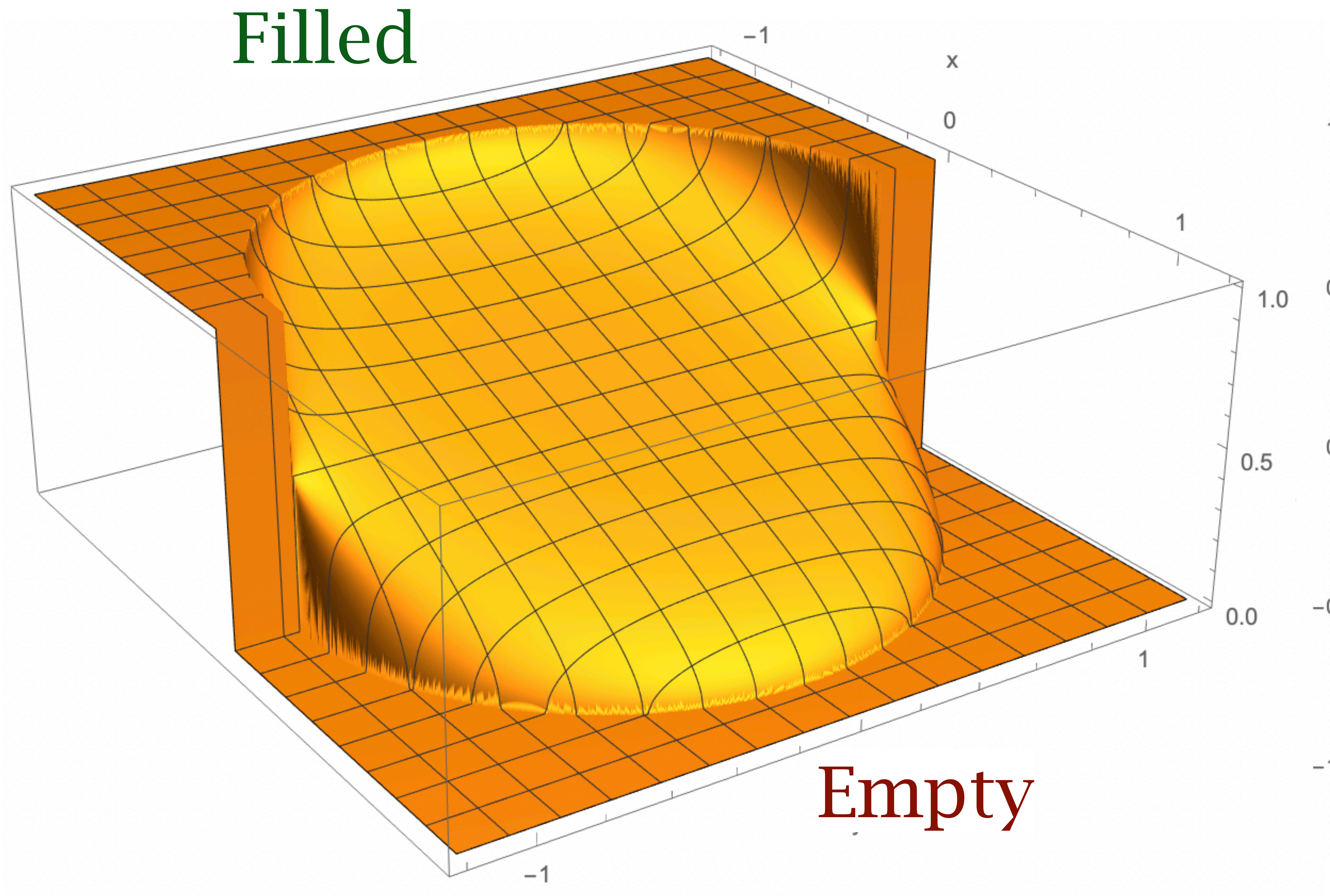
Optimal gas flow

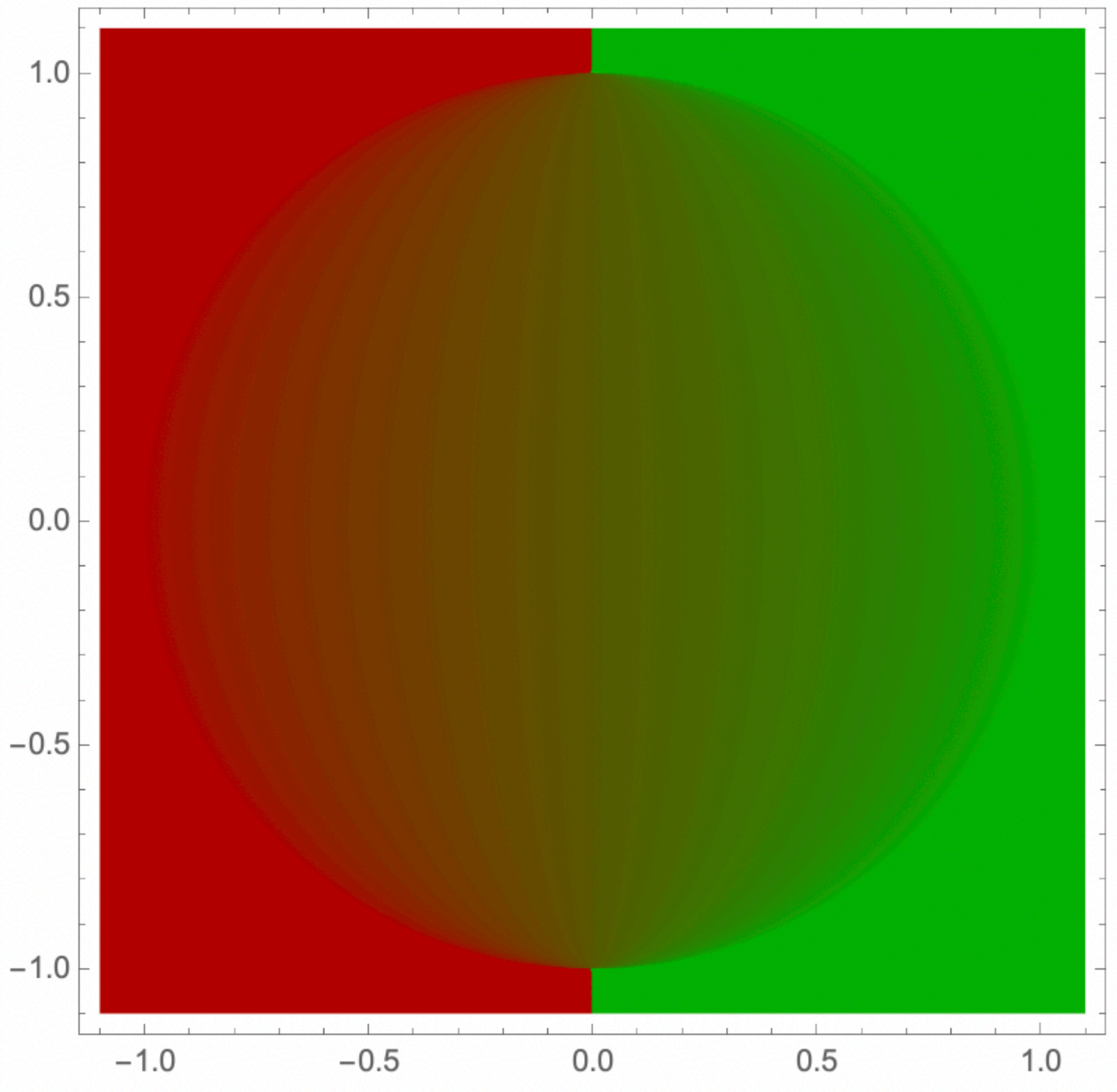


What is the optimal fluctuation of the gas in space and time so that at t=0 and at t=2R the left half line is empty and the right one is filled?

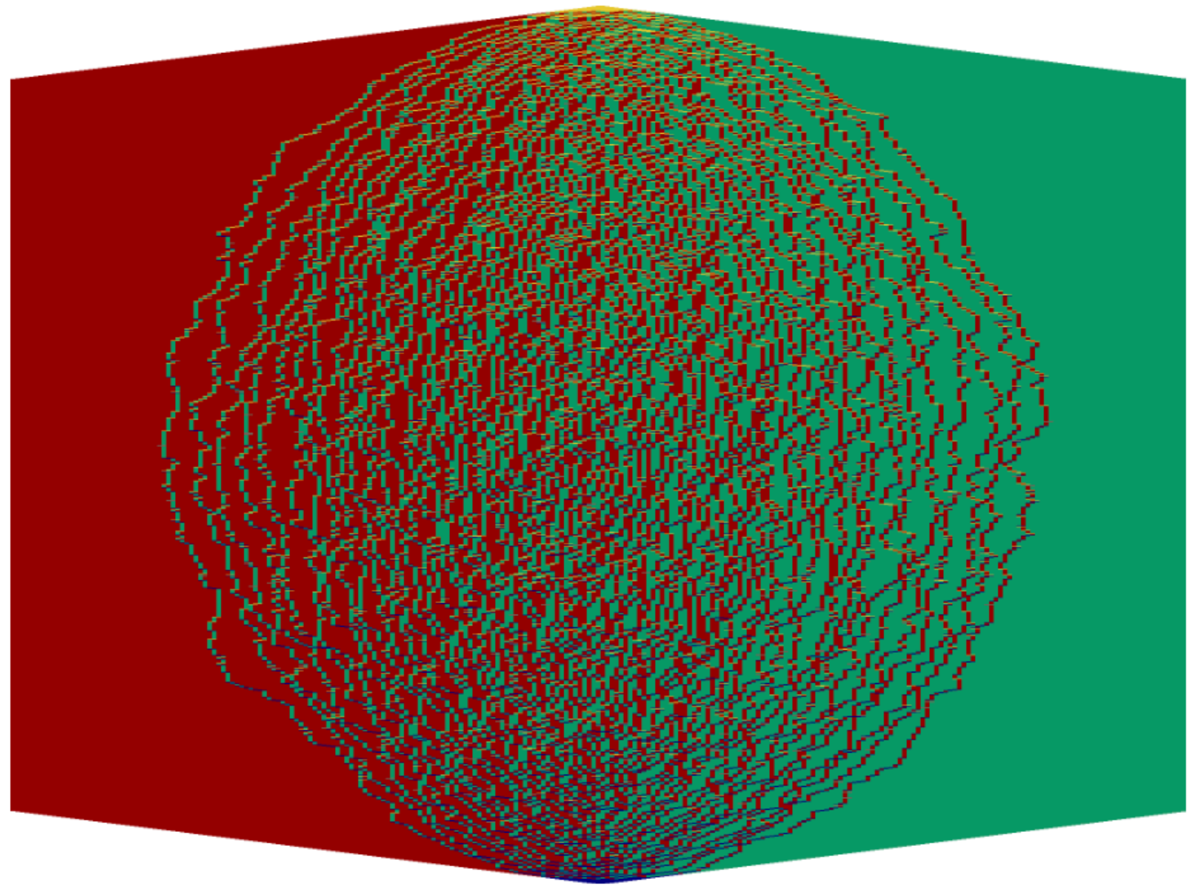
There are many particles (R is big), and one can think of gas as of continuous fluid and use hydrodynamic equations.

