Correlation functions for the doubly periodic Aztec diamond

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Based on joint work with Tomas Berggren, joint work with Alexei Borodin, and joint work with Arno Kuijlaars

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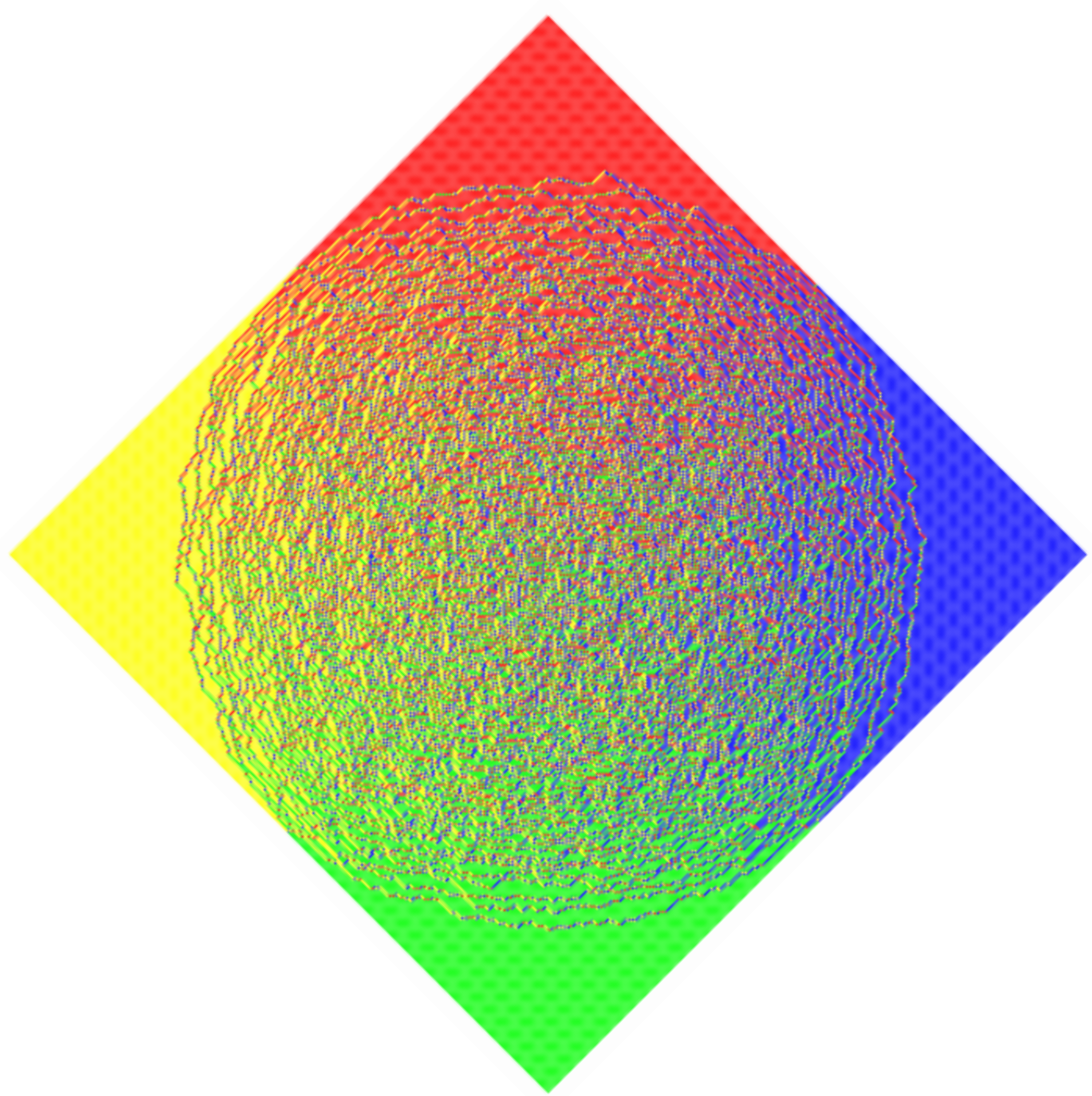
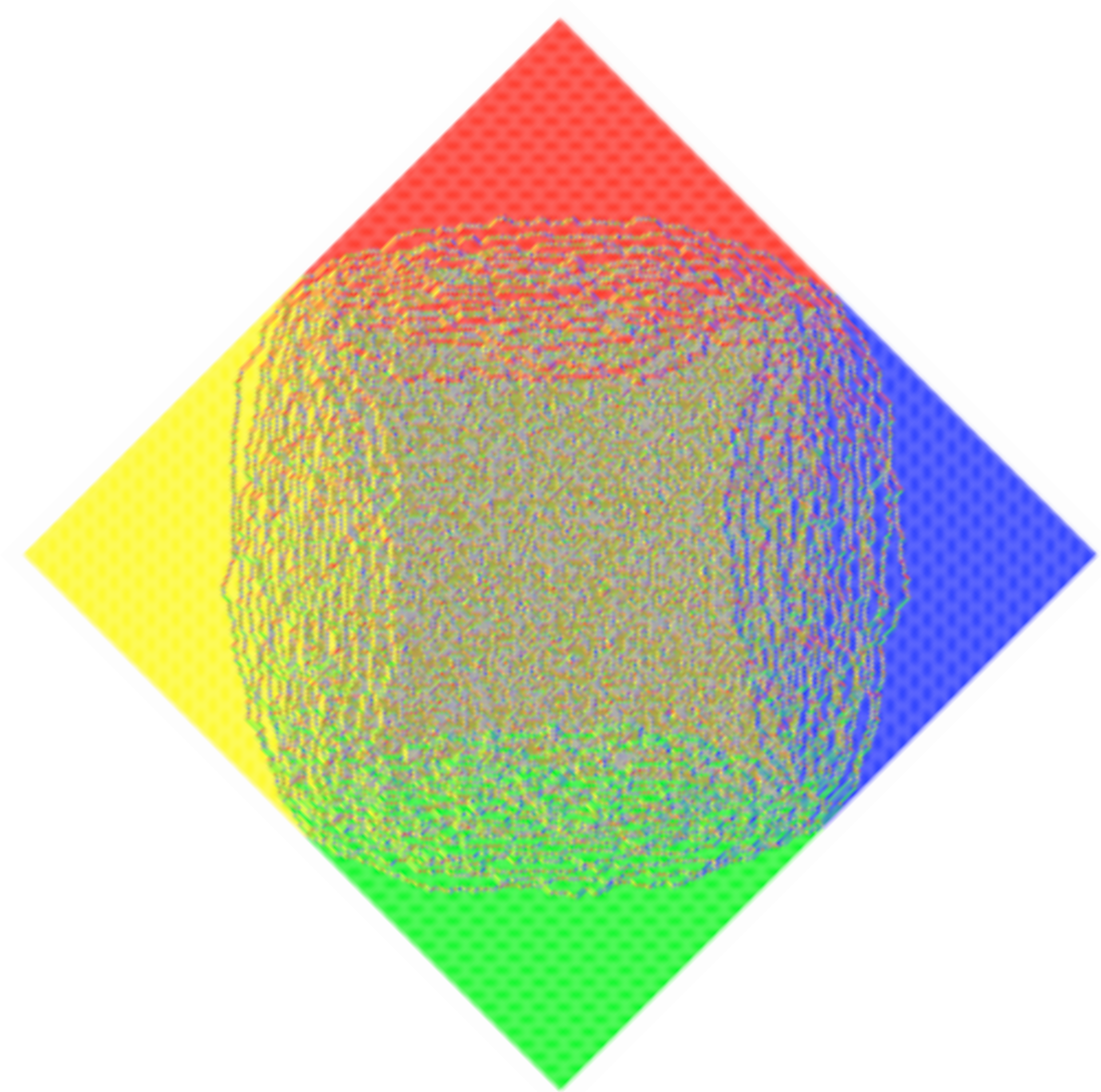
Domino tilings of large Aztec Diamonds





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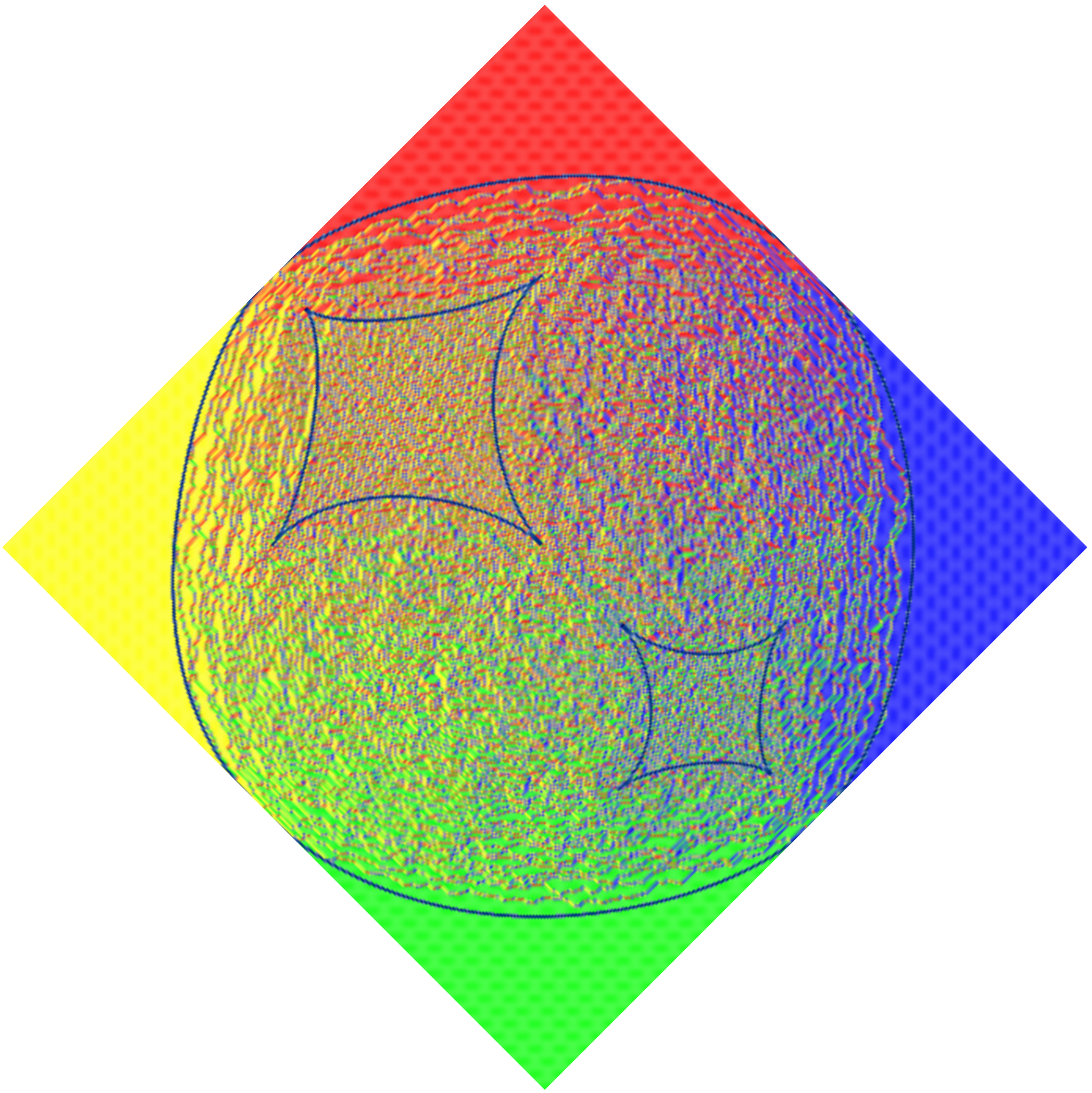
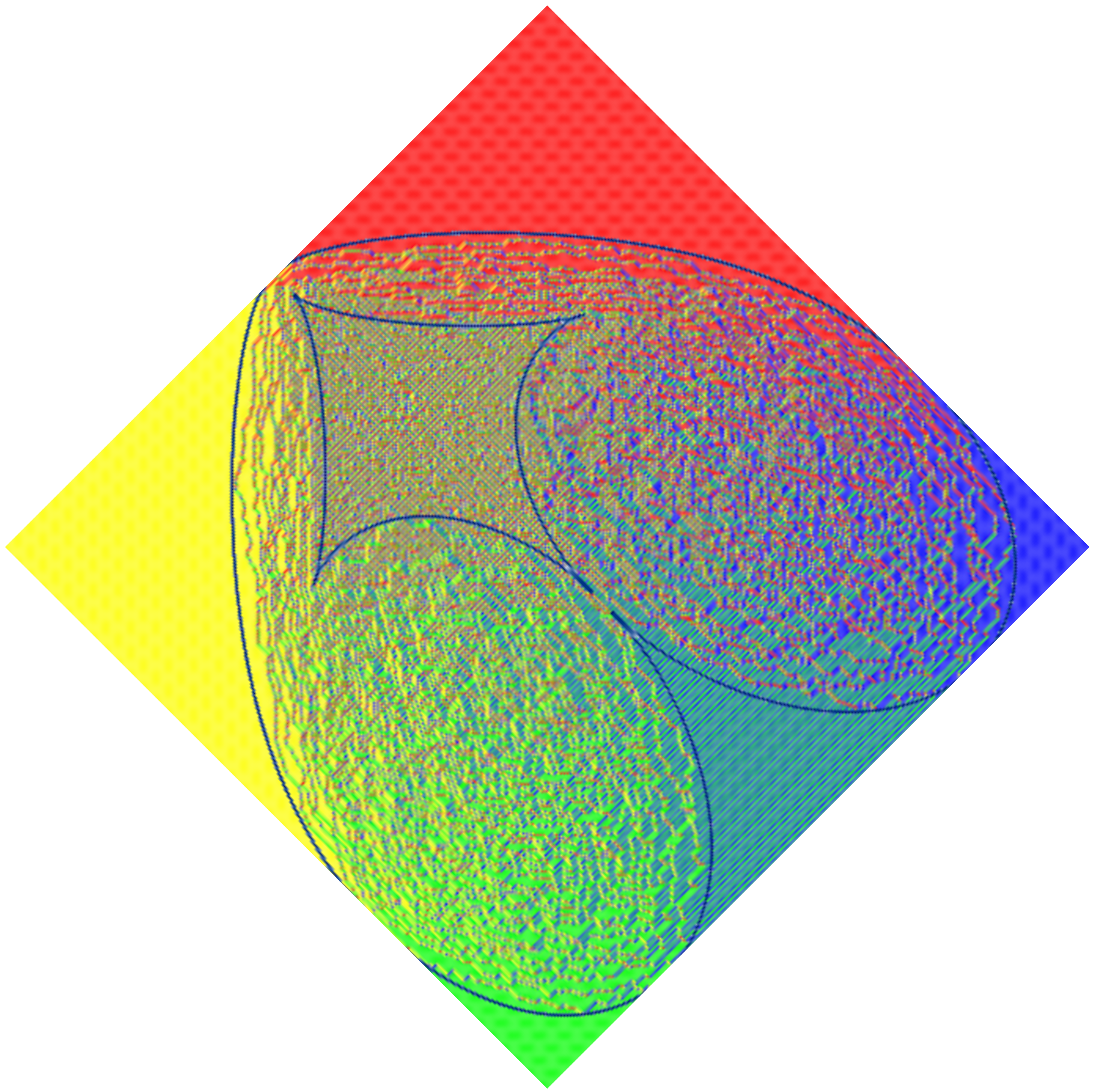
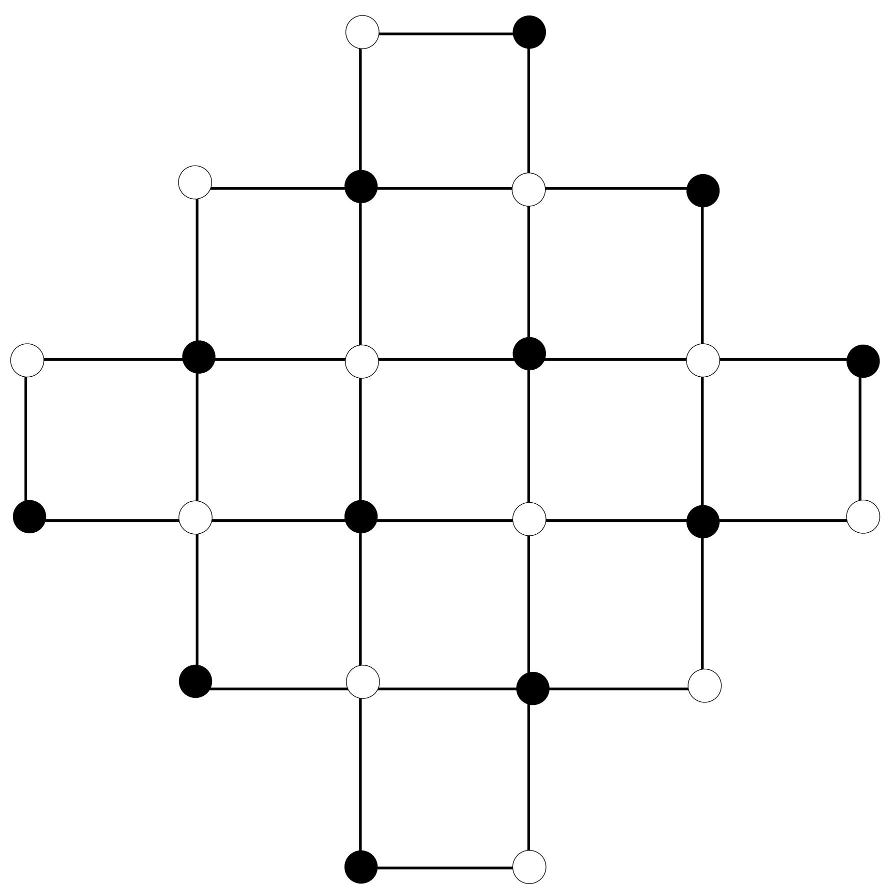
Domino tilings of large Aztec Diamonds

