GOE fluctuations for the maximum of the top path in alternating sign matrices

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April 2022.

Based on joint work with Arvind Ayyer (ICTS Bangalore) and Kurt Johansson (KTH).

Overview

- Some history on Alternating Sign Matrices/6 vertex model with domain wall boundary conditions
- Random Domino Tilings on the Aztec diamond,
- F_1 and F_2 Tracy Widom Distributions
- Large ASMs, formulation of theorem
- Sketch of the proof of the main theorem

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- Is there a formula for $\det_\lambda M$?

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Here are the seven alternating sign matrices of size 3:

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For A in ASM(n), let $n_-(A)$ be the number of -1's in A and $\mathrm{inv}(A) = \sum_{i<k, j>l} A_{i,j} A_{k,l}$ be the number of inversions of A. Robbins-Rumsey (1986) showed that for $M = (m_{i,j})_{i,j=1}^n$

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Question: how many ASMs of size n are there?

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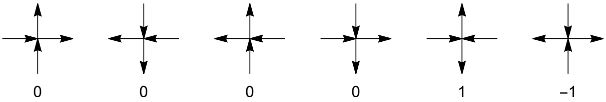

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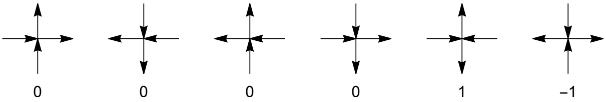

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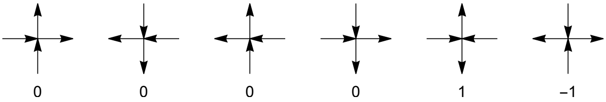

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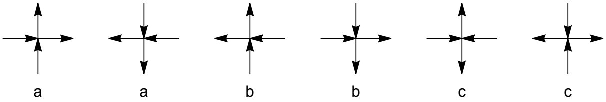

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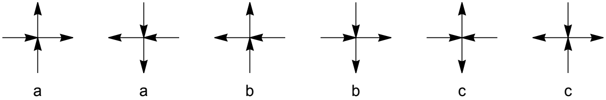

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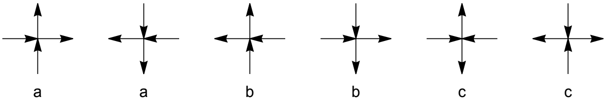

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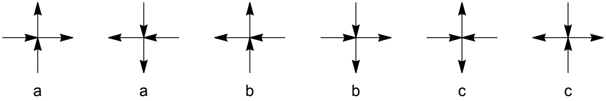

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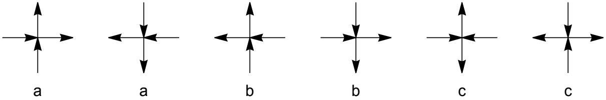

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- In this case, Lieb in 1967 computed the free energy.

Some History on ASMs: Combinatorics

- Mills-Robbins-Rumsey (1986) conjectured the number of ASMs is $\prod_{i=0}^{n-1} \frac{(3i+1)!}{(n+i)!} = 1, 2, 7, 42, 429, \ldots$

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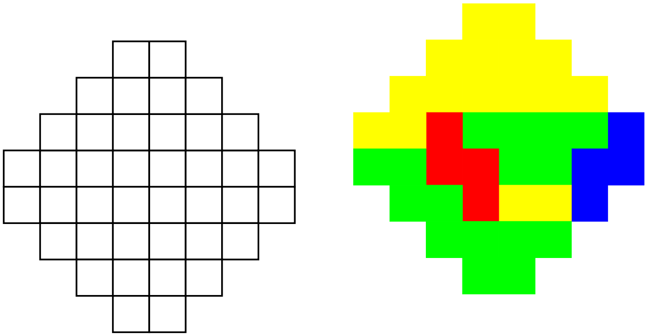
- There is no bijective proof between TSSCPPs and ASMs that preserve statistics!
- However, Fischer-Konvalinka (2019-20) gave a complicated bijective proof between ASMs and Descending plane partitions. This is highly nontrivial.

Random tilings

Random domino tilings of the Aztec diamond provide a good source for intuition for large uniformly random ASMs.