The solution for the optimal motion





Empty Filled



The density of the gas
inside the Arctic circle

Arctic Circle hydro solution

$$ik_\tau + \epsilon'(k)k_x = 0$$

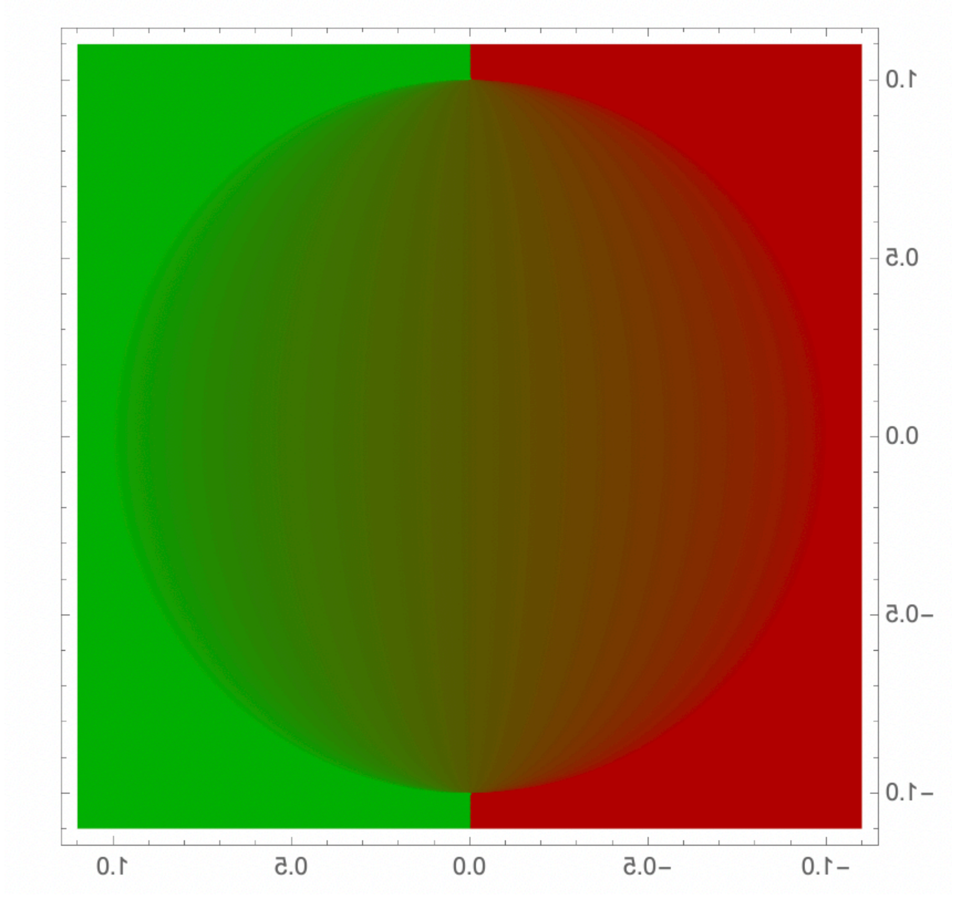
Free fermion hydro

$$z \equiv x + i\epsilon'(k)\tau = F(k)$$

General solution

$$\epsilon(k) = -\cos k$$

Dispersion



$$F(k) = R\cos k$$

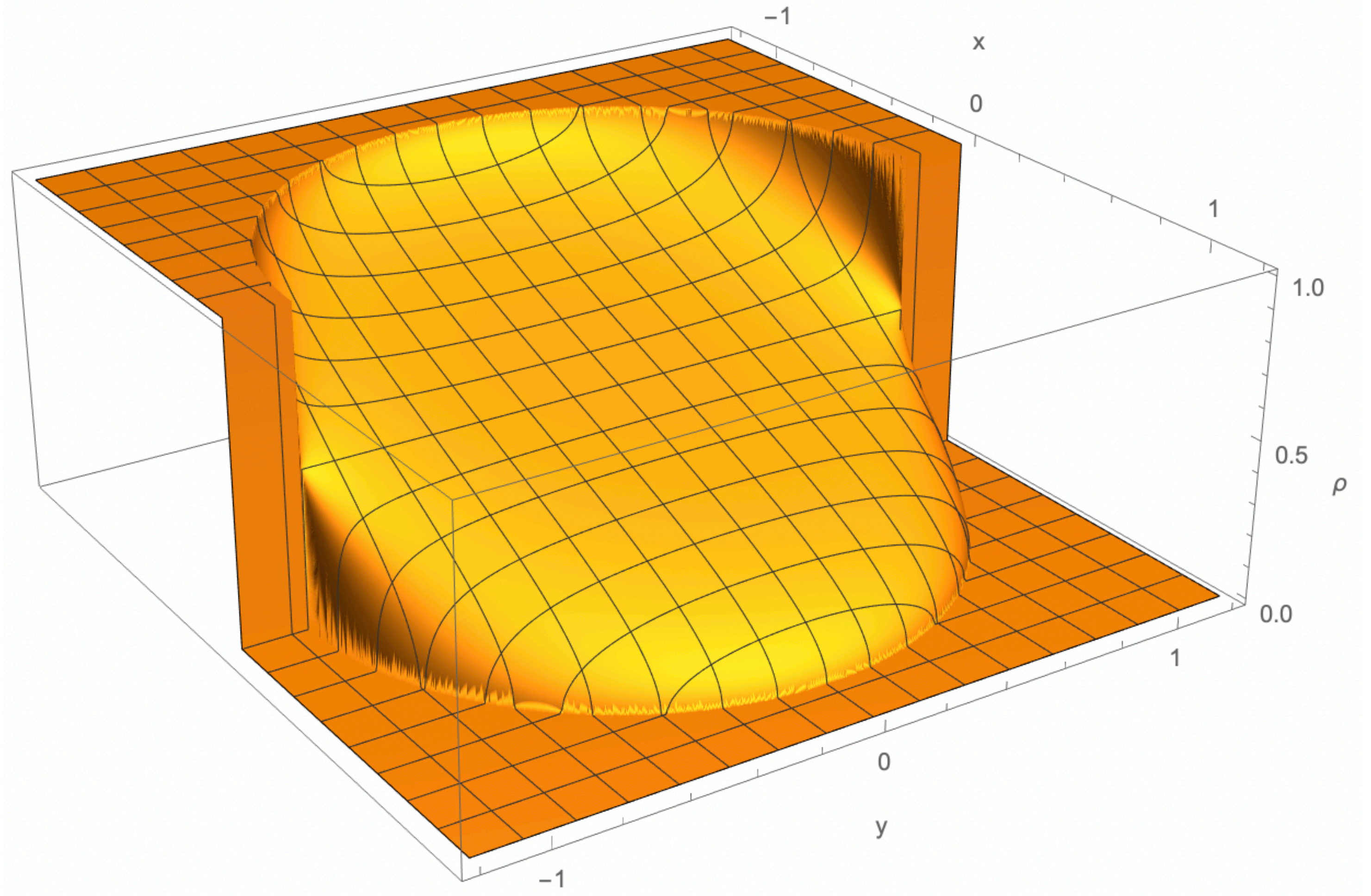
Arctic Circle solution

$$\boxed{\begin{aligned} x + i\tau\sin k &= R\cos k \\ k &= \pi\rho + iv \end{aligned}}$$

$$\tau = -R$$

$$x = Re^{i\pi\rho - v}$$

$$\rho = \theta(-x)$$



see: Allegra, Dubail, Stéphan, Viti, 2016 for general dispersion

Limit shape phase transitions Merging Arctic Circles

with James Pallister and Dimitri Gangardt

based on: https://arxiv.org/abs/2203.05269

Gross-Witten-Wadia model

Gross, Witten, 1980
Wadia, 1980

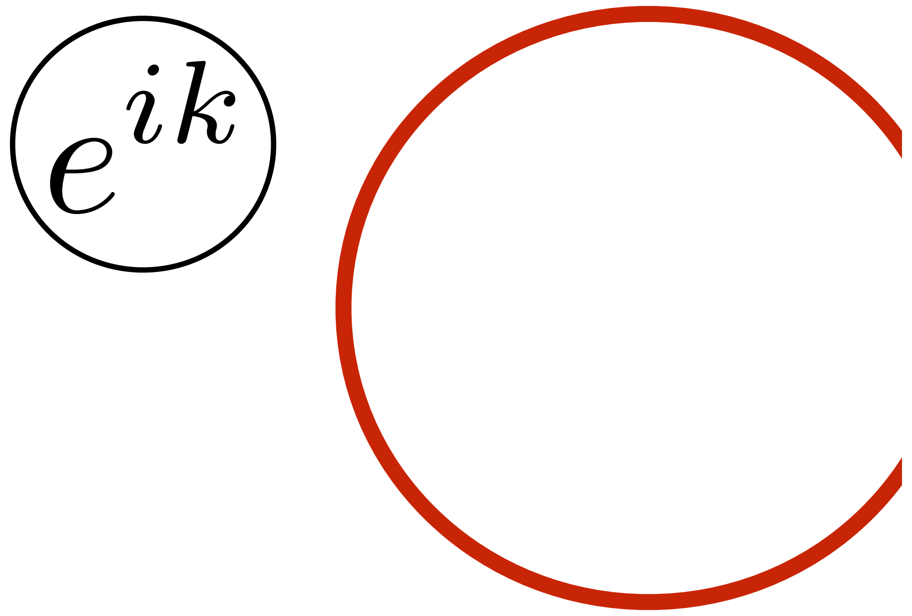
Partition function for 2d U(N) lattice gauge theory was reduced to:

$$Z_N = \int dU \; \exp\left\{\frac{1}{\lambda}\mathrm{Tr}\left(U + U^\dagger\right)\right\}$$

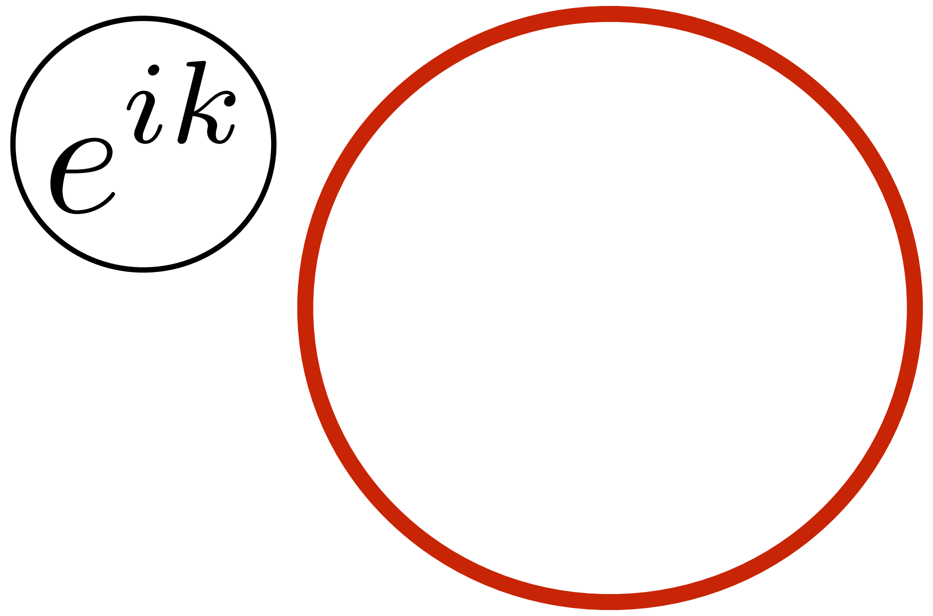
$$U = V\,\mathrm{diag}\left\{e^{ik_j}\right\}V^\dagger$$

$$Z_N = \prod_{i=1}^{N}\int_{-\pi}^{\pi} dk_i \prod_{i<j}\sin^2\left(\frac{k_i - k_j}{2}\right)\exp\left(\frac{2N}{\lambda}\cos k_i\right)$$