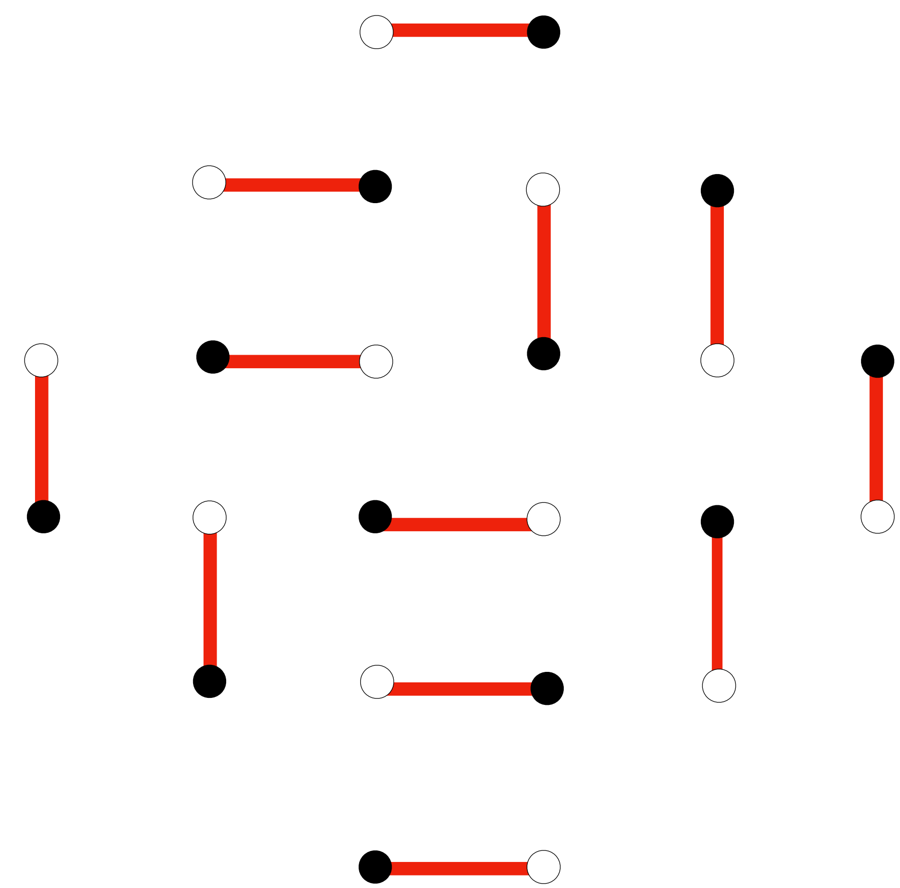




Dimer model



The Aztec Diamond graph



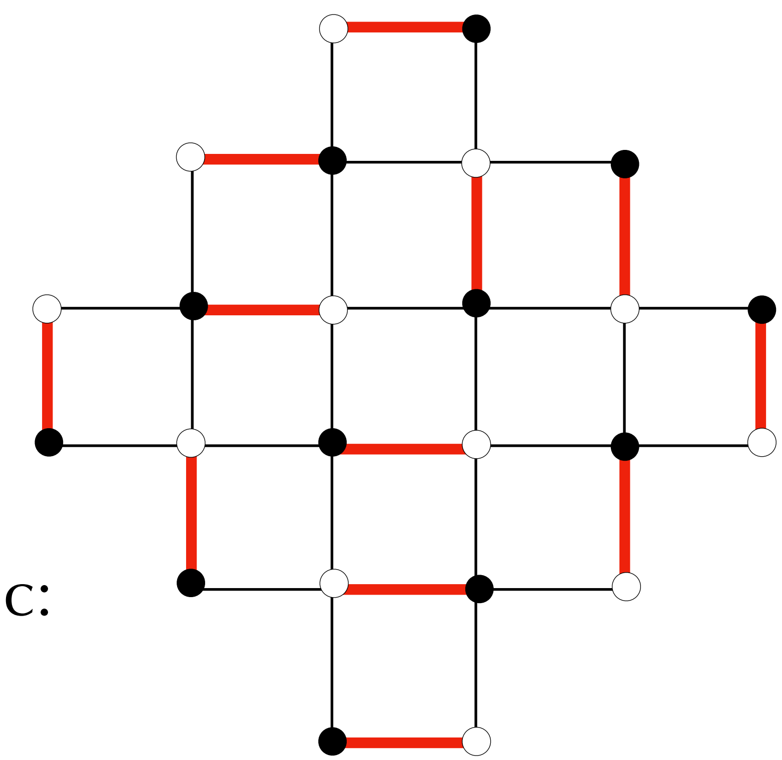
A perfect matching,
or dimer configuration

Dimer model

- **Random perfect matchings:**

  Let $w : E \to (0,\infty)$ be a weight function on the edges of the Aztec diamond graph. We define the probability of a given matching M by

$$\mathbb{P}(M) \sim \prod_{e \in M} w(e)$$



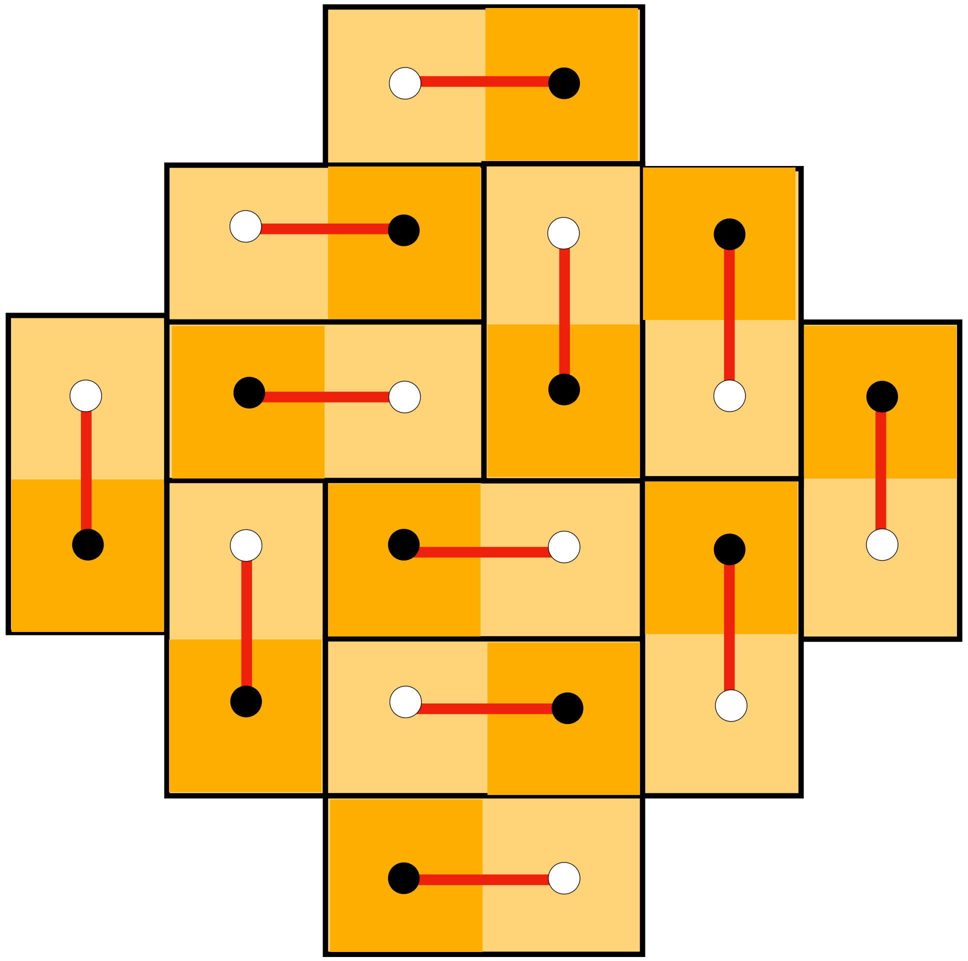
- **Doubly periodic weights**

  We will be interested in weights that are doubly periodic:

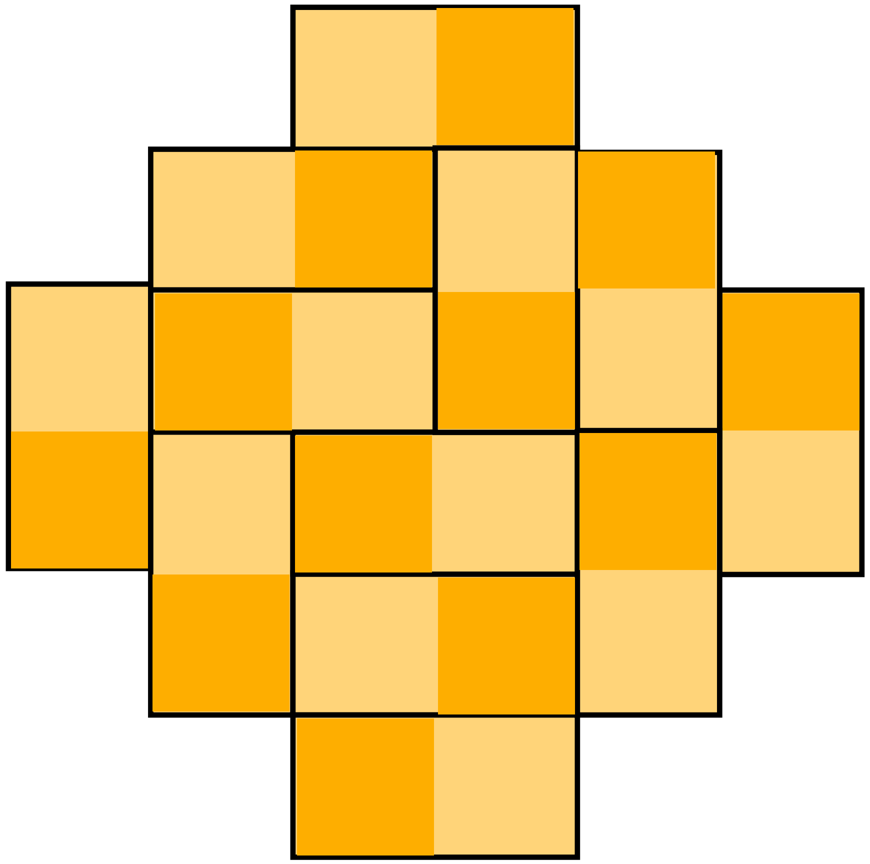
$$w(e + rT_1) = w(e) = w(e + sT_2)$$

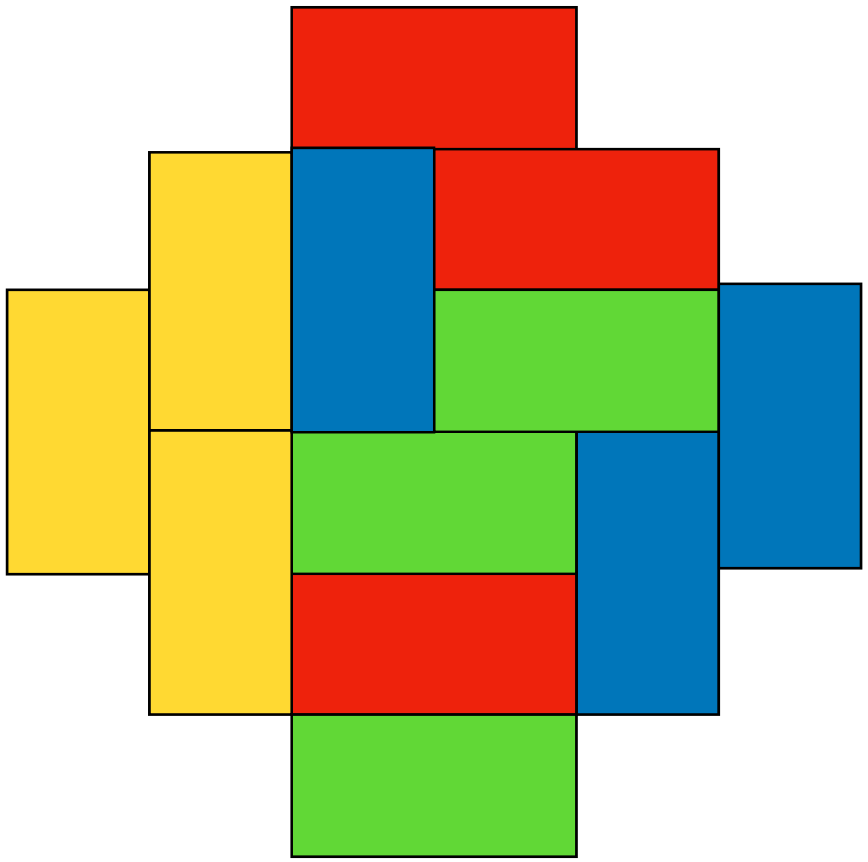
where $T_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $T_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and p, s integers.