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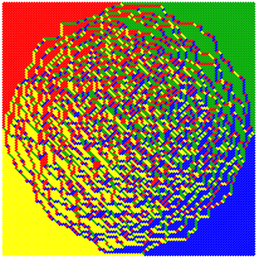

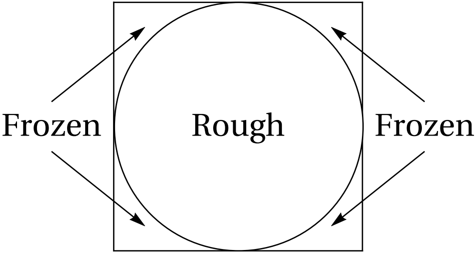

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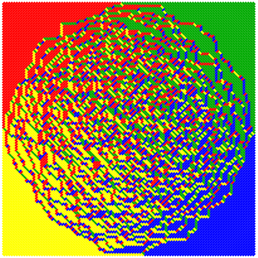

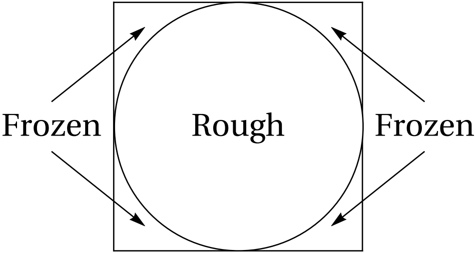

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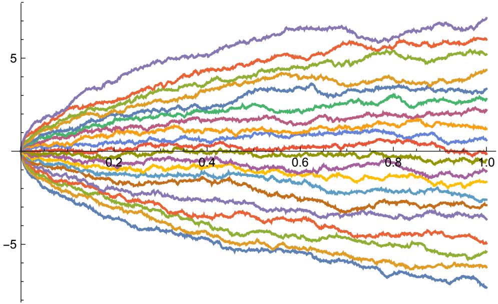
- At the tangency points Johansson-Nordenstam (2006) showed that the GUE corner process appears at the tangency points.
- General results on limit shapes due to Cohn-Kenyon-Propp (2000), Kenyon-Okounkov (2007), and Astala-Duse-Prause-Zhong (2020+).

Airy process, F_1 and F_2

Consider N independent Brownian motions, starting from the origin and ending after time N that are conditioned not to intersect.



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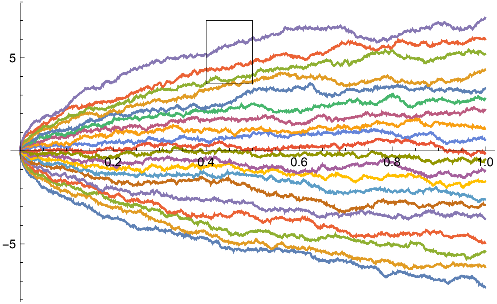

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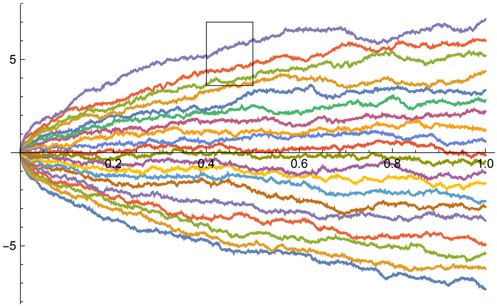

Rescale (in time and space) away from $0 < t < N$, top path converges to the Airy-2-process, A, as $N \to \infty$. Here, at one-time point, the top path has Tracy-Widom GUE Fluctuations, F_2.

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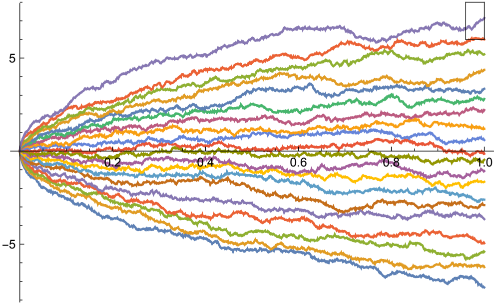

At $t = N$, after suitable centering and rescaling, the top path has Tracy Widom GOE fluctuations, F_1.