Third-order weak-strong coupling phase transition at the t'Hooft coupling $\lambda = 2$ in $N \to \infty$ limit



weak coupling $\lambda < 2$



strong coupling $\lambda > 2$

Electrostatic interpretation

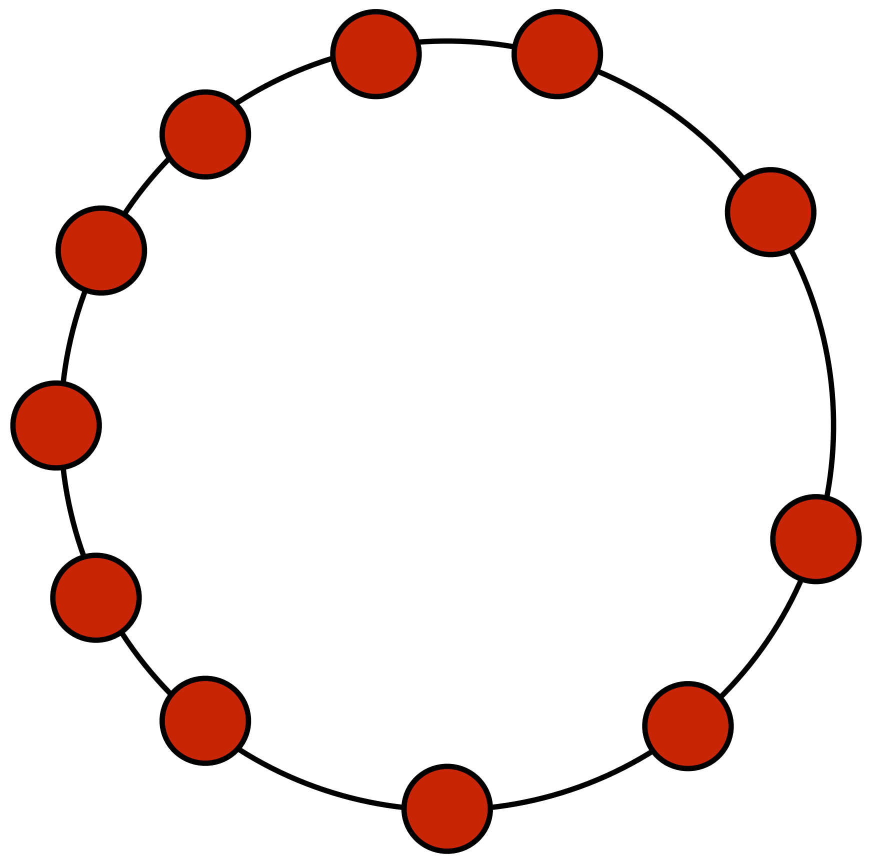
Partition function for charges on unit circle

$$Z_N = \prod_{i=1}^{N} \int_{-\pi}^{+\pi} dk_i \, e^{-E_N}$$

logarithmic repulsion — external potential

$$E_N = -2 \sum_{i<j} \ln \left| e^{ik_i} - e^{ik_j} \right| - \sum_i \frac{2N}{\lambda} \cos k_i$$

charges



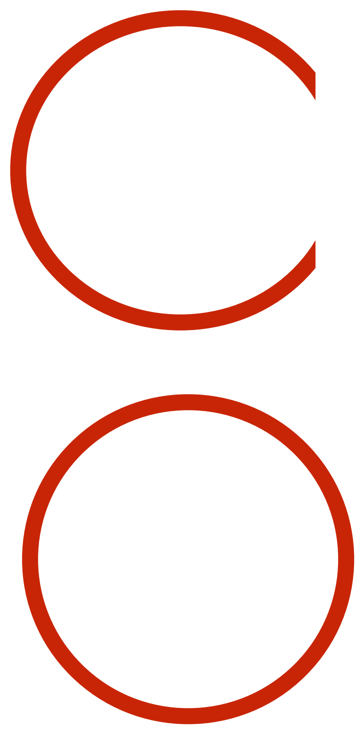
external electric field E

small λ means large E

Large N solution

Density of charges: $$\sigma(k) = \frac{1}{N}\sum_i \delta(k-k_i)$$

$$2\pi\sigma(k) = \begin{cases} \frac{4}{\lambda}\cos\frac{k}{2}\sqrt{\frac{\lambda}{2}-\sin^2\frac{k}{2}} & \lambda \le 2 \\ 1+\frac{2}{\lambda}\cos k & \lambda \ge 2 \end{cases}$$



$$-E(\lambda) = N^2 \begin{cases} \frac{2}{\lambda}+\frac{1}{2}\ln\frac{\lambda}{2}-\frac{3}{4} & \lambda \le 2 \\ \frac{1}{\lambda^2} & \lambda \ge 2 \end{cases}$$

$\frac{\partial^3 E}{\partial \lambda^3}$ is discontinuous at $\lambda = 2$

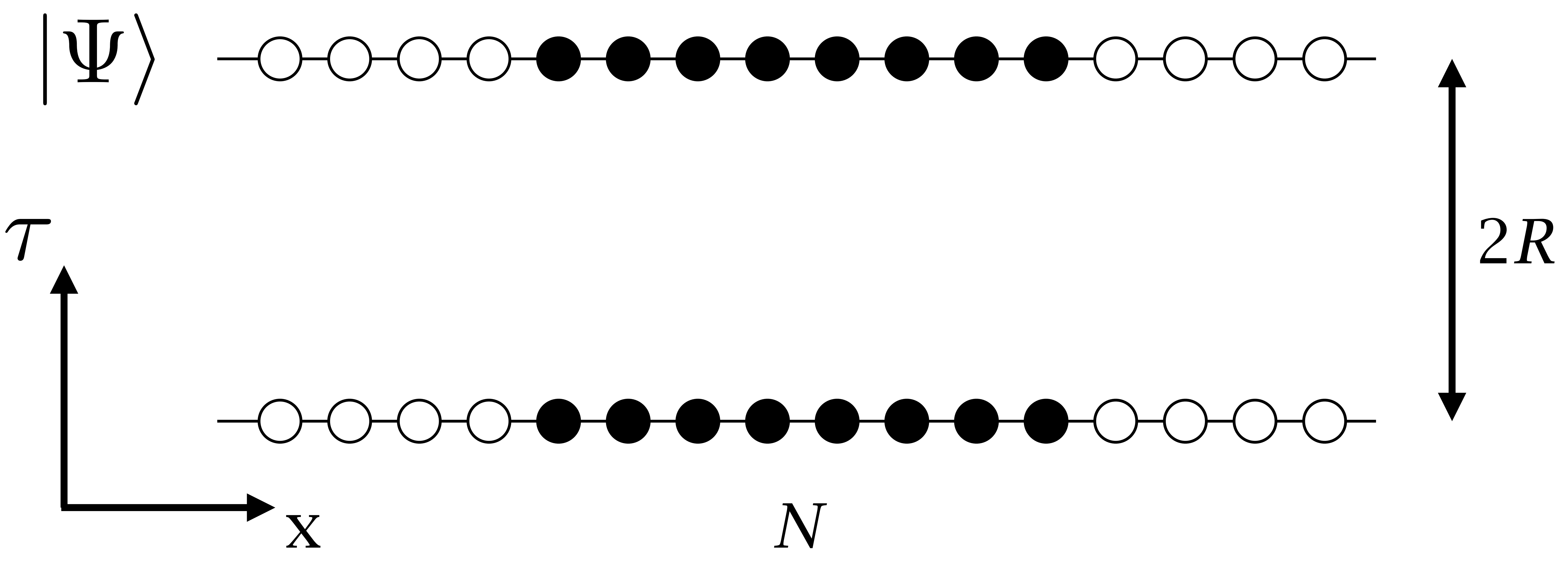
Gross, Witten, 1980
Wadia, 1980

Free fermions

Hamiltonian of free fermions $\quad H = \int_{-\pi}^{\pi} \frac{dk}{2\pi} \varepsilon(k) c^{\dagger}(k) c(k) \quad$ Diagonal in k-space

$$Z_N = \langle \Psi | e^{-2RH} | \Psi \rangle$$

$|\Psi\rangle$ Diagonal in x-space



$N,\ R \to \infty$

$\lambda = \frac{N}{R}$ - fixed

Partition function

$$Z_N(R) = \langle N | e^{-2RH} | N \rangle$$

$$|N\rangle = c_1^\dagger \dots c_N^\dagger |0\rangle$$

$$Z_N(R) = \frac{1}{N!} \int_{-\pi}^{\pi} \prod_{i=1}^{N} \frac{dk_i}{2\pi} \, \langle N | \{k_i\} \rangle e^{-2R \sum_l \varepsilon(k_l)} \langle \{k_i\} | N \rangle$$

$$Z_N = \frac{1}{N!} \int \frac{d^N k}{(2\pi)^N} \, |\Delta(e^{ik})|^2 \, e^{-2R \sum_l \varepsilon(k_l)}$$

$$\Delta(e^{ik}) = \prod_{i<j} (e^{ik_i} - e^{ik_j})$$ - Vandermonde determinant

Electrostatic interpretation

for the dispersion $\varepsilon(k) = -\cos k$ and $\lambda = \frac{N}{R}$

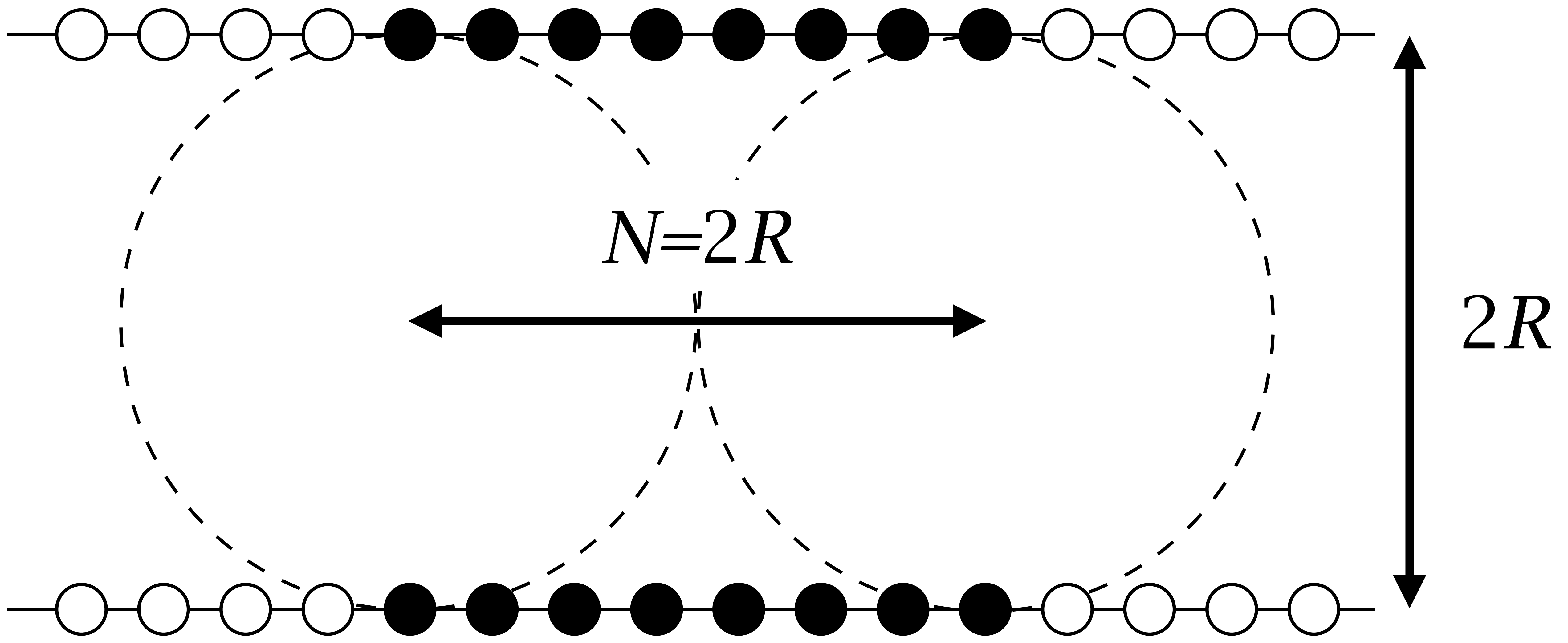
$$Z_N = \prod_{i=1}^{N} \int_{-\pi}^{+\pi} dk_i \, e^{-E_N}$$

$$E_N = -2 \sum_{i<j} \ln \left| e^{ik_i} - e^{ik_j} \right| - \sum_i \frac{2N}{\lambda} \cos k_i$$

$\lambda = 2$ - phase transition (as in Gross-Witten-Wadia model)

What is the picture of this transition in spacetime?