Dimers vs dominos

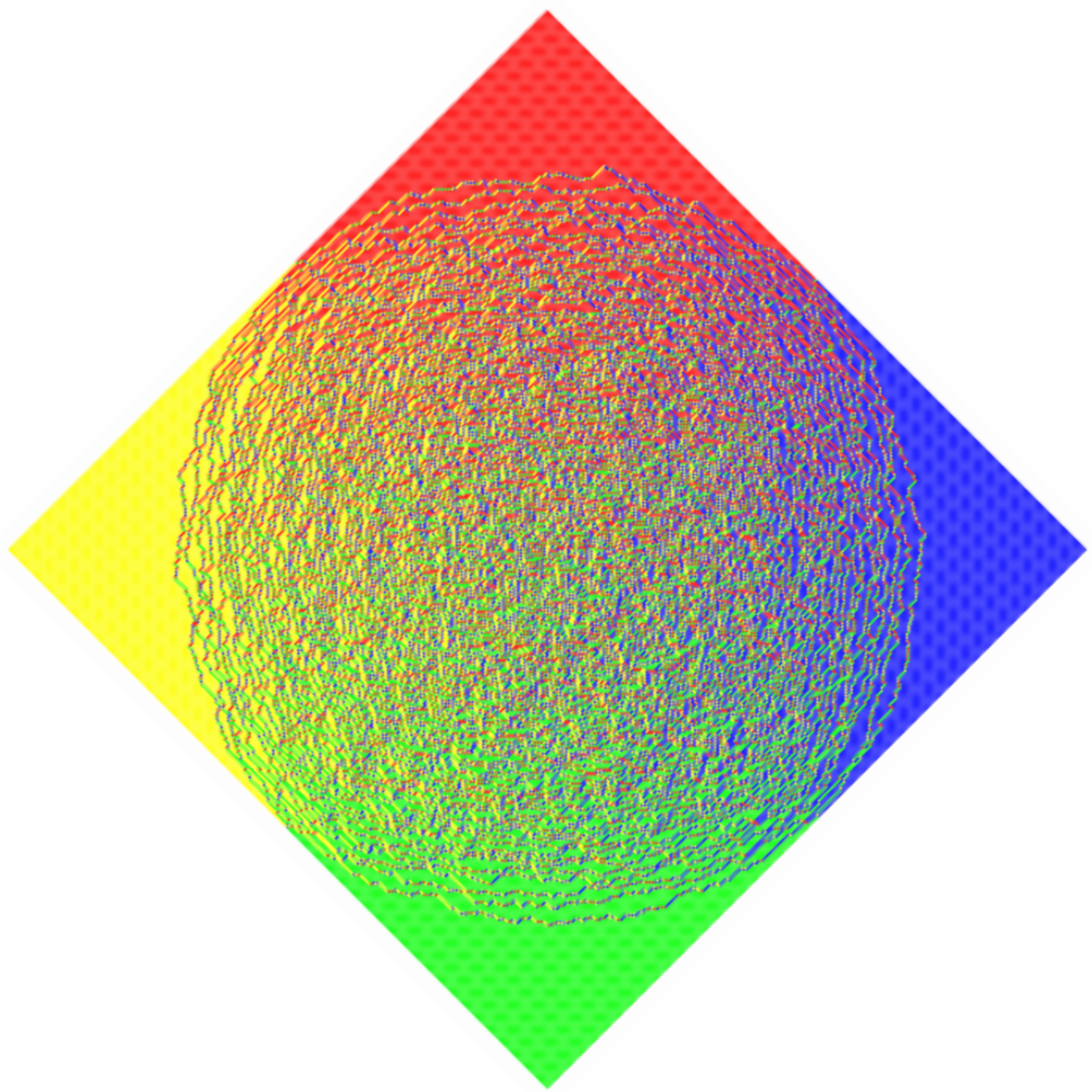


Domino tilings of the Aztec Diamond

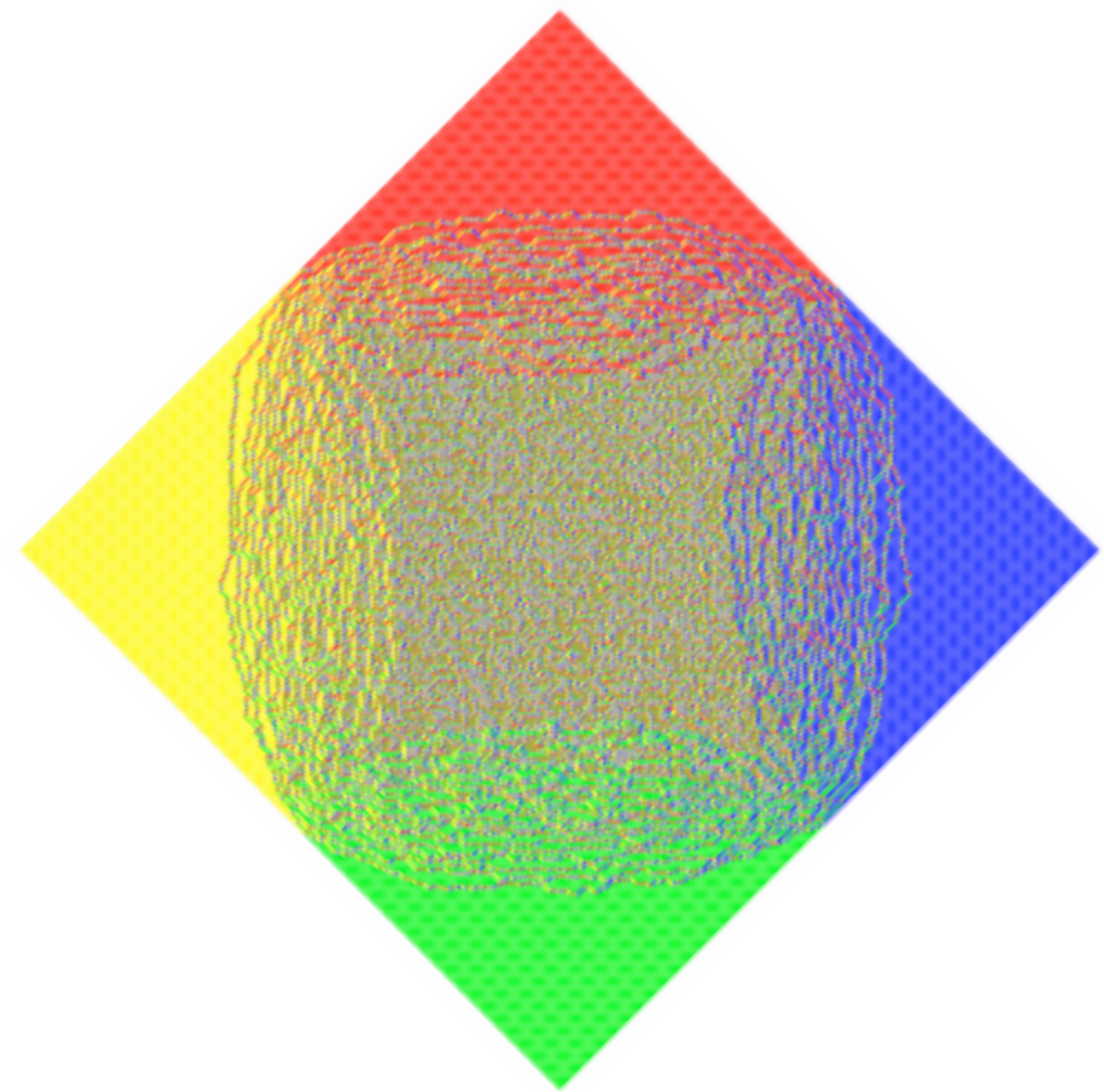




Domino tilings of large Aztec Diamonds

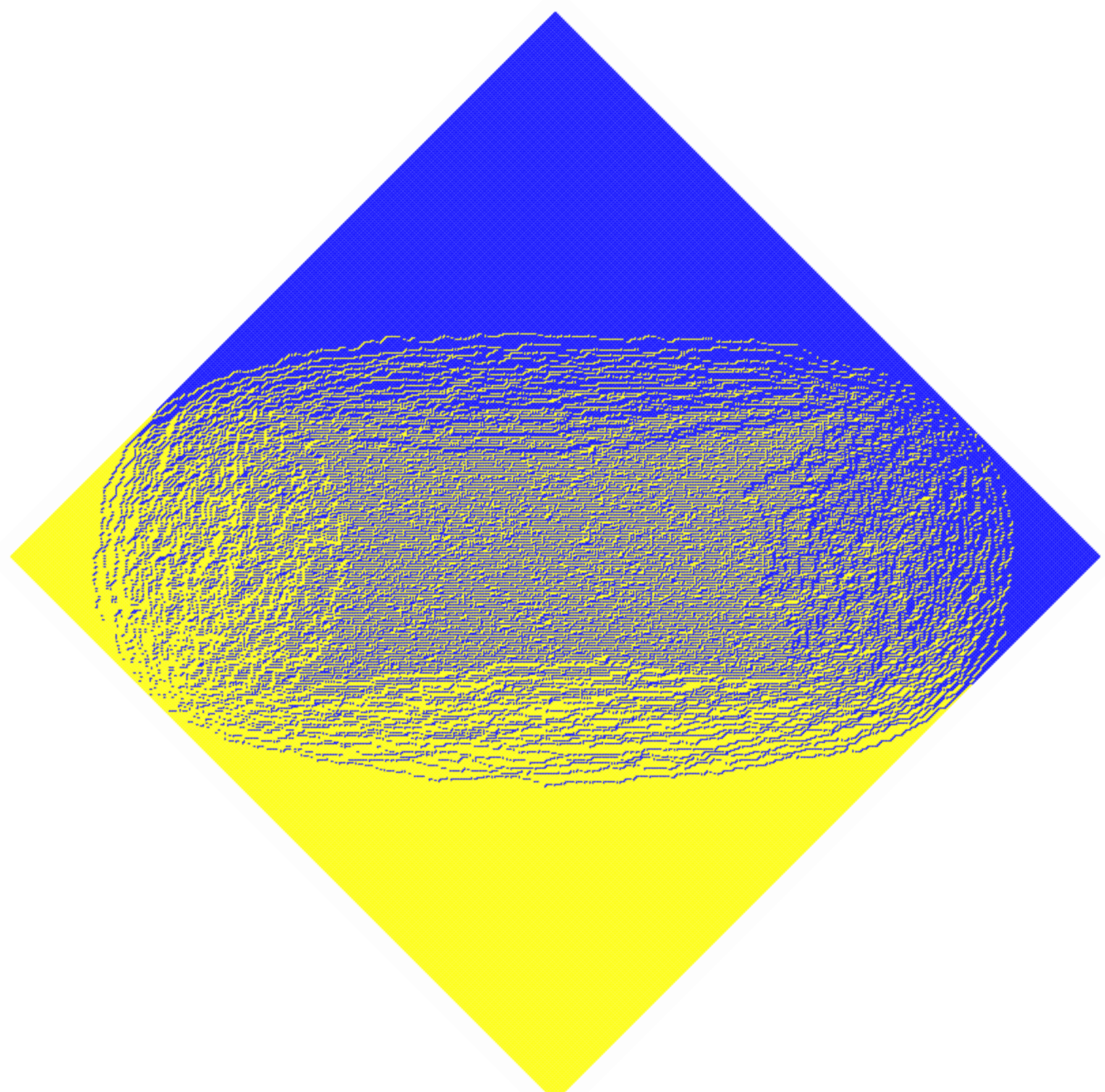


Uniform distribution
($p = q = 1$)



Unbiased 2×2 periodic weight

Domino tilings of large Aztec Diamonds



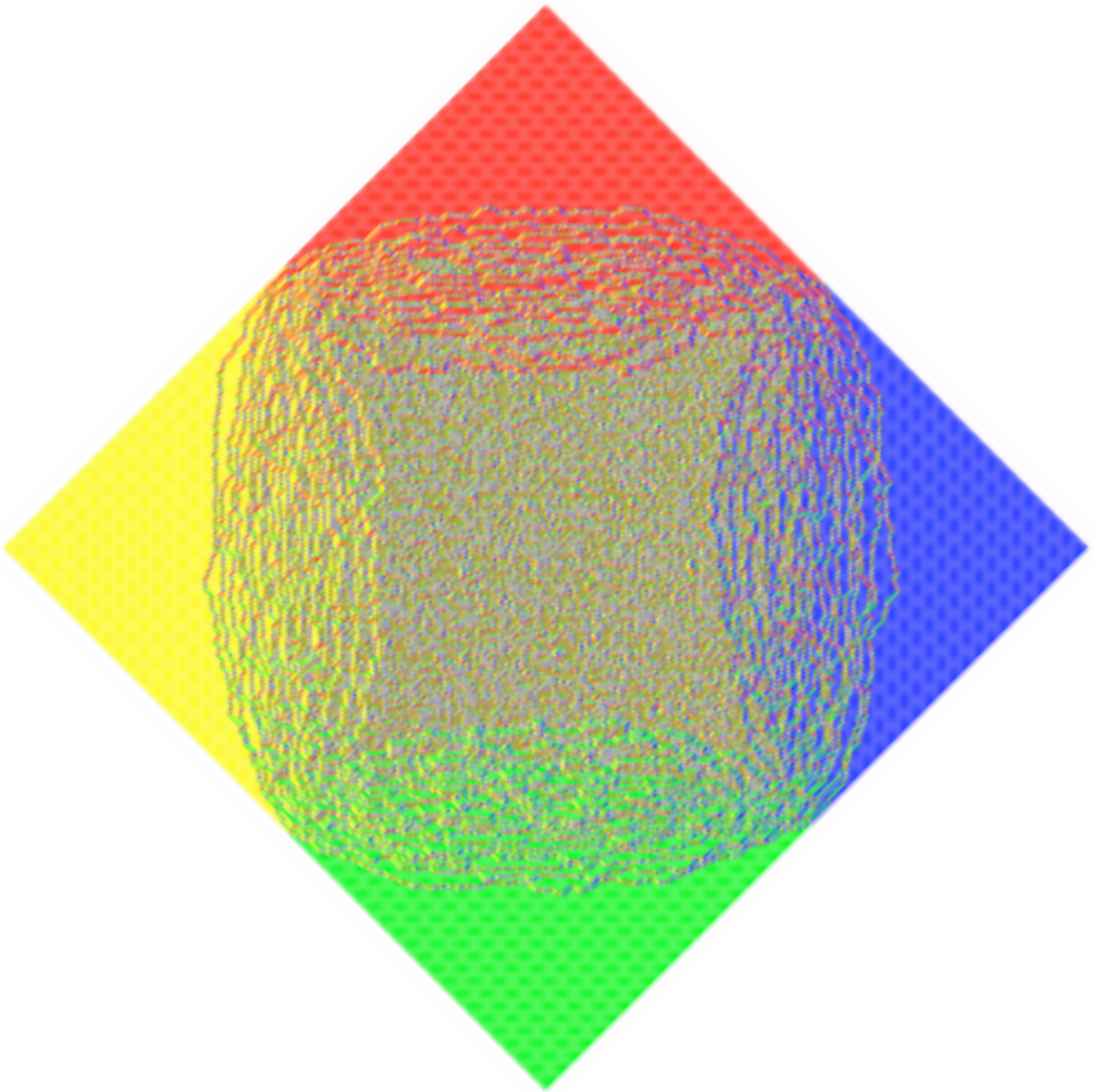
Biased 2×2 periodic Aztec diamond

Domino tilings of large Aztec Diamonds

- For doubly periodic weights, in addition to the frozen and rough disordered (or liquid) regions, also smooth disordered (or gaseous) region can appear *Kenyon-Okounkov-Sheffield '06*

- A first asymptotic analysis was performed by *Chhita-Johansson '14,* (see also *Chhita-Young '12*) for the unbiased two periodic Aztec diamond.

- The boundary between the rough and smooth disordered region has been discussed in *Beffara-Chhita-Johansson '16+'20, Johansson-Mason '21*.



Goal of the talk

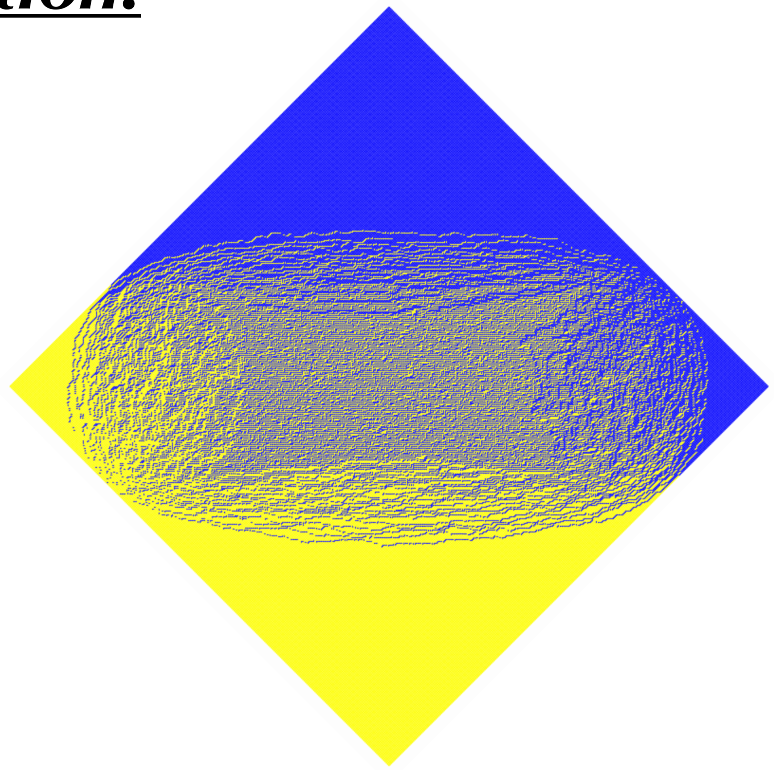
- In this talk I will discuss two approaches for studying correlation functions of doubly periodic weights for the Aztec diamond:

    - In *D-Kuijlaars '17* we showed a connection between doubly periodic tilings of planar domains and ***matrix-valued orthogonal polynomials***. This can be used for asymptotic studies.

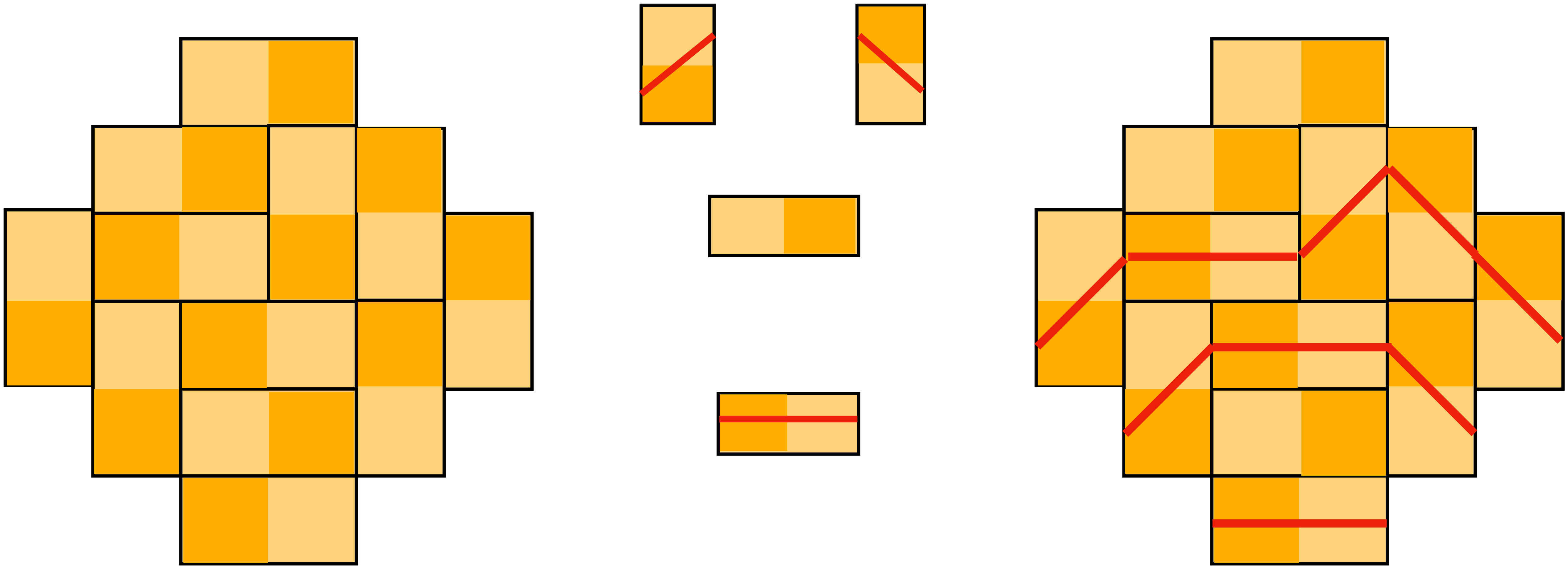
    - In *Berggren-D '19* we studied a generalization of the Schur process that includes the doubly periodic weights of the Aztec diamond. A double integral formula for the correlation kernel can be derived in terms of the solution to ***a Wiener-Hopf factorization.***

- Both approaches apply to more general models.

