On maps with tight boundaries

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Based on joint work with Emmanuel Guitter and Grégory Miermont
arXiv:2104.10084, arXiv:2203.14796 and work in progress

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GGI workshop on Randomness, Integrability and Universality
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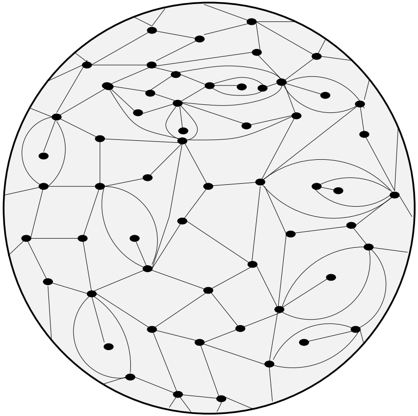
Outline

1. Introduction: definitions, motivations, previous results
2. Enumeration of pairs of pants with tight boundaries (arXiv:2104.10084)
3. On quasi-polynomials counting planar tight maps (arXiv:2203.14796)
4. Conclusion: future directions

Outline

Definitions



A map is a discrete surface formed by gluing polygons.

It consists of faces (the polygons), edges and vertices. Some faces may be distinguished as boundaries.

In this talk we restrict to orientable maps. The topology of a map is characterized by the pair (g, n) where:

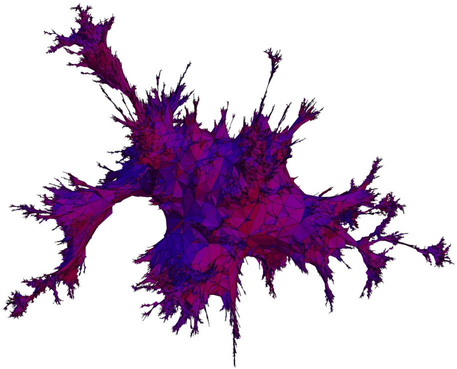
- g is the genus,
- n is the number of boundaries.

We mostly restrict to the planar ($g = 0$) case here:

$n = 0$	$n = 1$	$n = 2$	$n = 3$
sphere	disk	cylinder	pair of pants

A map is essentially bipartite if every inner (non-boundary) face has even degree (i.e., the universal cover of the map is bipartite). A quadrangulation has inner faces of degree 4 only.

Randomness, integrability and universality?



A random triangulation of the sphere with 50k triangles, by J. Bettinelli

Maps appear under various names (fatgraphs, planar diagrams...) in the theoretical physics and mathematics literature. The study of random maps has been a particularly active topic since the 1980's, as it is related with 2D quantum gravity, matrix models, etc. So many authors have been involved that I won't attempt to name them all here.

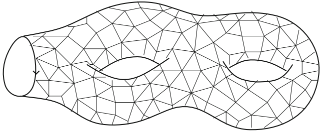
Generating functions of maps and related objects (Hurwitz numbers...) are known to be related to tau-functions of classical integrable hierarchies. A famous instance of this connection is Kontsevich's proof of Witten's conjecture.

Random maps admit universal scaling limits. Since 2000, many rigorous results have been obtained. Most of the focus has been on the "Brownian" (pure gravity) case but there were recent exciting developments for other universality classes too...

Topological recursion

Another "universal" phenomenon, more algebraic in nature, is the formalism of topological recursion.

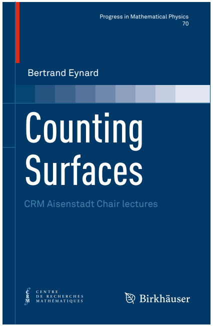
[Ambjørn-Chekhov-Kristjansen-Makeenko 1993, Akemann 1996, Alexandrov-Mironov-Morozov 2005, Eynard-Orantin 2007, Eynard 2016...]



Intuitively, it gives a recursive scheme to compute the generating function $W_{g,n}$ of maps of genus g with n boundaries, for any g, n. All we need to find is the initial data:

- $W_{0,1}$ (disks): this defines the "spectral curve",
- $W_{0,2}$ (cylinders): "fundamental second kind differential".

This scheme applies to problems other than map enumeration, e.g. Weil-Petersson volumes of hyperbolic surfaces. [Mirzakhani 2006]



Our long-term goal

Is there a combinatorial, possibly bijective, approach to topological recursion? What would be its geometric implications?