Spacetime picture



$$N,\ R \to \infty$$

$$\lambda = \frac{N}{R}$$ - fixed

Arctic circles touch at $\lambda = 2$

From k-space to spacetime

Density of charges

$$\sigma(k) = \frac{1}{N}\sum_i \delta(k - k_i)$$

$$2\pi\sigma(k) = \begin{cases} \frac{4}{\lambda}\cos\frac{k}{2}\sqrt{\frac{\lambda}{2} - \sin^2\frac{k}{2}} & \lambda \leq 2 \\ 1 + \frac{2}{\lambda}\cos k & \lambda \geq 2 \end{cases}$$

$$F(\pi\lambda\sigma(k), k) = 0$$

$$F(z,k) = \left(z - \frac{\lambda}{2} - \cos k\right)\left(z + \frac{\lambda}{2} + \cos k\right) + \left(1 - \frac{\lambda}{2}\right)^2 \Theta(2 - \lambda)$$

complex curve encodes large N solution!

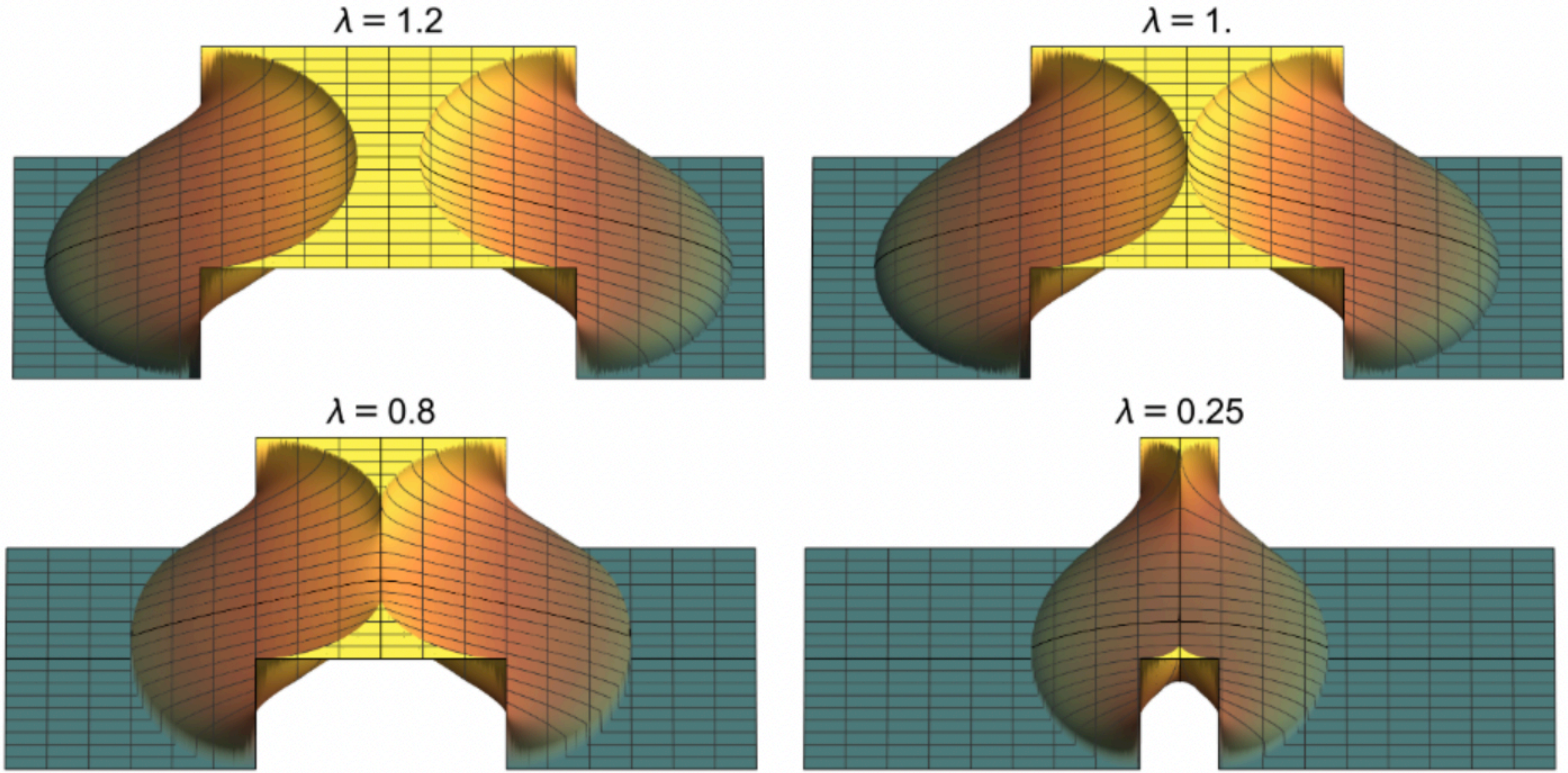
$$F(x + it\sin k, k) = 0 \longrightarrow k(x,t)$$

- complex function of x, t

$$k(x,t) = \pi\rho(x,t) + iv(x,t)$$

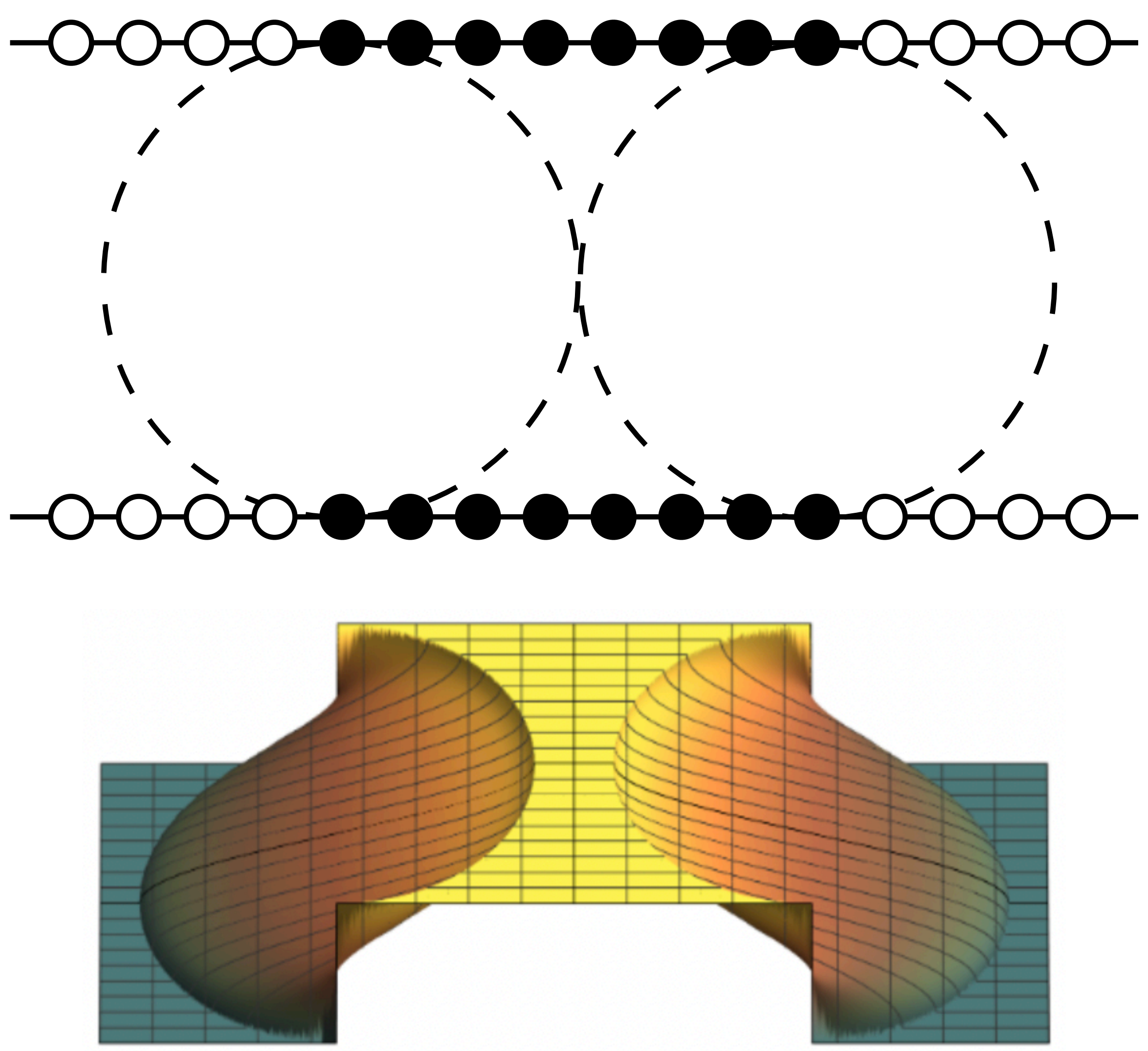
gives density and velocity of fluid for optimal fluctuation