- We will discuss the second approach in more detail for the biased 2×2 Aztec Diamond. *Borodin-D '22*



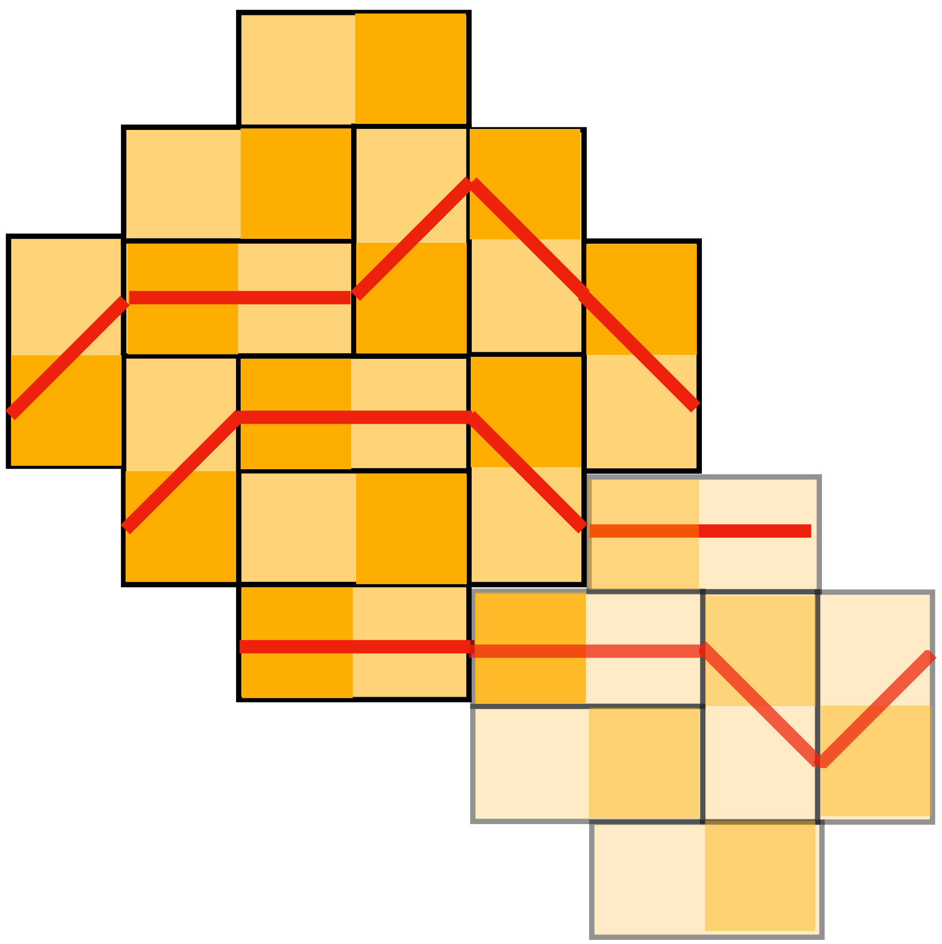
Domino tilings vs DR-paths



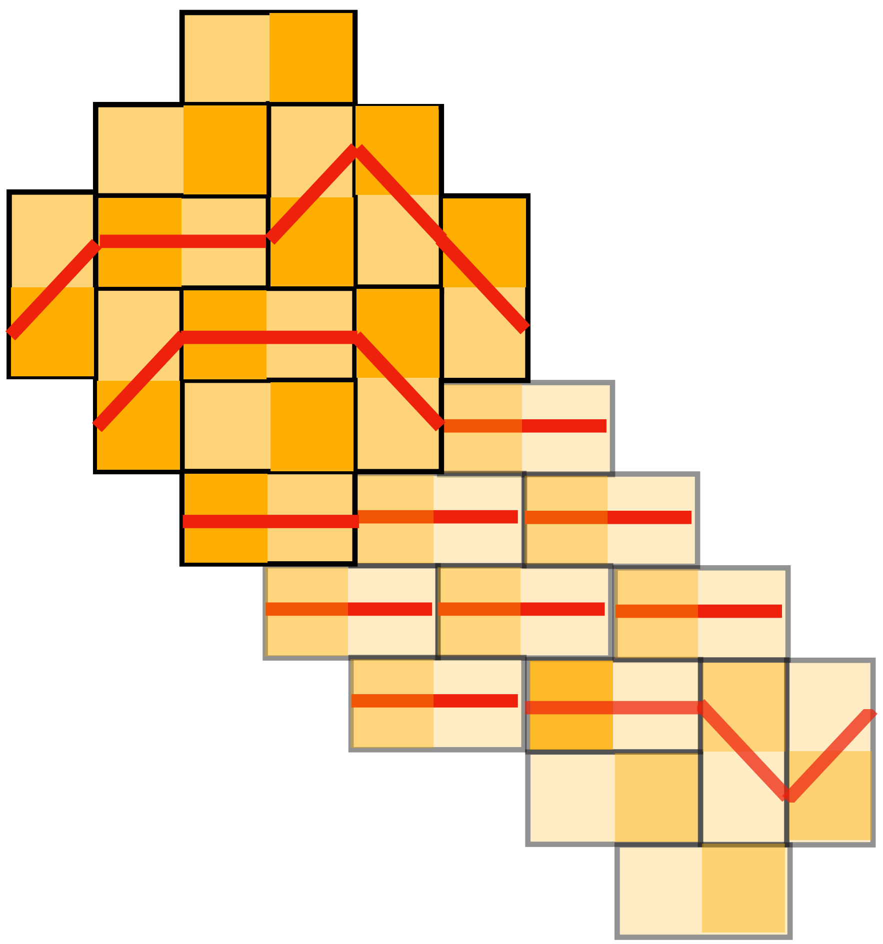
The tiling is completely determined by the
configuration of non-intersecting paths, often referred to as DR paths.

Domino tilings vs non-intersecting paths

- DR paths have the disadvantage of unequal length. Instead one can tile a larger domain.



Two Aztec diamond glued together. It is not possible to tile the bigger domain such that there is a domino that has parts in both diamonds.



Two Aztec diamonds, with a corridor in between (of arbitrary length). The tiling in the corridor must be trivial.

Non-intersecting paths on a

doubly periodic weighted directed graph.

A weighted directed graph

- Consider the graph $(\mathscr{V}, \mathscr{E})$ where

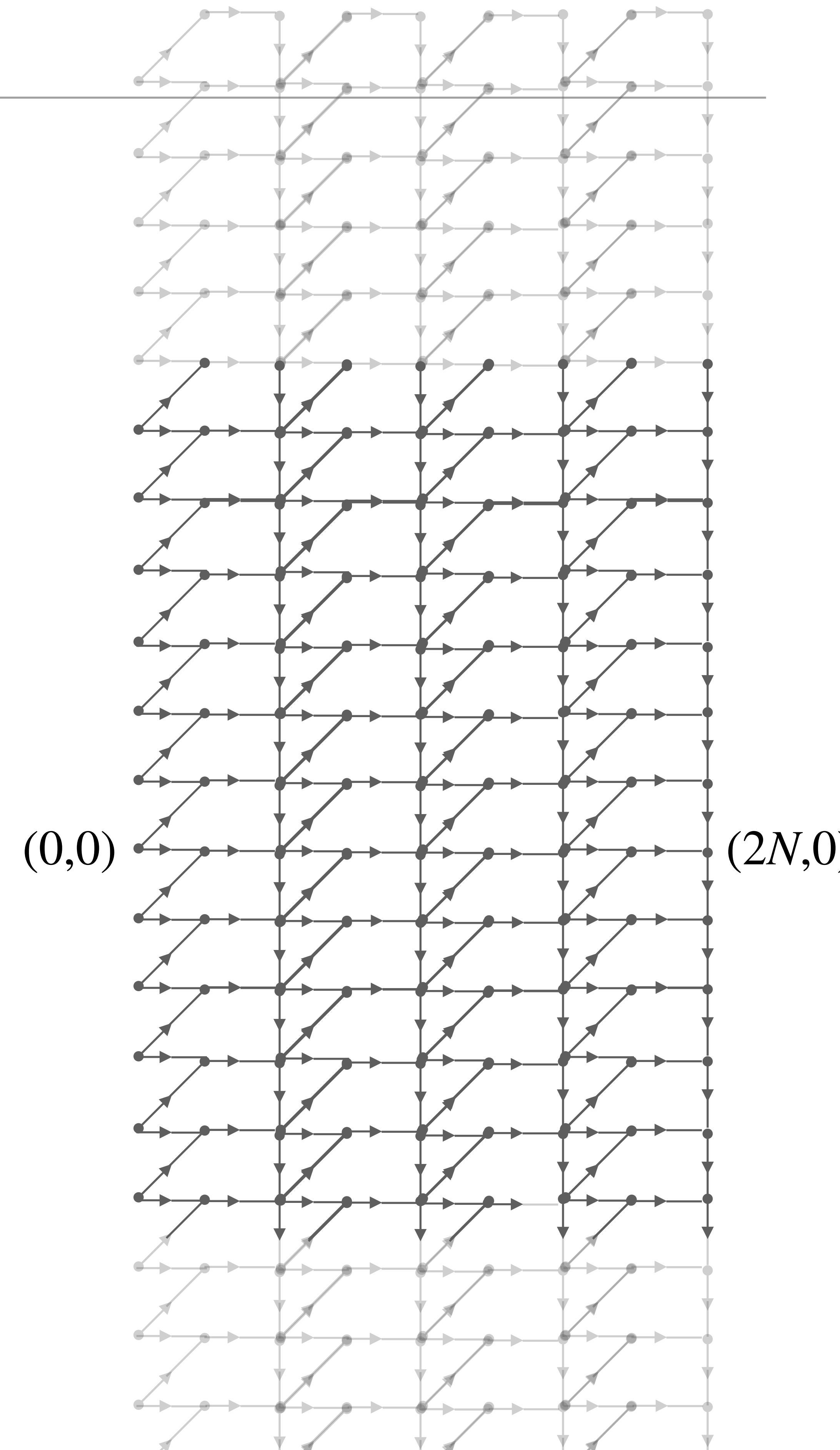
$$\mathscr{V} = \{0,\ldots,2N\} \times \mathbb{Z}$$

and the edges $\mathscr{E}$ are as illustrated in the figure.

- Let $\omega : \mathscr{E} \to (0,\infty)$ be a weight function on the edges of the graph. We will be interested in weights that are doubly periodic:

$$\omega(e + q\tau_1) = \omega(e) = \omega(e + p\tau_2)$$

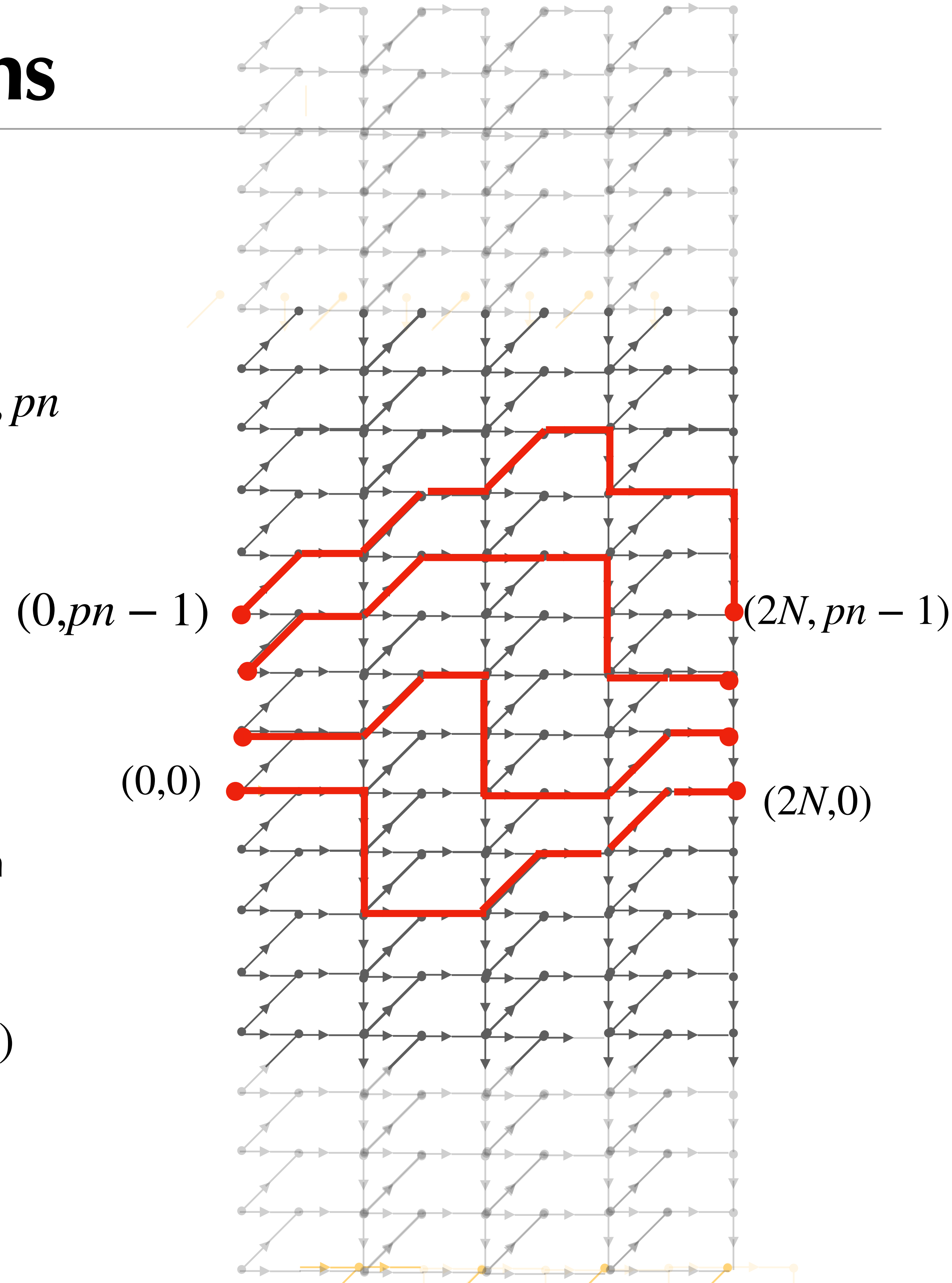
where $\tau_1 = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$ and $\tau_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and p, q positive integers.