Warm-up: enumeration of pointed rooted planar maps

We consider maps with controlled face degrees:

$$\text{weight(map)} = t^{\#\text{vertices}} \prod_{f \text{ inner face}} g_{\text{degree}(f)}.$$

This corresponds to the Hermitian one-matrix model: $\int_{\mathcal{H}_N} dM\, e^{\frac{N}{t}\mathrm{Tr}\left(-\frac{M^2}{2}+\sum_{k\geq 1} g_k \frac{M^k}{k}\right)}$.

For simplicity we restrict to essentially bipartite maps: $g_1 = g_3 = g_5 = \cdots = 0$.
Quadrangulations are obtained by taking only g_4 nonzero.

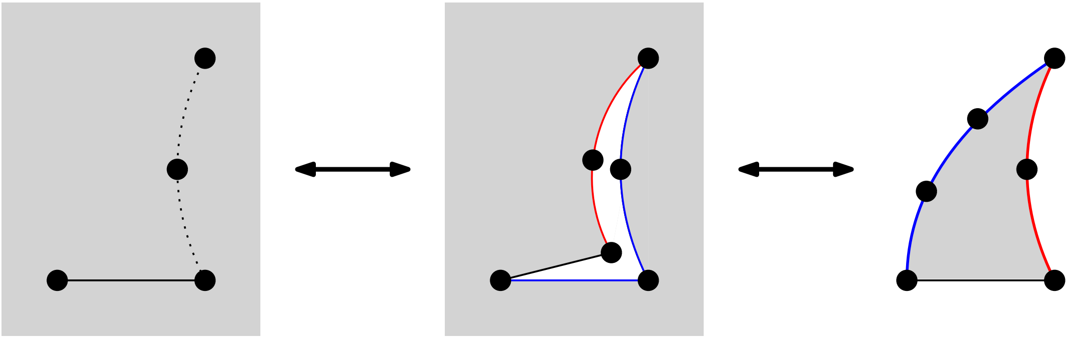
Theorem (reformulation of Tutte's census of slicings, 1962)

The generating function R of planar bipartite maps with one marked edge and one marked vertex (i.e. pointed rooted maps) satisfies

$$R = t + \sum_{k\geq 1} \binom{2k-1}{k} g_{2k} R^k.$$

Let's give an elementary proof of this, via the method of slice decomposition. [B.-Guitter, 2012]

From pointed rooted planar maps to slices



More generally, slice decomposition has been successfully applied to disks and cylinders.

Can we extend it to other topologies? Next in line are pairs of pants.

Outline

Theorem (Ambjørn-Jurkiewicz-Makeenko 1990, Eynard 2016, Collet-Fusy 2012)

The generating function of bipartite planar maps with three boundaries of lengths $2a, 2b, 2c$ is equal to

$$P(a,b,c) = \binom{2a-1}{a}\binom{2b-1}{b}\binom{2c-1}{c} R^{a+b+c}\frac{d\ln R}{dt} - t^{-1}\mathbf{1}_{a+b+c=0}.$$

We say that a boundary is tight if it has minimal length in its homotopy class.

Theorem (B.-Guitter-Miermont, arXiv:2104.10084)

The generating function of essentially bipartite planar maps with three tight boundaries of lengths $2a, 2b, 2c$ is equal to

$$T(a,b,c) = R^{a+b+c}\frac{d\ln R}{dt} - t^{-1}\mathbf{1}_{a+b+c=0}.$$

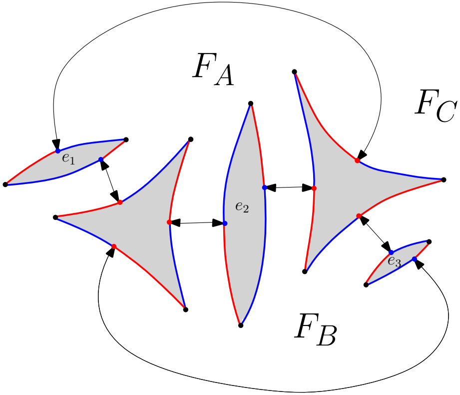
Idea of the proof

Even though our formula is logically equivalent to the previous one (decompose along outermost minimal separating cycles), its simplicity calls for a direct bijective explanation.

We provide such an explanation, by introducing new objects called geodesic triangles and diangles. Gluing these objects in an appropriate way produces a pair of pants with tight boundaries.