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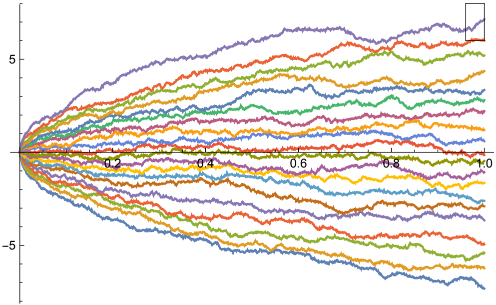

At $t = N$, after suitable centering and rescaling, the top path has Tracy Widom GOE fluctuations, F_1.
In fact, Johansson (2003) showed that $\max A(t) - t^2$ has distribution F_1.

Universality?

These random matrix theory limit laws appear in many other models analyzed by using *determinantal point processes*. Some examples include

- $\Delta = 0$ six-vertex model with domain wall boundary conditions (equivalent to uniformly random domino tilings of the Aztec diamond)
- Directed last passage percolation in 2D with geometric weights
- Polynuclear Growth Models
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Huge amount of progress on understanding the algebraic structure by Aggarwal, Borodin, Bufetov, Corwin, Gorin, Petrov, Wheeler,....

Back to ASMs

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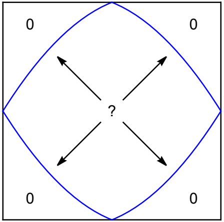

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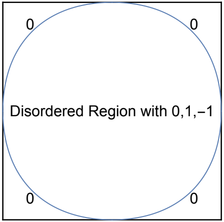

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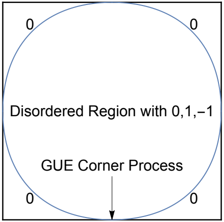

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Heights from ASMs

We want to give a good description of the boundary.

- For each ASM of size $n+1$, $A = (a_{i,j})_{1\le i,j\le n+1}$ construct a new matrix $C = (c_{i,j})_{1\le i,j\le n}$ by

$$c_{i,j} = n - \sum_{\substack{1\le r\le i \\ 1\le s\le n+1-j}} a_{r,s}, \quad 1 \le i, j \le n.$$

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$$c_{i,j} = n - \sum_{\substack{1 \le r \le i \\ 1 \le s \le n+1-j}} a_{r,s}, \quad 1 \le i,j \le n.$$

- The matrices of size 2 are

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$$

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Heights from ASMs

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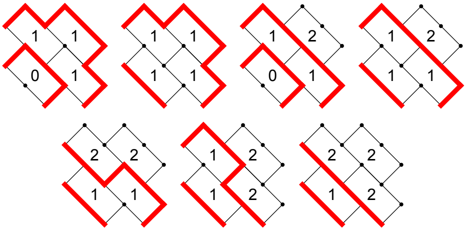
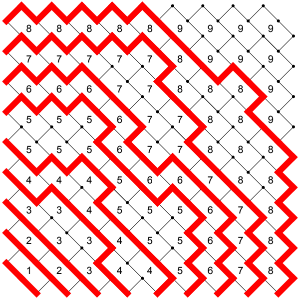


Maximum of the Top Path

Consider the level lines of the heights.



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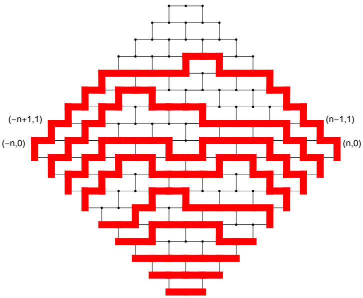

Remove the entries of the matrix and rotate by $\pi/4$ counterclockwise. The x-coordinate marks time and the y-coordinate marks the height, with the highest leftmost and rightmost vertices having coordinates $(-n, 0)$ and $(n, 0)$.