Non-intersecting paths

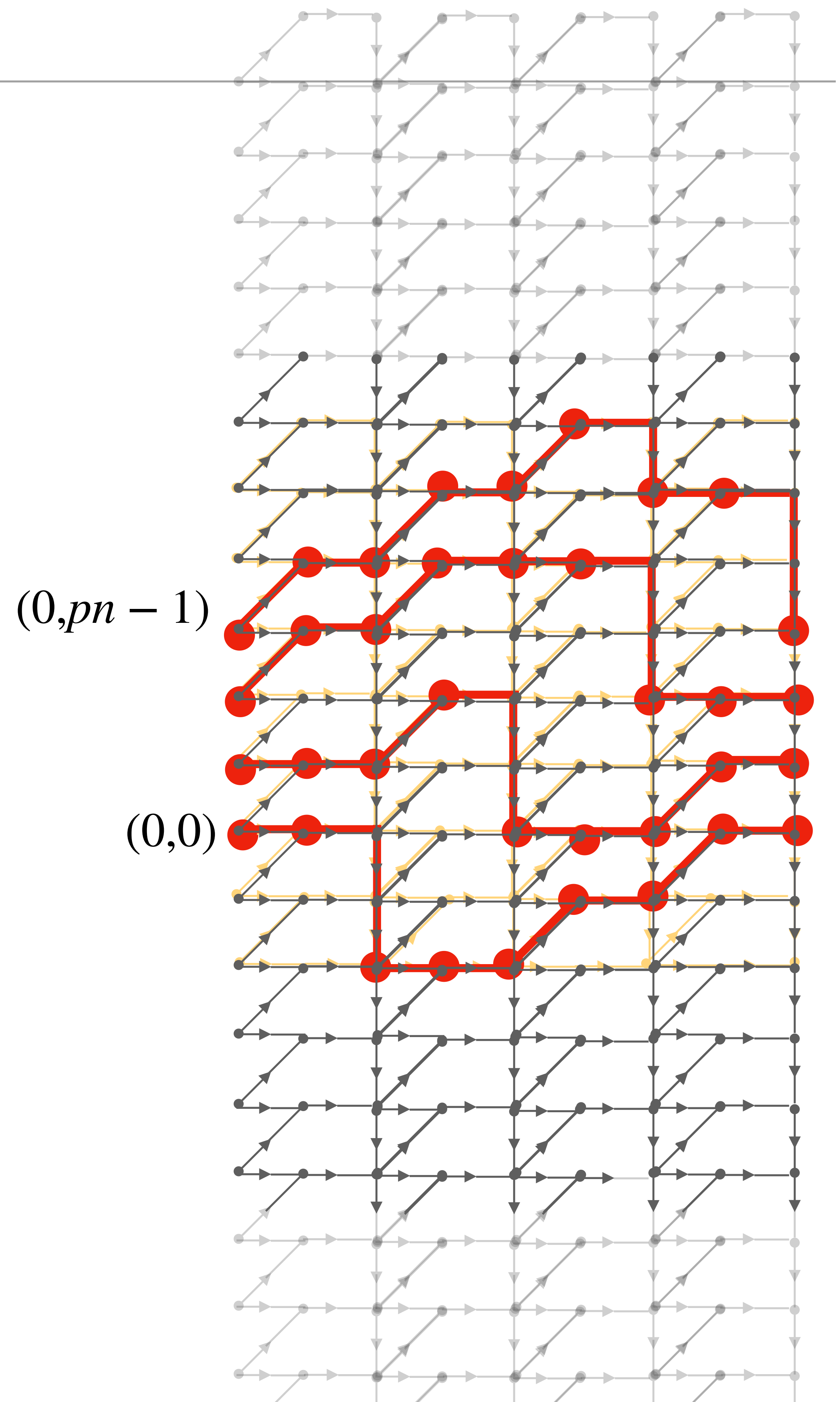
- Let $n \in \mathbb{N}$.
- Define the set $\Pi_{n.i.}$ as the set of all collection of pn paths that for $j = 1,\ldots,pn$
  - start at $(0, j-1)$
  - end at $(2N, j-1)$
  - never intersect
- Then define the probability measure on collections $\Pi_{n.i}$ by

$$\mathbb{P}((\pi_1, \ldots \pi_{pn}) \in \Pi_{n.i}) \sim \prod_{j=1}^{pn} \prod_{e \in \pi_j} \omega(e)$$



A point process

- Let (m, x_j^m) for $j = 1,\ldots,pn$ and $m = 0,\ldots,2N$ be the coordinates of the j-th path
- In case of ambiguity: choose the lowest point for graphs that are directed downwards and the highest point for points that are direct upwards
- These points uniquely determine the paths.

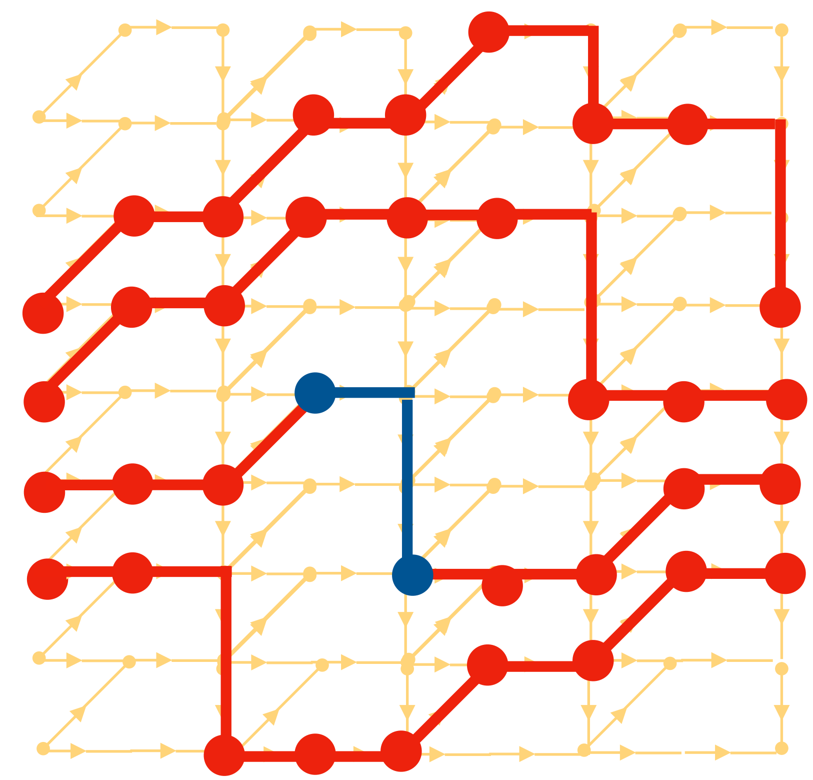


Transition matrices

- For any (m, x) and $(m+1, y)$ in the graph there is a unique path $\pi_m(x, y)$ connecting the two points and we set

$$T_m(x, y) = \prod_{e \in \pi_m(x,y)} \omega(e)$$

The matrices T_m are called ***transition matrices.***



- Periodicity: since ω is doubly periodic we have:
  - Vertical periodicity:

$$T_m(x + p, y + p) = T_m(x, y)$$

  - Horizontal periodicity:

$$T_{m+2q} = T_m$$

Transition matrices

- By the periodicity in the vertical direction, the transition matrices have a block form

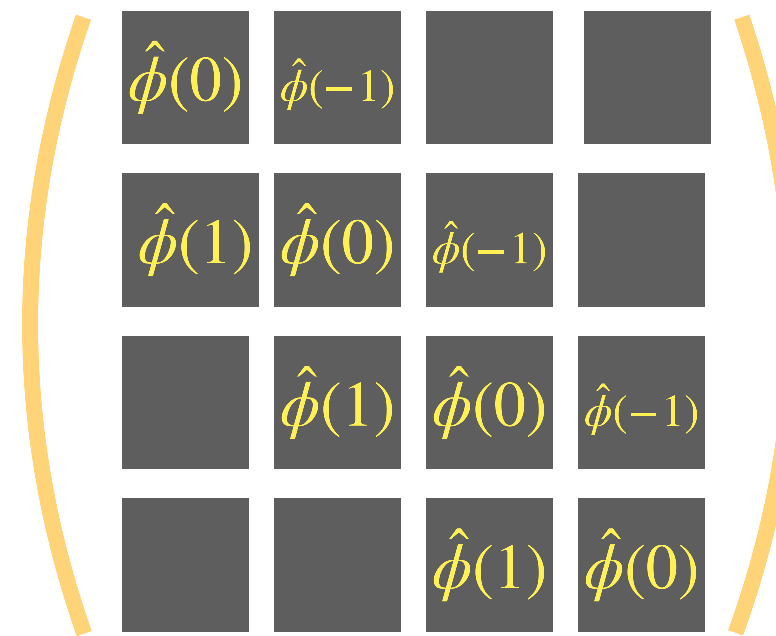
$$T_m(px + r, py + s) = \left(\hat{\phi}_m(x - y)\right)_{r,s}$$

In other words, the transition matrices are ***doubly infinite block Toeplitz matrices***

$$T_m = T(\phi_m)$$

with symbol

$$\phi_m(z) = \sum_x \hat{\phi}_m(x) z^x$$



- Note that for doubly infinite Toeplitz matrices we have

$$T_1 T_2 \cdots T_{2N} = T(\phi_1 \cdots \phi_{2N})$$

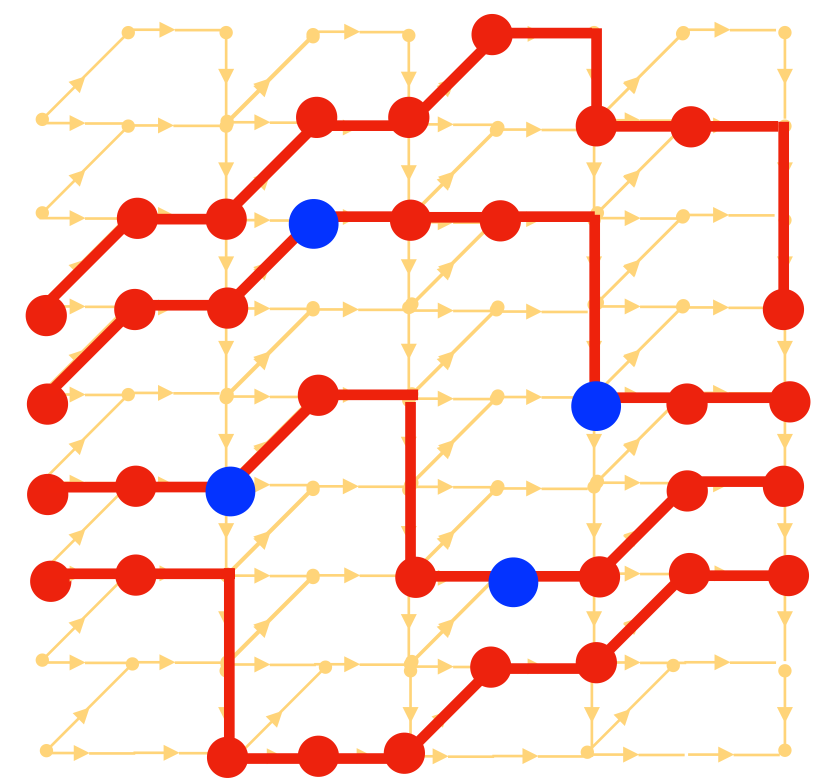
Determinantal correlation functions

- By the LGV theorem, the probability measure for the point process is a product of determinants

$$\mathbb{P}(\vec{\pi} \in \Pi_{n.i.}) \sim \prod_{m=1}^{2N} \det \left(T_m(x_i^m, x_j^{m+1}) \right)_{j,k=1}^{pn}$$

where $T_m = T(\phi_m)$ is a block Toeplitz matrices that is of the above form.