Density profiles



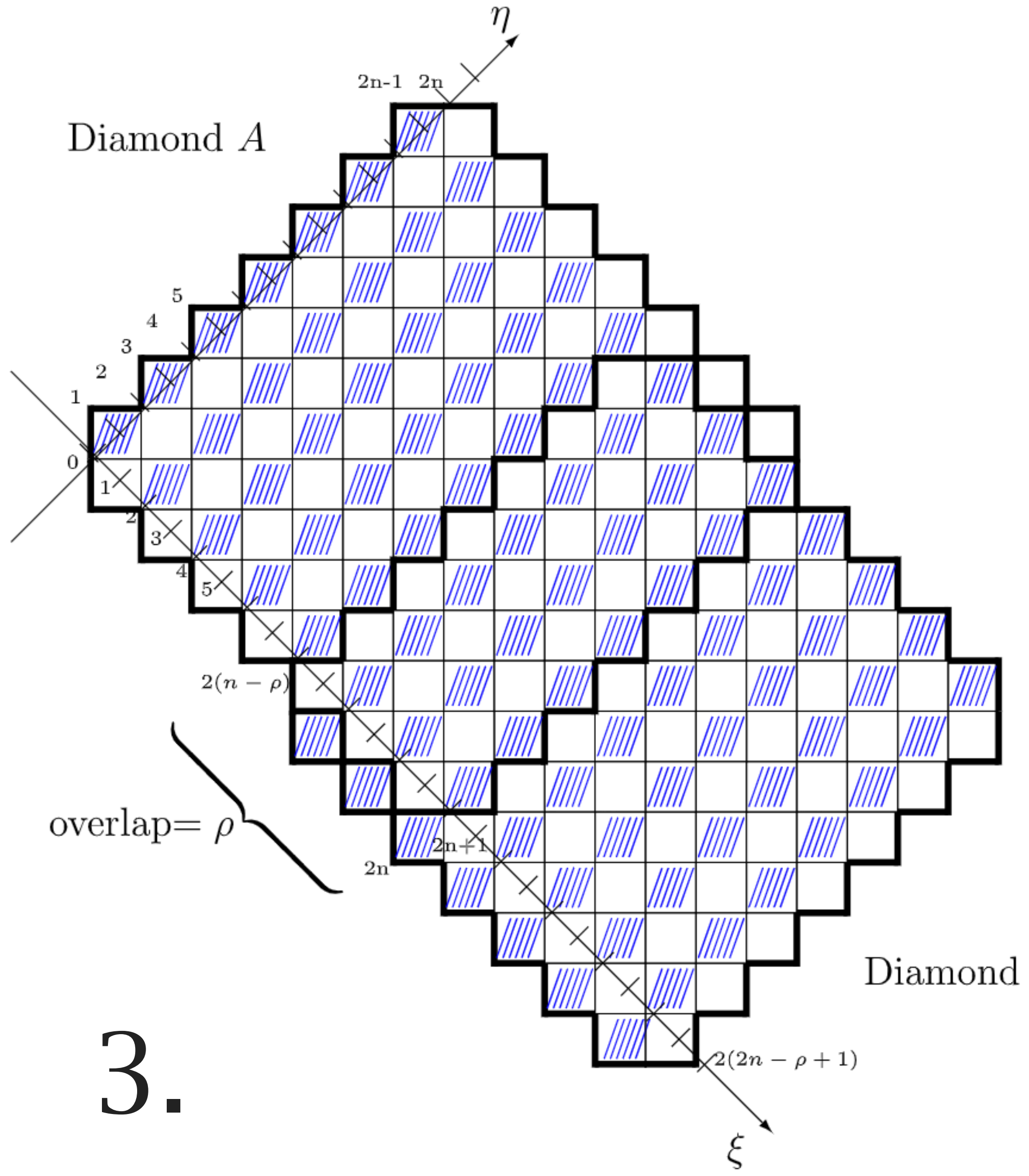
Other equivalent systems

1.



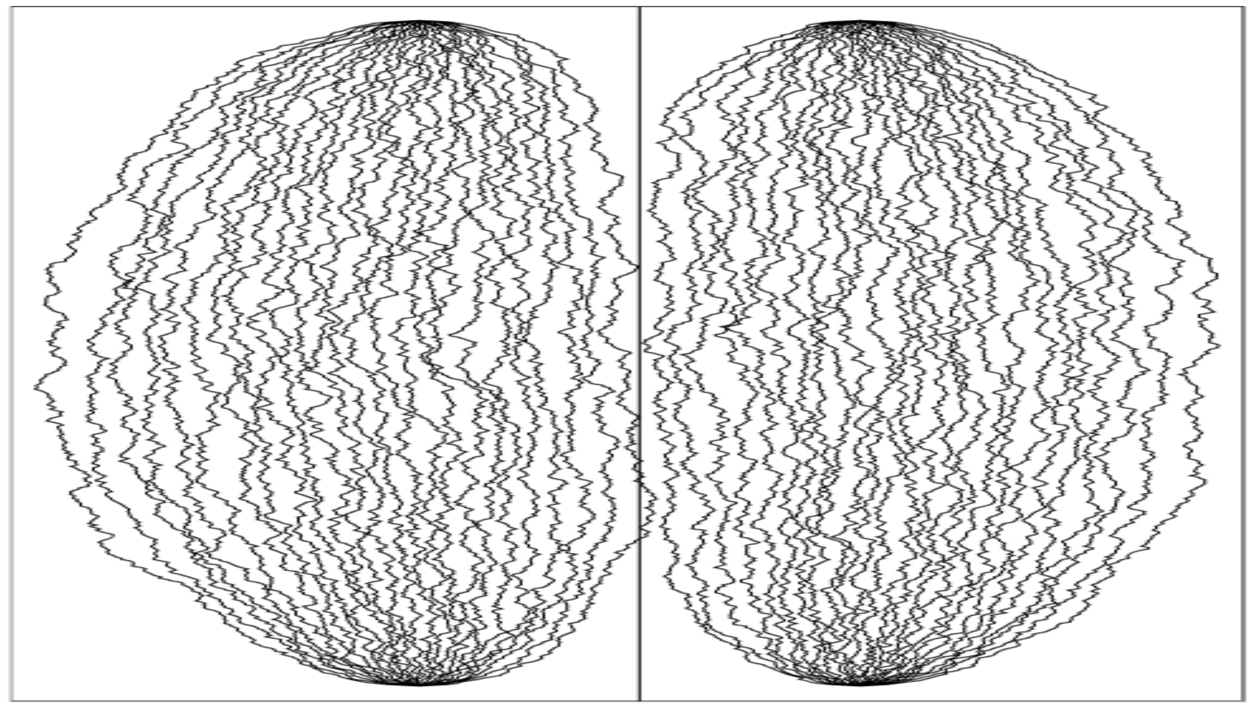
1. Free fermion (+hydro)
2. Nonintersecting random walkers
3. Domino tilings
4. Six-vertex models with domain wall bc
5. Random Young diagrams

Adler, Johanson, van Moerbeke, 2014



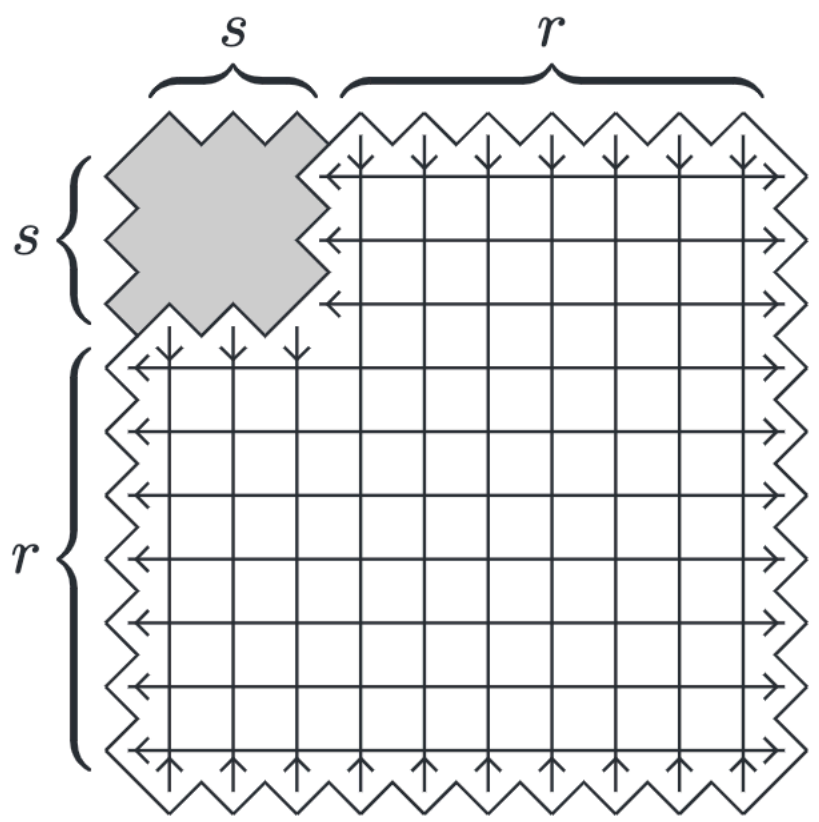
3.

2.