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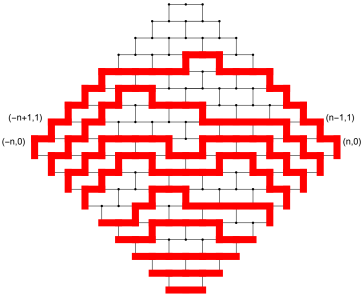

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$$T_n = (T_n(-n), \dots, T_n(-1), T_n(0), T_n(1), \dots, T_n(n)).$$

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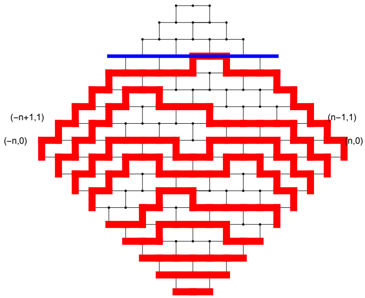

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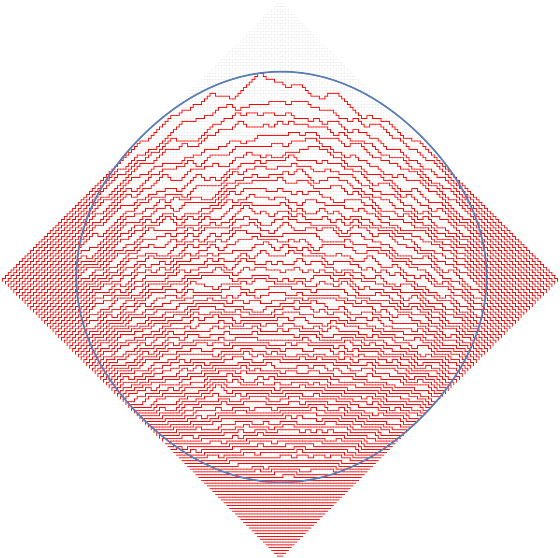
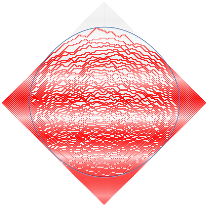
Our theorem concerns fluctuations of $\max T_n$.

Simulation



Main Theorem



Introduce the constants $\alpha = 2 - \sqrt{3}$ and $c_0 = \frac{1}{2 \cdot 3^{1/6}}$. Let F_1 and F_2 be the GOE and GUE Tracy-Widom distributions.

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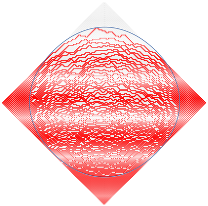

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Theorem (Ayyer-C.-Johansson (2021+))

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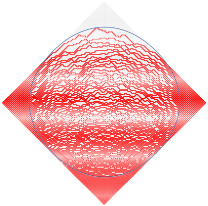

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Conjecture (Ayyer-C.-Johansson (2021+))

After rescaling, T_n converges to the Airy-2-process. In particular

$$\lim_{n\to\infty} \mathbb{P}\left[\frac{T_n(0) - (1-\alpha)n}{4^{\frac{1}{3}} c_0 n^{\frac{1}{3}}} \leq s\right] = F_2(s).$$

GOE kernel

- Let $\mathrm{Ai}(x)$ denote the Airy function, that is,

$$\mathrm{Ai}(x) = \frac{1}{\pi} \int_0^\infty \mathrm{d}t \, \cos\left(\frac{t^3}{3} + xt\right),$$

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- Introduce the following 2 by 2 block kernel

$$\mathbf{K}_{\mathrm{GOE}}(x,y) = \begin{pmatrix} K_{\mathrm{GOE}}^{11}(x,y) & K_{\mathrm{GOE}}^{12}(x,y) \\ K_{\mathrm{GOE}}^{21}(x,y) & K_{\mathrm{GOE}}^{22}(x,y) \end{pmatrix}$$

GOE block kernel

$$K^{11}_{\mathrm{GOE}}(x,y) = \frac{1}{4}\int_0^\infty \mathrm{d}\lambda\ \left(\mathrm{Ai}(x+\lambda)\mathrm{Ai}'(y+\lambda) - \mathrm{Ai}'(x+\lambda)\mathrm{Ai}(y+\lambda)\right),$$

$$K^{12}_{\mathrm{GOE}}(x,y) = \int_0^\infty \mathrm{d}\lambda\ \mathrm{Ai}(x+\lambda)\mathrm{Ai}(y+\lambda) + \frac{1}{2}\mathrm{Ai}(x)\int_0^\infty \mathrm{d}\lambda\ \mathrm{Ai}(y-\lambda)$$

$$K^{21}_{\mathrm{GOE}}(x,y) = -K^{12}_{\mathrm{GOE}}(y,x),$$

$$K^{22}_{\mathrm{GOE}}(x,y) = \int_0^\infty \mathrm{d}\lambda \int_\lambda^\infty \mathrm{d}\mu\ \mathrm{Ai}(x+\lambda)\mathrm{Ai}(y+\mu) - \mathrm{Ai}(x+\mu)\mathrm{Ai}(y+\lambda)$$
$$- \int_0^\infty \mathrm{d}\mu\ \mathrm{Ai}(x+\mu) + \int_0^\infty \mathrm{d}\mu\ \mathrm{Ai}(y+\mu) - \mathrm{sgn}(x-y).$$