- By the Eynard-Mehta theorem the point process is determinantal, meaning that there exists a function $K_{n,N}$ such that



$$\mathbb{P}(\textit{points at } (m_j, x_j) \textit{ for } j = 1, \ldots, m)$$

$$= \det K_{n,N} \Big((m_j, x_j), (m_k, x_k) \Big)_{j,k=1}^{m}$$

Determinantal correlation functions

- The Eynard-Mehta theorem only gives a general formula for the correlation kernel $K_{n,N}$, involving the inverse of a large $p \times p$ block matrix G of size $n \times n$:

$$G_{jk} = (T_1 \ldots T_{2N})_{k,j} = \frac{1}{2\pi i} \oint \left(\prod_{m=1}^{2N} \phi_m(z) \right) z^{j-k} \frac{dz}{z}, \qquad j,k = 1,\ldots,n$$

  To be able to start an asymptotic analysis we need alternative and more explicit expressions for the correlation kernel. In other words, we need to control G^{-1}

- We recently achieved this with two different approaches:

  1. *Diagonalizing G by orthogonalization.*
     Introduced in D-Kuijlaars '17 and used further in Charlier-D-Kuijlaars-Lenells, Charlier, in the context of lozenge tilings of the hexagon.

  2. *Computing the inverse by a Wiener-hopf factorization (when $n \to \infty$).*
     Studied by Berggren-D '19, Berggren '19, Borodin-D '22

Approach with matrix-valued orthogonal polynomials

Matrix-valued orthogonal polynomials

- In *D-Kuijlaars '17* we showed the correlation kernel $K_{n,N}$ can be expressed using $p \times p$ matrix-valued biorthogonal polynomials.

- For $j, k = 0, \ldots, n-1$ there exist matrix valued polynomials P_j and Q_k of degree j and k respectively such that

$$\frac{1}{2\pi i} \oint_{|z|=1} P_j(z) \frac{\prod_{m=1}^{2N} \phi_m(z)}{z^{n+1}} Q_k(z)^t dz = \delta_{jk} I_p \,.$$

  The orthogonality is non-hermitian.

- We then define the reproducing kernel of order n by

$$R_n(w, z) = \sum_{j=0}^{n-1} Q_j(w)^t P_j(z)$$

Matrix-valued orthogonal polynomials

- The kernel can then be written as

$$\left[K_{n,N}(m, px+j, m', py+i)\right]_{i,j=0}^{p-1} = -\frac{\chi_{m>m'}}{2\pi i}\oint \prod_{k=m'+1}^{m} \phi_k(z)\frac{dz}{z^{x-y+1}}$$

$$+\frac{1}{(2\pi i)^2}\oint\oint\left(\prod_{k=m'+1}^{2N}\phi_k(w)\right)R_n(w,z)\left(\prod_{\ell=1}^{m}\phi_\ell(z)\right)\frac{w^y}{z^{x+1}w^n}dzdw$$

  where $R_n(w,z)$ is the reproducing kernel of order n.

- To study the correlation kernel one would need to perform the two step procedure:

  - study the asymptotic behavior of $R_{n,N}$ as both $n \to \infty$ and $N \to \infty$ simultaneously, for instance using Riemann-Hilbert methods

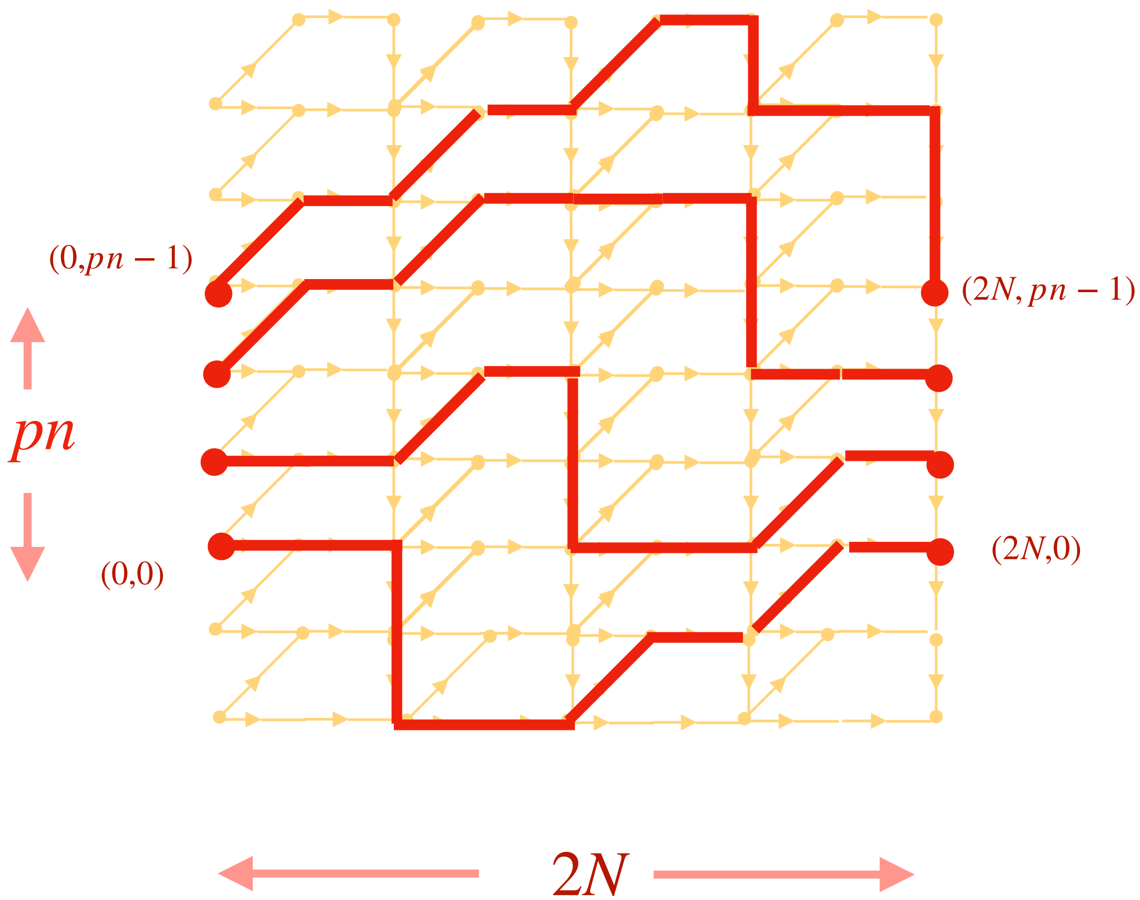
  - Insert this asymptotic expressions into the double integral formula and perform a steepest descent analysis

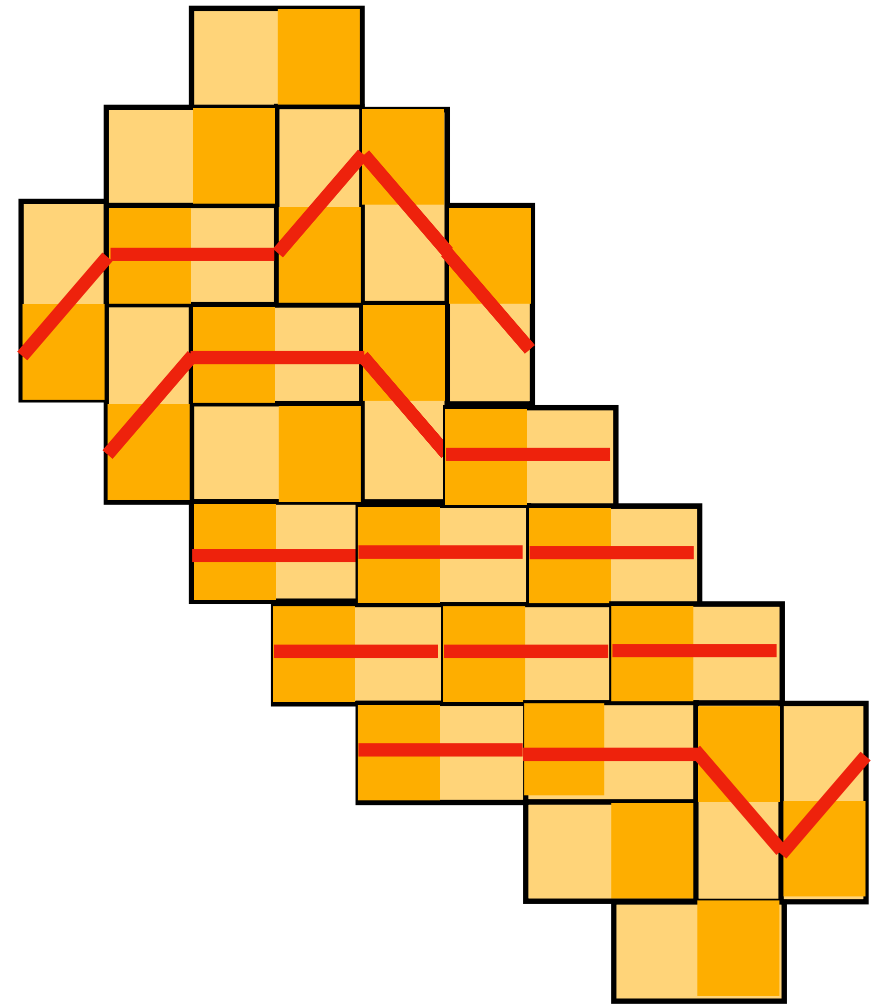
Matrix-valued orthogonal polynomials

- In *D-Kuijlaars '17* we were able to perform an asymptotic analysis for the two-periodic Aztec diamond and reproduce and extend the results of *Chhita-Johansson '14*.
- An important feature is that the matrix-valued orthogonal polynomials were simple enough to have closed expressions for finite n
- The approach can also be applied for doubly periodic lozenge tilings of the hexagon. In that case, the polynomials do not simplify, but one can carry out the above program. *Charlier-D-Kuijlaars-Lenells '19, Charlier '20, '20*

Approach with Wiener-Hopf factorization in the limit $n \to \infty$

Taking the limit $n \to \infty$





- One can take the limit $n \to \infty$, before studying $N \to \infty$
- In that limit many things simplify.

Correlation kernel — Factorizations

Suppose that the symbol

$$\phi(z) = \prod_{m=1}^{2N} \phi_m(z)$$

has two Wiener-hopf factorizations

$$\phi(z) = \phi_+(z)\phi_-(z) \qquad \text{and} \qquad \phi(z) = \tilde{\phi}_-(z)\tilde{\phi}_+(z)$$

where

$$\phi_+^{\pm 1}(z), \tilde{\phi}_+^{\pm 1}(z) \text{ are an analytic in } |z| < 1$$

$$\phi_-^{\pm 1}(z), \tilde{\phi}_-^{\pm 1}(z) \text{ are analytic in } |z| > 1$$

$$\tilde{\phi}_-(z), \phi_-(z) \sim I \text{ as } z \to \infty$$

Limiting process — Bottom paths

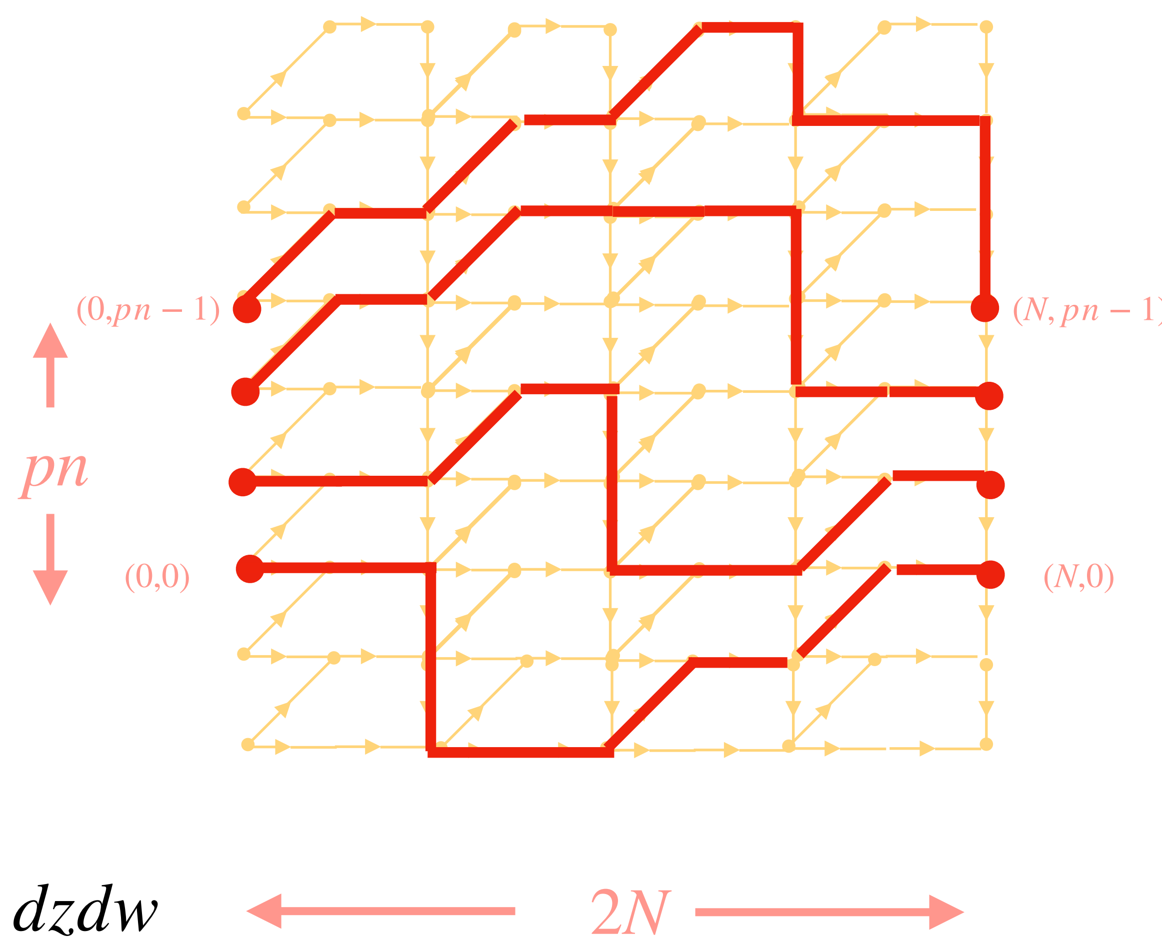
Theorem (*Berggren-D '19*)