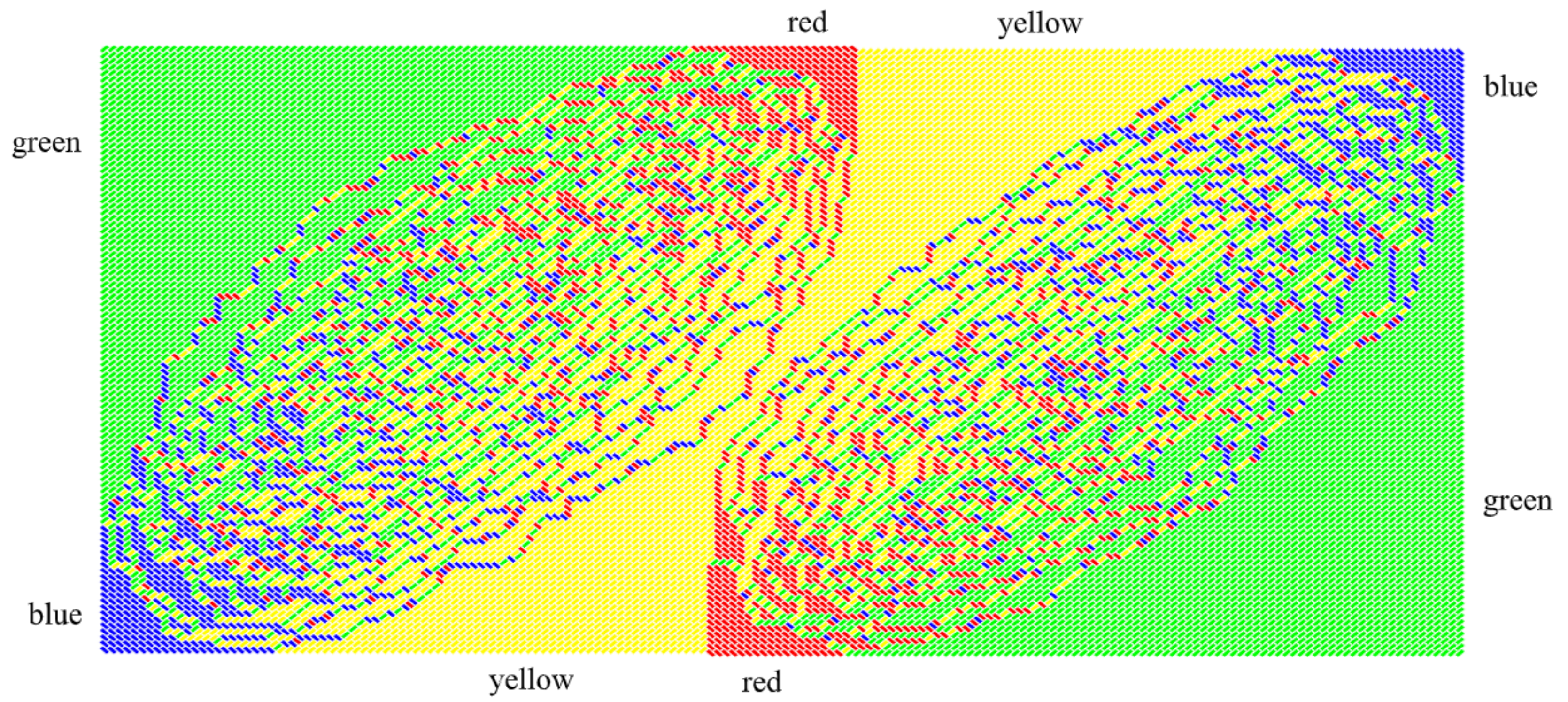


Adler, Ferrari, van Moerbeke, 2013

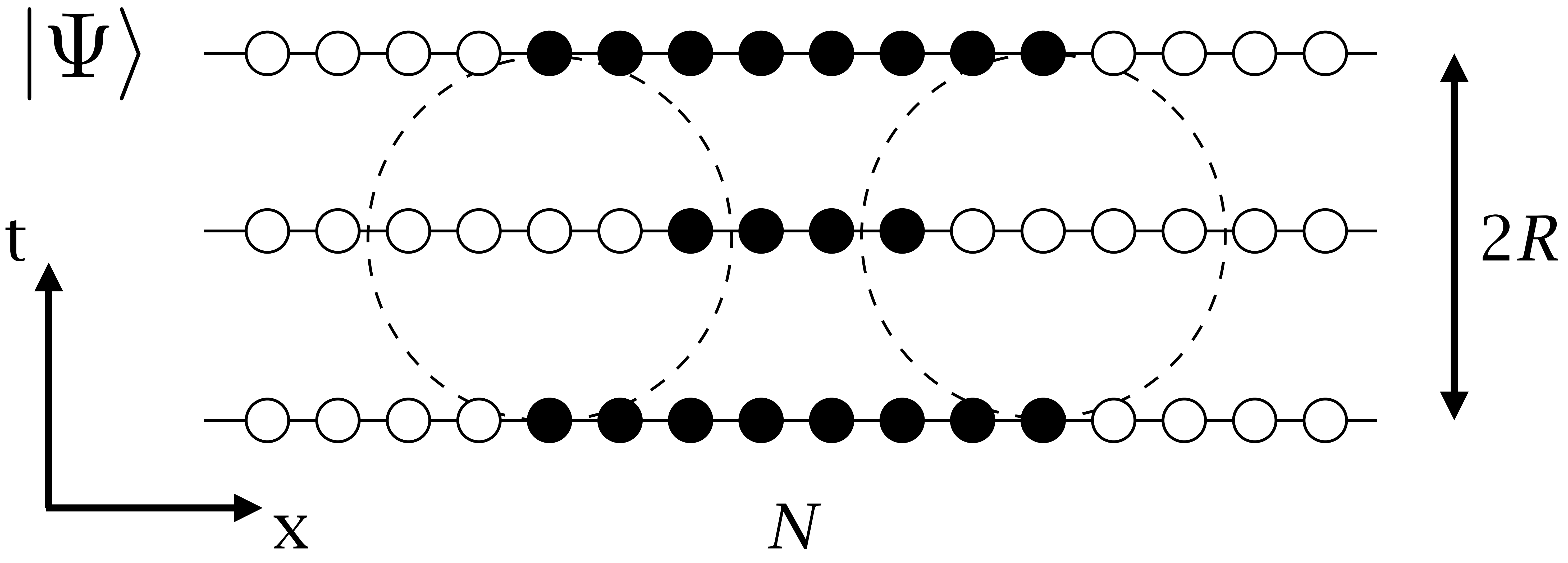
4.