This turns out to be a discrete analogue of a construction known in hyperbolic geometry: a pair of pants (with its hyperbolic metric) can be obtained by gluing two ideal triangles together, with shifts.

At this stage, we do not know an extension of our construction to other topologies (higher genus, more boundaries).



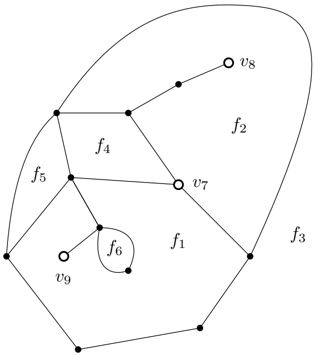
Outline

Tight maps

To find inspiration, let us do more enumeration!

A tight map is a map with some vertices marked, such that every vertex of degree 1 is marked. Equivalently, it is a map with tight boundaries, such that every face is a boundary and marked vertices are seen as boundaries of length 0.

Norbury encountered these objects in the study of lattice points in the moduli space of curves.



Theorem (Norbury, 2010)

The number $N_{g,n}(b_1, \ldots, b_n)$ of tight maps of genus g with n labelled boundaries of lengths $b_1, \ldots, b_n$ is a parity-dependent quasi-polynomial of degree $3g + n - 3$ in $b_1^2, \ldots, b_n^2$.

Its top degree homogenenous part is half the volume polynomial $V_{g,n}(b_1, \ldots, b_n)$ of Kontsevich.

Our result

We find an explicit expression for Norbury's lattice count quasi-polynomial $N_{g,n}(b_1,\dots,b_n)$ in the genus zero ($g=0$) case. For simplicity let me just state it in the bipartite case:

Theorem (B.-Guitter-Miermont, arXiv:2203.14796)

For $n \geq 3$, we have

$$N_{0,n}(2m_1,\dots,2m_n) = (n-3)! \sum_{\substack{k_1,k_2,\dots,k_n\geq 0\\ k_1+k_2+\cdots+k_n=n-3}} p_{k_1}(m_1)q_{k_2}(m_2)\cdots q_{k_n}(m_n)$$

where

$$p_k(m) := \frac{1}{(k!)^2}\prod_{i=1}^{k}\left(m^2-i^2\right), \qquad q_k(m) := \frac{1}{(k!)^2}\prod_{i=0}^{k-1}\left(m^2-i^2\right).$$

The top degree part gives an expression for Kontsevich's volume polynomial for $g=0$:

$$V_{0,n}(b_1,\dots,b_n) = \frac{(n-3)!}{2^{2n-7}} \sum_{\substack{k_1,\dots,k_n\geq 0\\ k_1+\cdots+k_n=n-3}} \left(\frac{b_1^{k_1}\cdots b_n^{k_n}}{k_1!\cdots k_n!}\right)^2.$$

Idea of the proof

We establish bijectively our formula for $N_{0,n}(2m_1, \ldots, 2m_n)$ in the following successive cases:

1. trees: only one m_i is nonzero,
2. necklaces: exactly two m_i are nonzero,
3. slices: one m_i is equal to 1 and another to 0 (pointed rooted map),
4. the general case follows by slice decomposition!

Outline

How about maps with tight boundaries having some inner faces?

Theorem (B.-Guitter-Miermont, work in progress)

The generating function $T_{g,n}(b_1, \dots, b_n)$ of maps of genus g with n labelled tight boundaries of lengths $b_1, \dots, b_n$ is, up to a power of R, a parity-dependent quasi-polynomial of degree $3g + n - 3$ in $b_1^2, \dots, b_n^2$, whose coefficients are rational in R and its derivatives w.r.t. t.

Our proof relies on the topological recursion and on a combinatorial interpretation of Zhukovsky's transformation. See also Norbury and Scott (2013) who observed the quasi-polynomiality phenomenon in a slightly more general class of spectral curves.

We also have a general expression for $T_{0,n}(b_1, \dots, b_n)$, at least when all b_i are even. It is an intriguing deformation of the lattice count polynomial $N_{0,n}(b_1, \dots, b_n)$.

Open question: can these results be understood bijectively?

Another intriguing direction to explore is the case of maps with irreducibility constraints, whose enumeration is related with Weil-Petersson volumes of hyperbolic surfaces, see Budd (2020).

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Thanks for your attention!