Limit at the bottom:

$$\lim_{n\to\infty}\left(K_{n,N}((m_1,px_1+r),(m_2,px_2+s))\right)_{r,s=1}^{p}$$

$$=\frac{\chi_{m_1>m_2}}{2\pi i}\oint\prod_{j=m_2}^{m_2}\phi_j(z)\,\frac{dz}{z^{x_1-x_2+1}}$$

$$-\frac{1}{(2\pi i)^2}\iint_{|z|<|w|}\prod_{j=m_2}^{2N}\phi_j(w)\phi_-^{-1}(w)\phi_+^{-1}(z)\prod_{j=1}^{m_1}\phi_j(z)\frac{w^{x_2}}{z^{x_1+1}}\frac{dzdw}{z-w}$$



In the scalar case $p=1$ one recovers known expressions for the Schur process *Okounkov '01, Okounkov-Reshitikhin '03*

Limiting process — Top paths

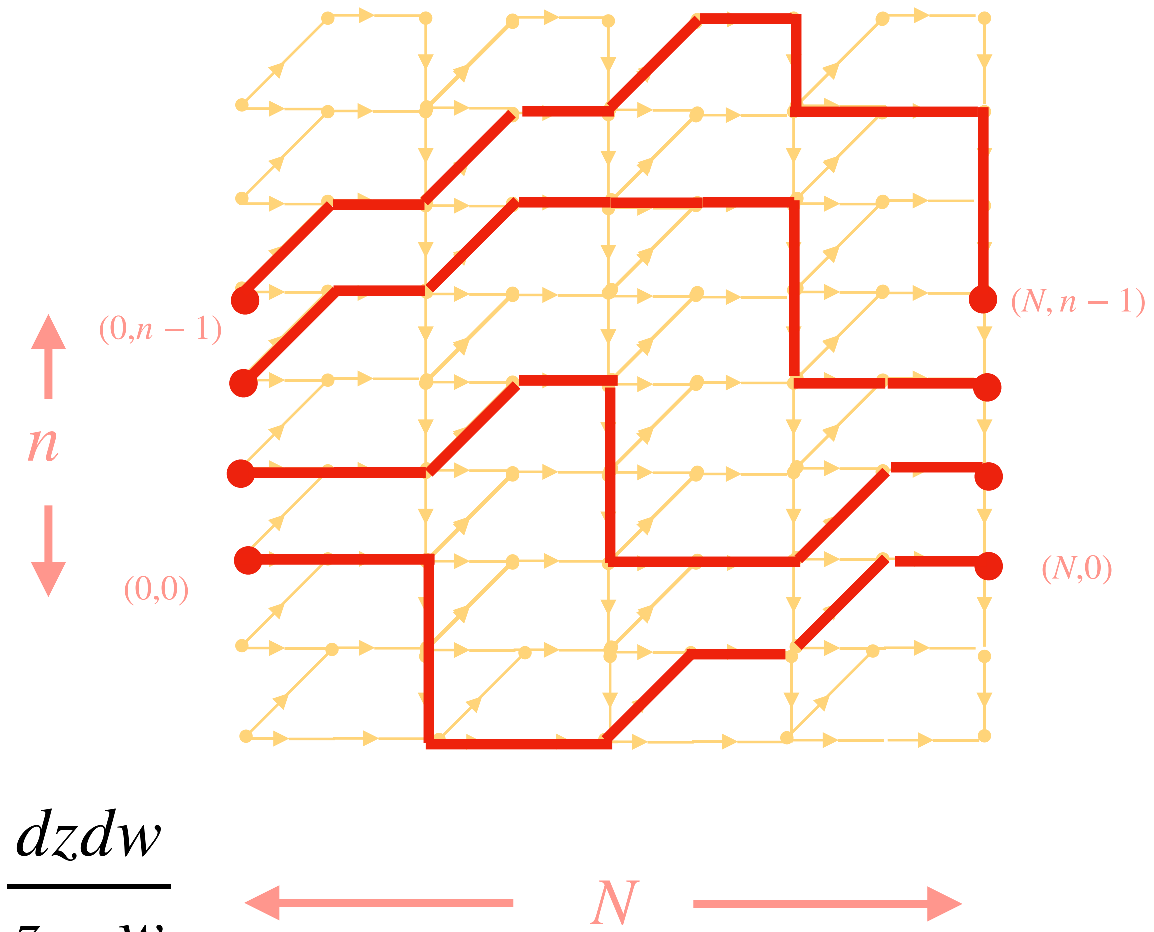
<u>Theorem (*Berggren-D '19*)</u>

Limit at the top::

$$\lim_{n\to\infty} \left(K_{n,N}((n+m_1, px_1+r),(n+m_2,px_2+s)\right)_{r,s=1}^{p}$$

$$= \frac{\chi_{m_1>m_2}}{2\pi i}\oint \prod_{j=m_2}^{m_2} \phi_j(z)\ \frac{dz}{z^{x_1-x_2+1}}$$

$$-\frac{1}{(2\pi i)^2}\iint_{|z|<|w|} \prod_{j=m_2}^{2N} \phi_j(w)\tilde{\phi}_{+}^{-1}(w)\tilde{\phi}_{\cdot}^{-1}(z)\prod_{j=1}^{m_1}\phi_j(z)\frac{w^{x_2}}{z^{x_1+1}}\frac{dzdw}{z-w}$$



Scalar case $p = 1$ — Schur process

- In the scalar case , the factorization problem is absolutely trivial, since each $\phi_m(z)$ and its inverse are either analytic inside or outside. To get the factorizations one simply reorders the terms:

- Example: to factorize

$$\phi(z) = \phi_1(z)\phi_2(z)\phi_3(z)\phi_4(z)\phi_5(z)\phi_6(z)$$

we first identify the terms analytic inside in red and analytic outside in blue:

$$\phi(z) = \phi_1(z)\phi_2(z)\phi_3(z)\phi_4(z)\phi_5(z)\phi_6(z)$$

then we reorder the terms

$$\phi(z) = \phi_1(z)\phi_3(z)\phi_4(z)\phi_2(z)\phi_5(z)\phi_6(z) = \phi_2(z)\phi_5(z)\phi_6(z)\phi_1(z)\phi_3(z)\phi_4(z)$$

the product in blue is $\phi_- = \tilde{\phi}_-$ and the product in red is $\phi_+ = \tilde{\phi}_+$

Block case —

- In the scalar case, the factorization problem is far from non-trivial because of lack of commutativity!

$$\phi_j(z)\phi_k(z) \neq \phi_k(z)\phi_j(z)$$

  One can no longer simply reorder terms!

- Recall that each transition matrix $T_m = T(\phi_m)$ has its own parameters $\phi_m(z) = \phi_m(z; \mathbf{a})$.

  There exists explicit but elaborate maps

$$(\mathbf{a}_j, \mathbf{a}_k) \mapsto (\mathbf{a}'_j, \mathbf{a}'_k)$$

  such that

$$\phi_j(z; \mathbf{a}_j)\phi_k(z; \mathbf{a}_k) = \phi_k(z; \mathbf{a}'_k,)\phi_j(z; \mathbf{a}'_j)$$

  and the ϕ_j, ϕ'_j s and ϕ_k, ϕ'_k have the same analyticity properties. *(Lam-Pylyavskyy '12, Kirillov '00, Etingof '03, Kashiwara-Nakashima-Okado '10)*

Doubly periodic weights

- One may hope that for ***doubly*** periodic weights some simplification arises. Recall that our $p \times p$ matrix valued symbols ϕ_m satisfy:

  $$\phi_{m+2q} = \phi_m$$

  We will for simplicity also assume that the width is $2qN$

- Then

  $$\phi(z) = \prod_{m=1}^{2qN} \phi_m(z) = (A(z))^N \qquad \text{with} \qquad A(z) = \phi_1(z) \cdots \phi_{2q}(z)$$

Doubly periodic weights

- Suppose we can find a factorization

$$A(z) = A_+(z)A_-(z)$$

Then

$$\phi(z) = (A(z))^N = (A_+(z)\ A_-(z))^N = A_+(z)\ (A_-(z)\ A_+(z))^{N-1}\ A_-(z)$$
$$= A_+(z)\ (\tilde{A}(z))^{N-1}\ A_-(z)$$

Factorize $A(z) = A_+(z)A_-(z)$

then compute $\tilde{A}(z) = A_-(z)A_+(z)$

…and continue find a factorization of $\tilde{A}(z)$ …. and so on..

Wiener-Hopf factorization via a flow on matrices

- Our weight is $\phi(z) = (A(z))^N$
- Algorithm:
  - Set $A^{(0)} = A(z)$
  - At step j compute a factorization

    $$A^{(j)}(z) = A_+^{(j)}(z)A_-^{(j)}(z)$$

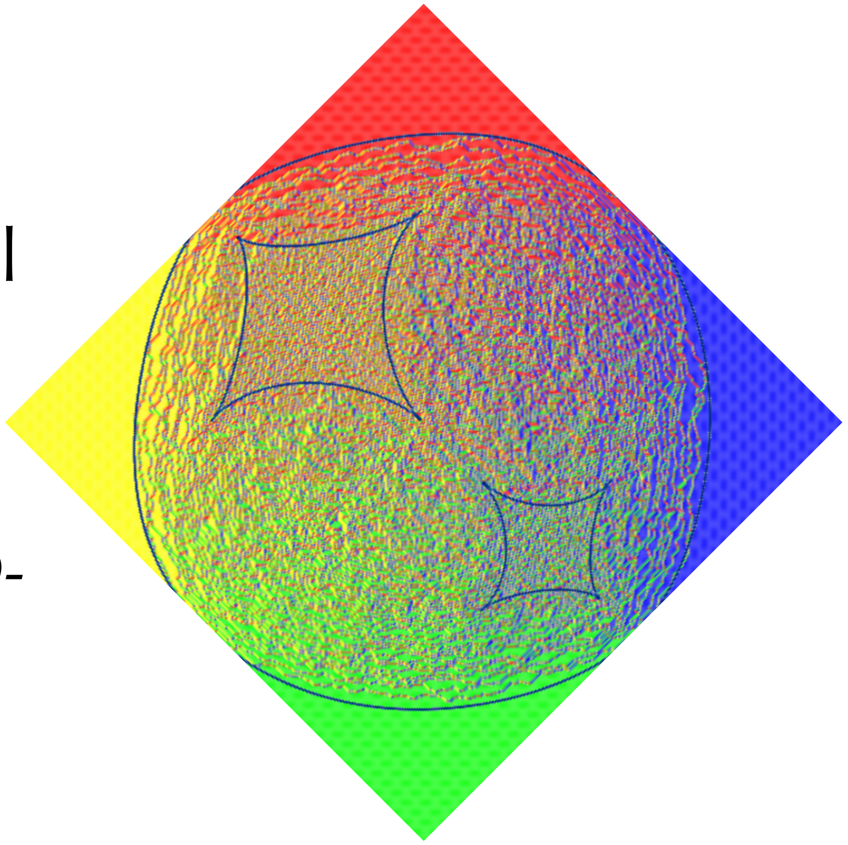
    and set

    $$A^{(j+1)}(z) = A_-^{(j)}(z)A_+^{(j)}(z).$$
- Then the factorization that we want is

$$\phi_+(z) = A_+^{(0)}(z)\cdots A_+^{(N-1)}(z) \quad \text{and} \quad \phi_-(z) = A_-^{(N-1)}(z)\cdots A_-^{(0)}(z)$$

Traceable flows

- It happens in various cases that this flow can be traced and is periodic, ie $A^{(T)} = A^{(0)}$ for some T. In that case, one obtains an explicit double integral formula that can be studied by a steepest descent analysis:



  - The *unbiased* 2×2 periodic Aztec diamond *D-Berggren '19*. One recovers results by *D-Kuijlaars '17, Chhita-Johansson '16*
  - Various examples, including lozenge tilings, *D-Berggren '19*
  - *A family of* $2 \times q$ periodic weights for the Aztec diamond — *Berggren '19*

- For the biased Aztec diamond with $a < 1$, the flow **can be linearized.** *Borodin-D-'22*