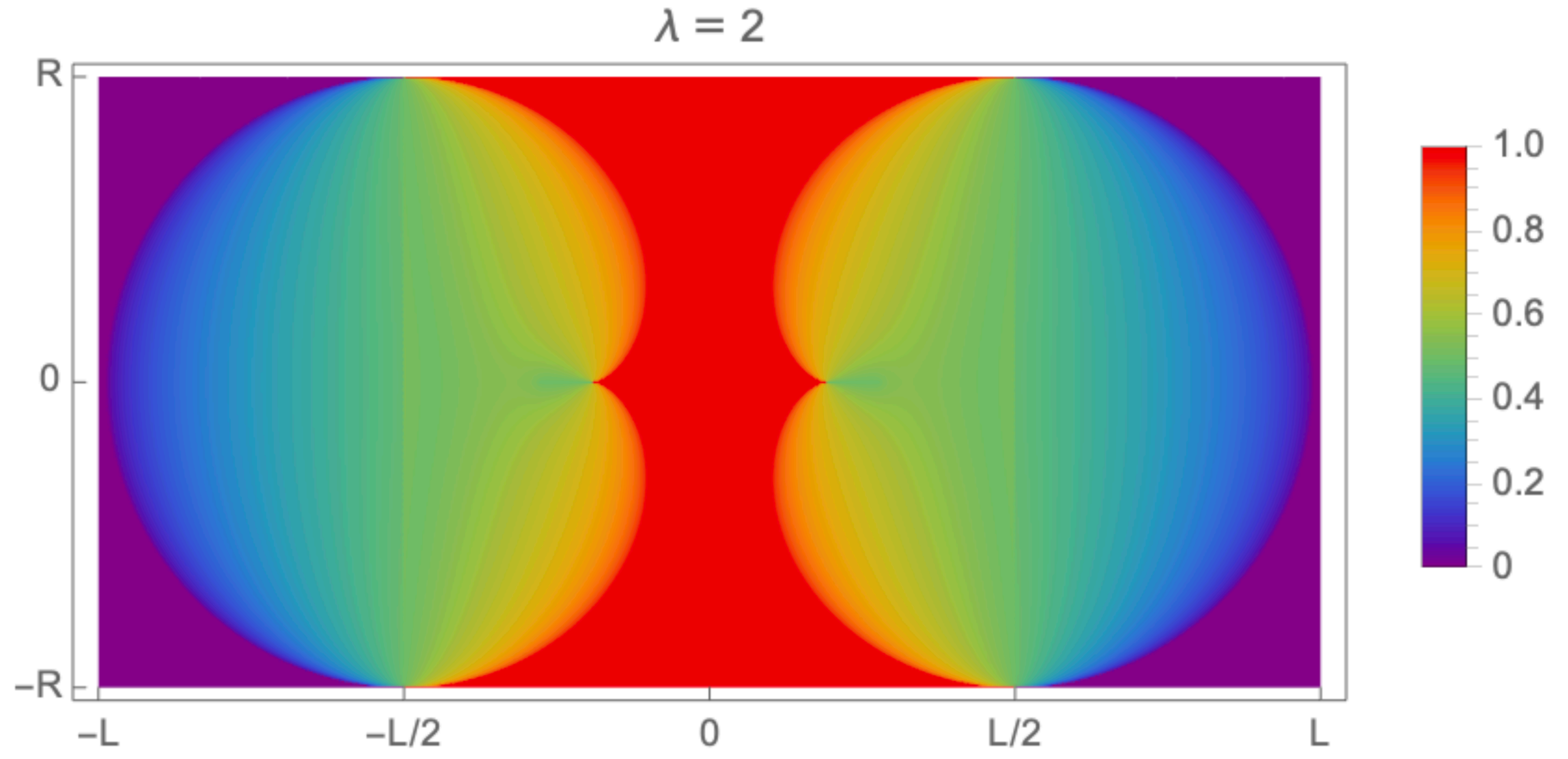
Colomo, Pronko, 2013

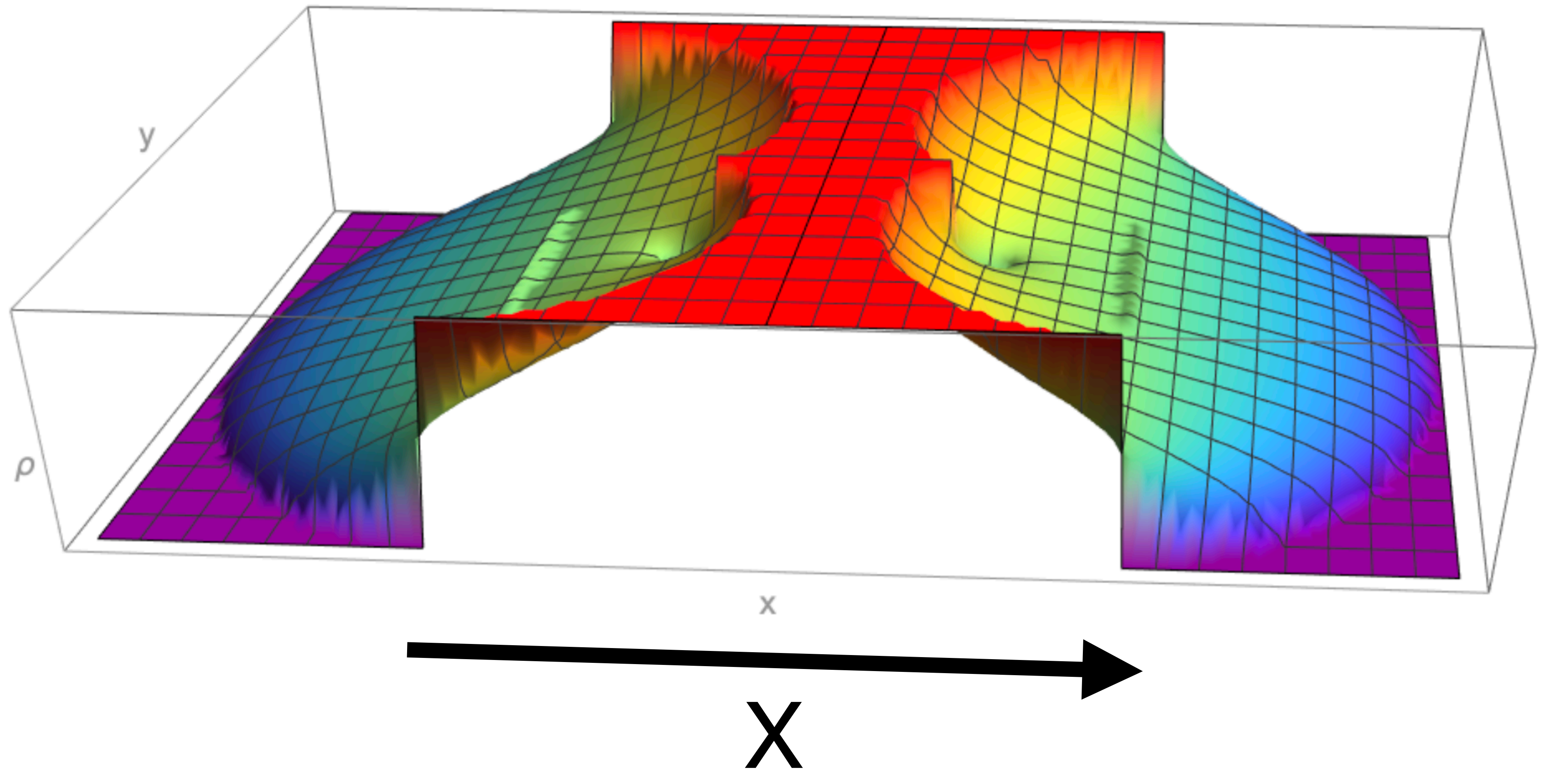


Other boundary conditions



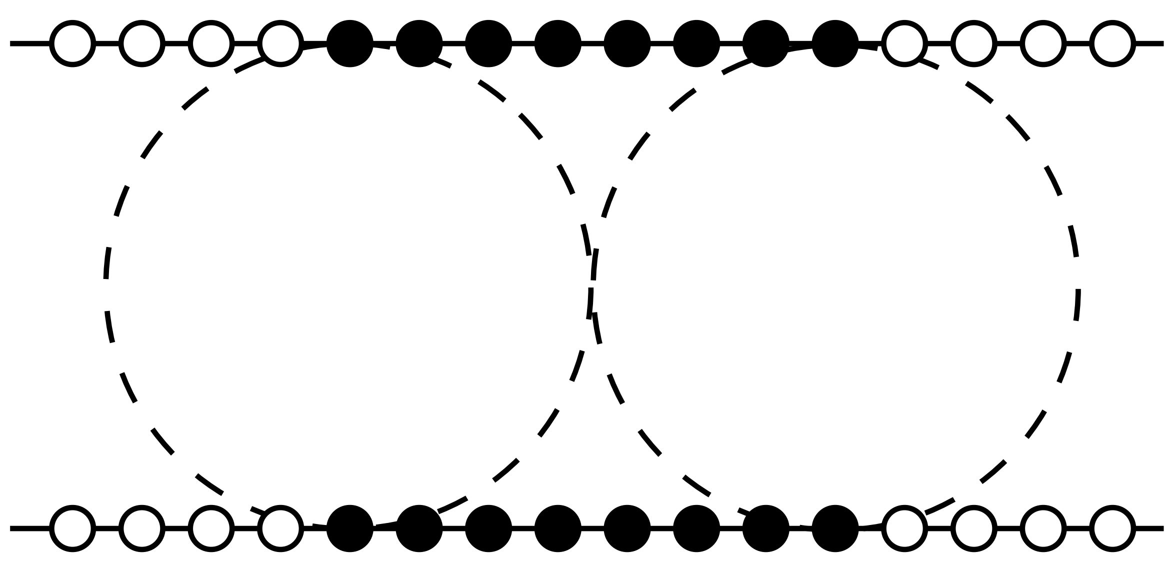
Insertion of
“Fullness” at t=0





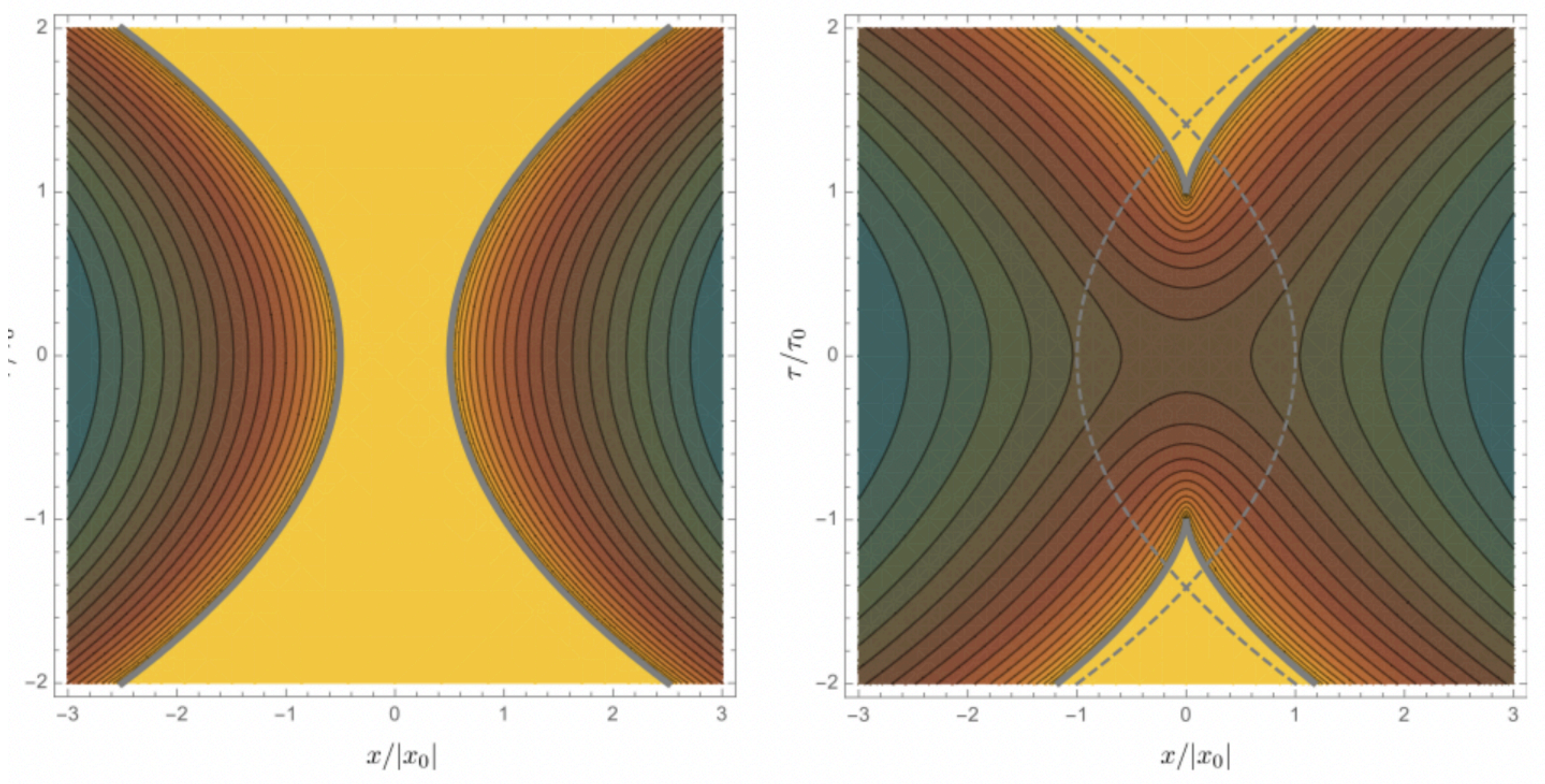
On the universality of the third-order phase transition

Interacting problems



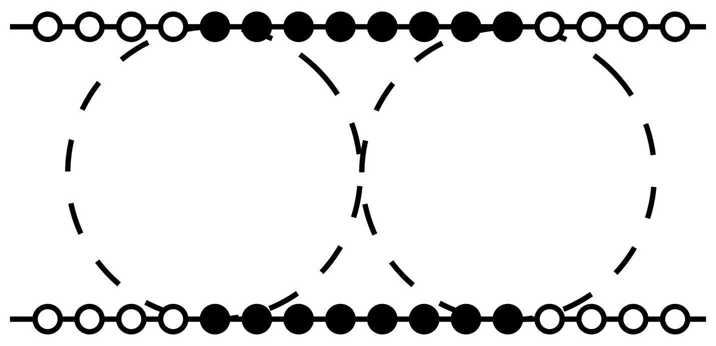
Lattice fermions with interactions or XXZ-type models

- Hydrodynamic approach is not limited to free or even integrable systems
- The difficulty is in solving PDEs
- At the point of Arctic Circles merging the density of particles (or holes) is small
- Fermion interactions are not important for the transition
- The transition is of the third order and fluid profiles in the vicinity of the merging point the shape are universal



Density profiles

Universal profiles



Lattice fermions with interactions or
XXZ-type models. P and T symmetries.

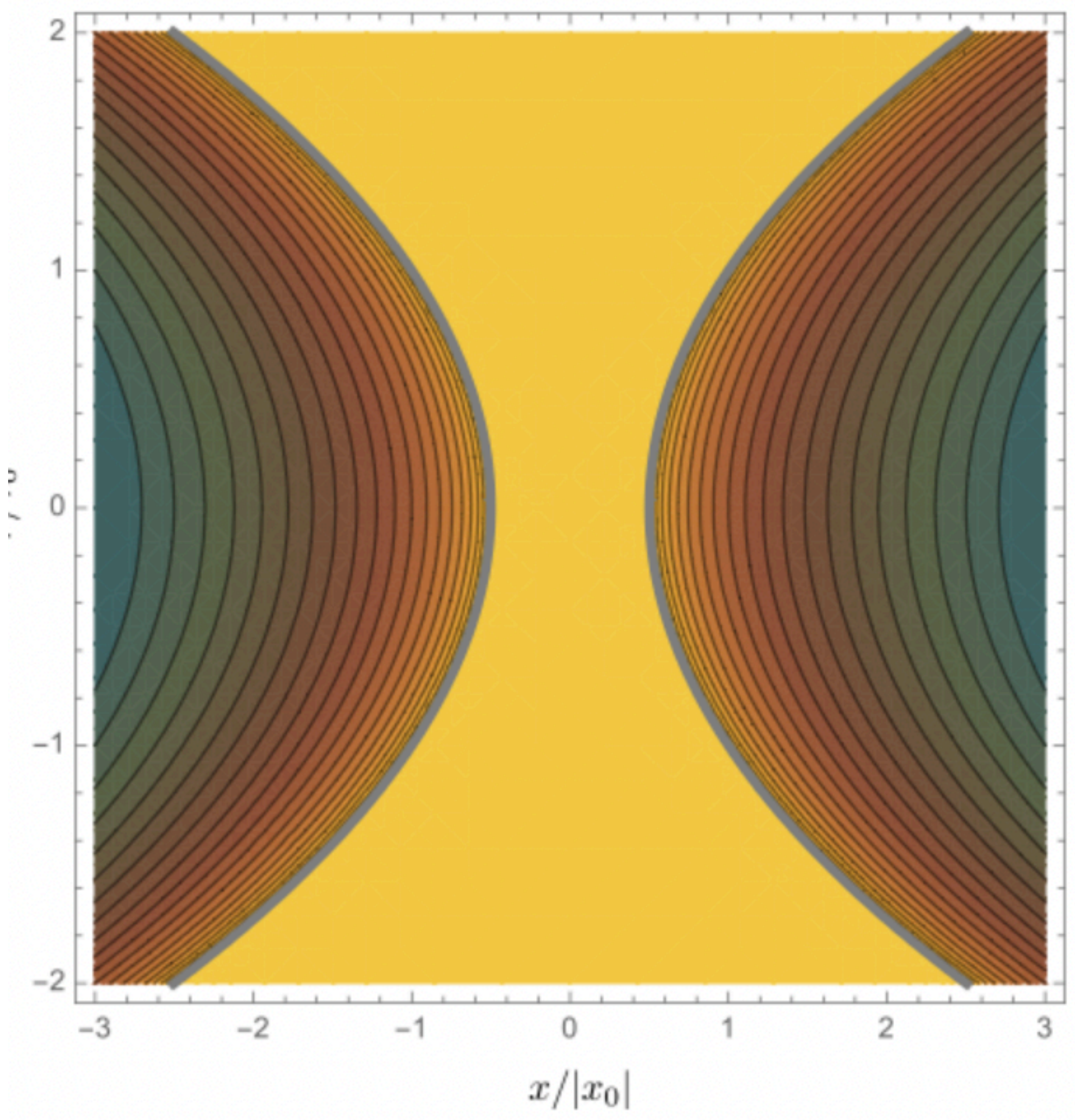
$$X^2 = (q^2 + \delta_1)(q^2 + \delta_2)$$

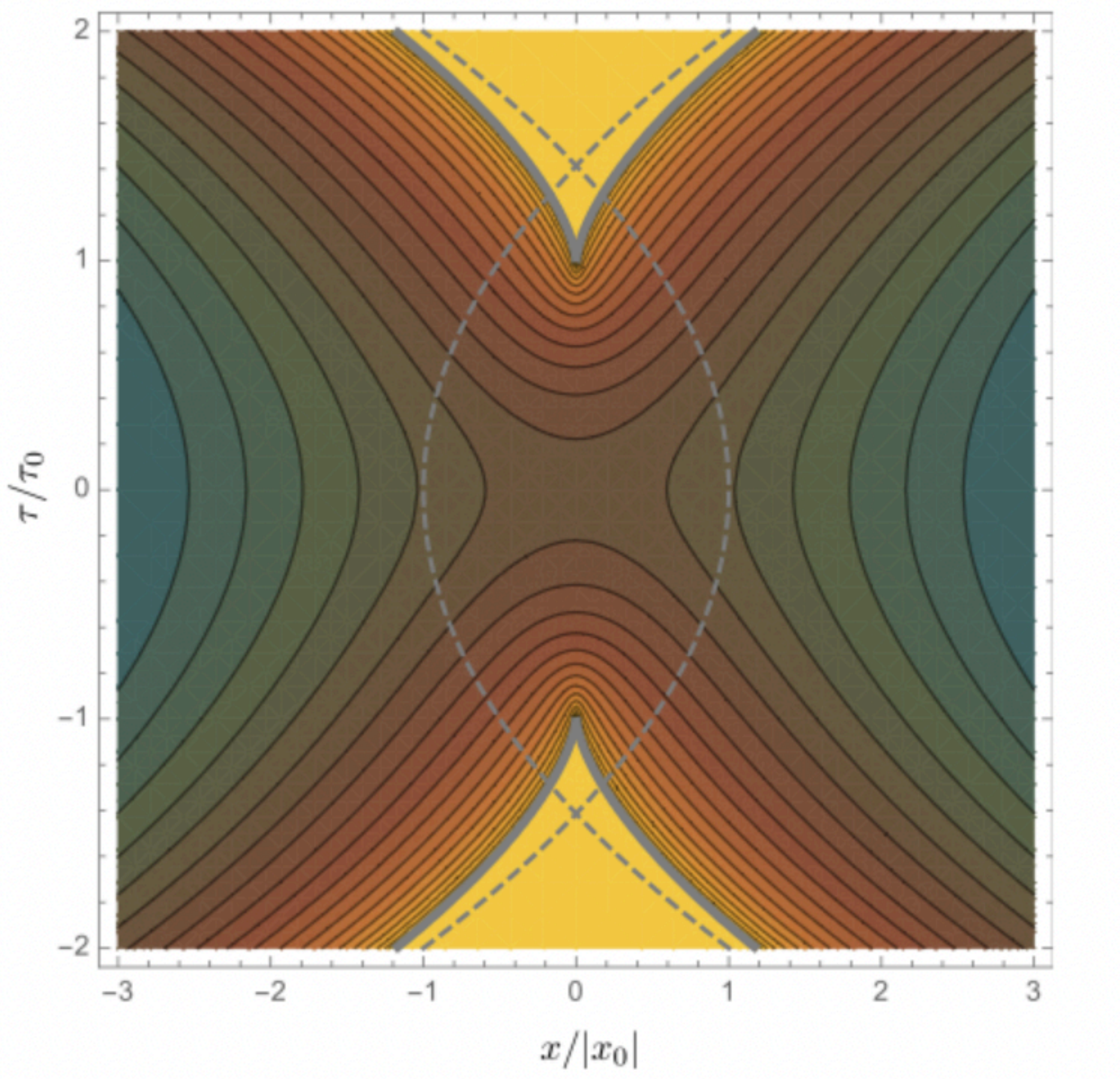
$$X = x - i\tau q \qquad q = \pi\rho - iv$$

Universal curve

$$X^2 = \begin{cases} (q^2+\delta)^2\,, & \delta > 0\,, \\ q^4\,, & \delta = 0\,, \\ q^2(q^2+2\delta)\,, & \delta < 0\,. \end{cases}$$

$\delta > 0$





$\delta < 0$

Free energy estimate

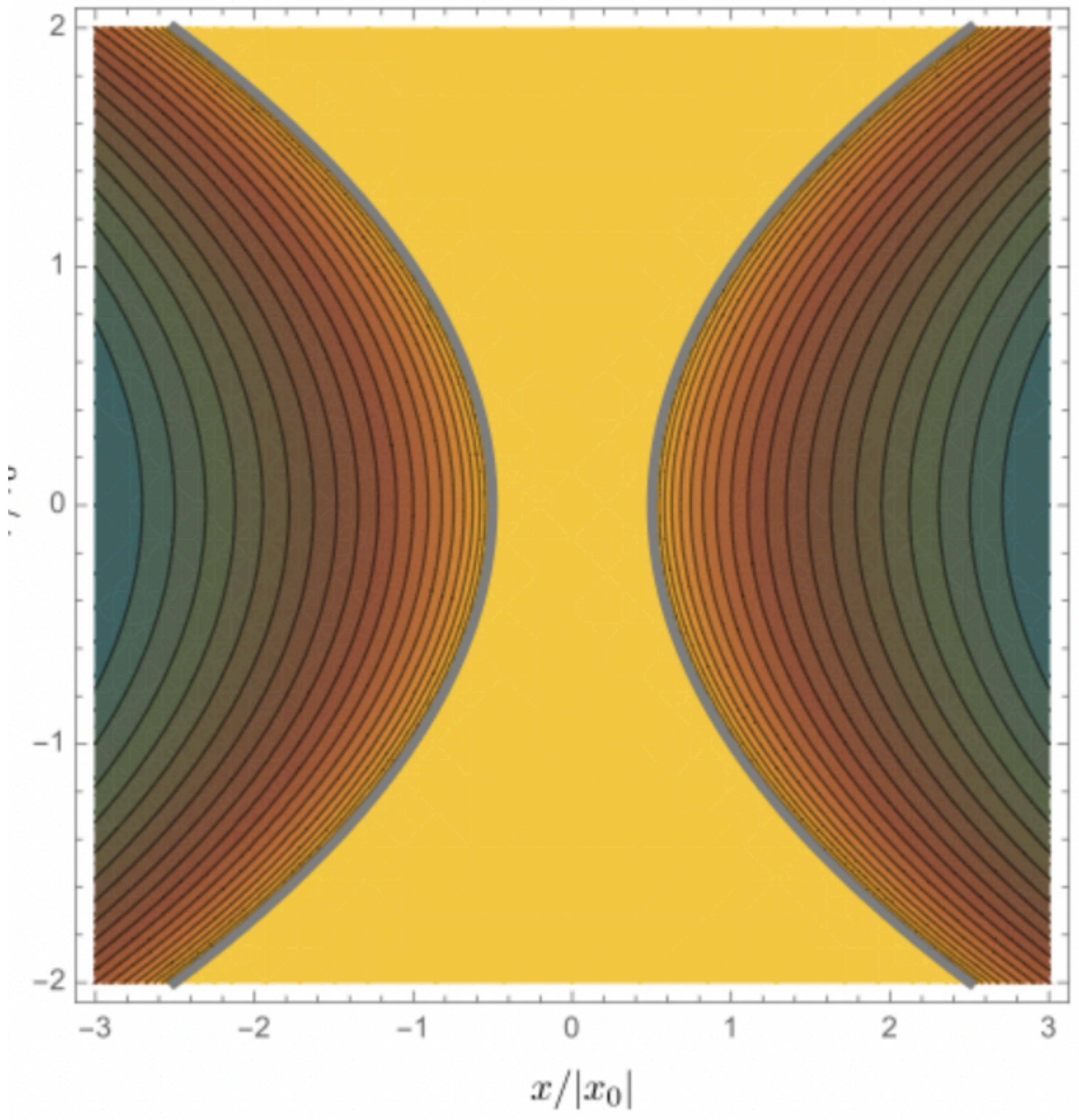
$$\delta\rho \sim q \sim \delta^{1/2} \qquad \delta E \sim \delta\rho^3 \sim \delta^{3/2}$$

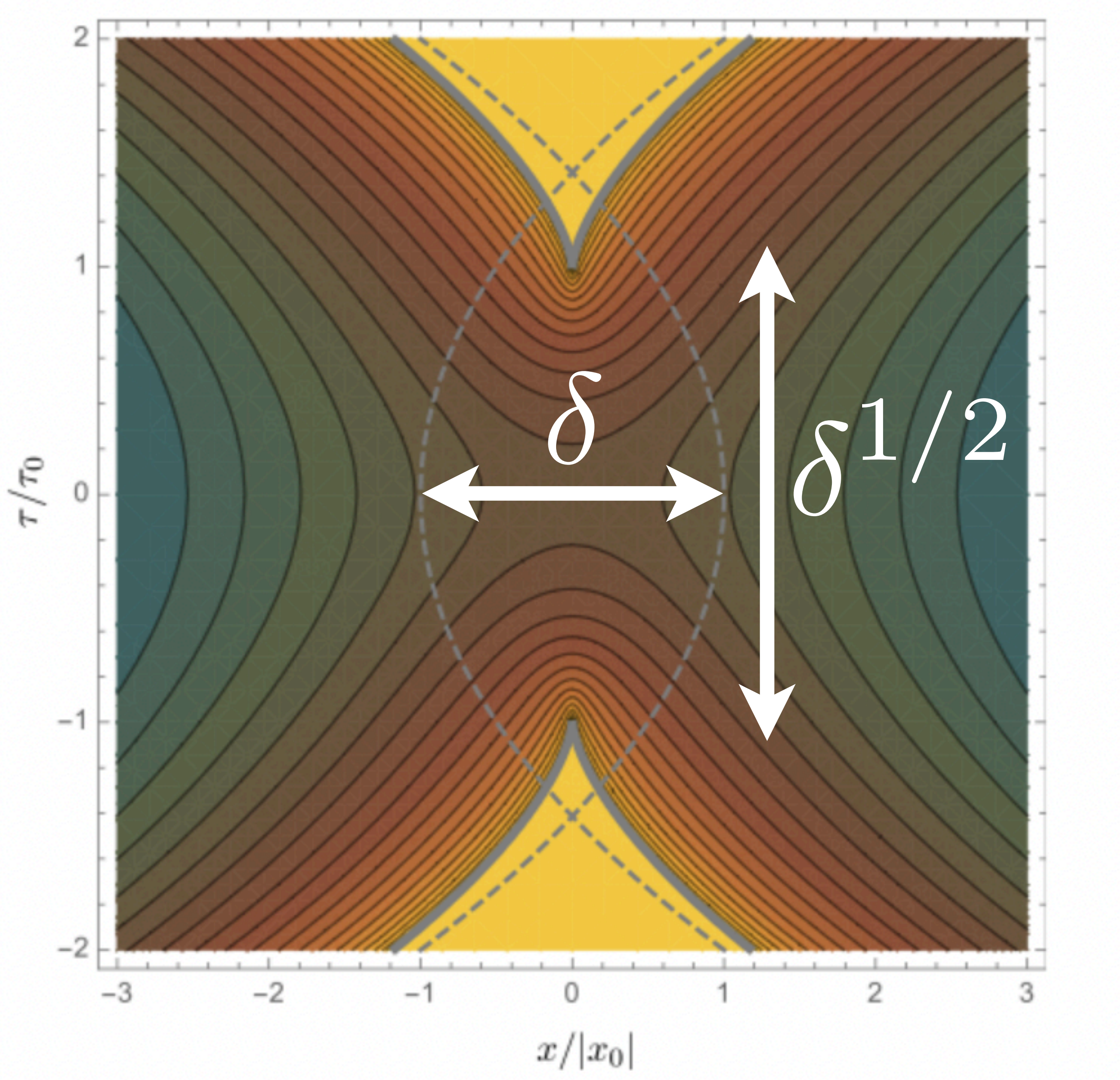
$$\delta S \sim \delta\tau\,\delta x\,\delta E \sim \delta^{1/2}\delta\,\delta^{3/2} \sim \boxed{\delta^3}$$

Discontinuity of third derivative

Third order!

$\delta > 0$





$\delta < 0$

Conclusions

- Limit shape problems occur in many random/statistical systems
- The Gross-Witten-Wadia transition has been mapped to the problem of Arctic circle merging in dimer and free fermion models.
- The hydrodynamic picture can be generalized for interacting particles.
- Universal (stable to interactions) density and velocity profiles corresponding to the merging of Arctic circles have been found. They are given by solutions of fermion hydrodynamic equations.
- It is conjectured that the transition is of the third order even in the presence of interactions (protected by P and T symmetries).

Remarks

- Only mean-field (large N) limit has been considered in this talk, not fluctuations.
- In proper scaling limits one can study Tracy-Widom, Airy, tachnoid, … statistics.
- A lot of connections with algebraic geometry, combinatorics, and representation theory.
- Integrable interacting models (6-vertex etc).

Reviews

1. R. Kenyon, Lectures on Dimers, arXiv:0910.3129v1 [math.PR]
2. J.-M. Stéphan, Extreme boundary conditions and random tilings, arXiv:2003.06339v2 [cond-mat.stat-mech]
3. YouTube: (4 lectures in Russian) Н. Решетихин, “Предельные формы в статистической механике” (N. Reshetikhin, “Limit shapes in statistical mechanics”).
4. YouTube (popular): Mathologer (B. Polster), “The ARCTIC CIRCLE THEOREM or Why do physicists play dominoes?”.