Fredholm Pfaffian

- The Pfaffian of a $2k \times 2k$ anti-symmetric matrix A is given by

$$\mathrm{Pf}(A) = \frac{1}{2^k k!} \sum_{\sigma \in \mathcal{S}_{2k}} \mathrm{sgn}(\sigma) A_{\sigma(1),\sigma(2)} \cdots A_{\sigma(2k-1),\sigma(2k)},$$

where $\mathcal{S}_{2k}$ is the set of permutations of $\{1, \dots, 2k\}$.

- The GOE Tracy–Widom distribution is defined through a Fredholm Pfaffian by

$$\begin{aligned} F_1(s) =& \mathrm{Pf}(\mathbb{J} - \mathbf{K}_{\mathrm{GOE}})_{L^2(s,\infty)} \\ =& 1 + \sum_{k=1}^{\infty} \frac{(-1)^k}{k!} \int_s^{\infty} \mathrm{d}x_1 \cdots \int_s^{\infty} \mathrm{d}x_k \, \mathrm{Pf}(\mathbf{K}_{\mathrm{GOE}}(x_i, x_j))_{1 \le i,j \le k}, \end{aligned}$$

where

$$\mathbb{J}(x,y) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \mathbb{I}_{x=y}.$$

Overview of the proof

Unfortunately, there are no amenable formulas for ASMs. Colomo-Pronko had a series of works (including those with Di Giulio (2021), Cantini (2019), Noferini(2010), Zinn-Justin(2010)) on the "emptiness formulation probability" but these formulas are difficult to analyze. Fischer (2018) also has similar formulas.

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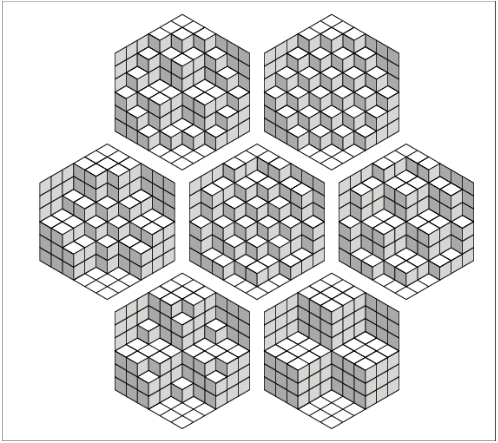
- Use Zeilberger's results to relate statistics of ASMs to statistics of TSSCPPs.
- Use formulas in Ayyer-C. (2021) to analyze the relevant events in TSSCPPs.
- Perform an asymptotic analysis to obtain the result.

TSSCPPs

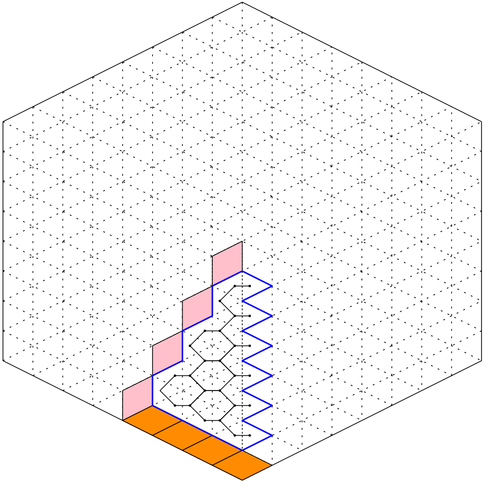
A totally symmetric self complementary plane partition is a boxed plane partition with maximum symmetry.



Credit: Bressoud's book Andrews (1994) computed the number of TSSCPPs of size *n*.

TSSCPPs to dimers

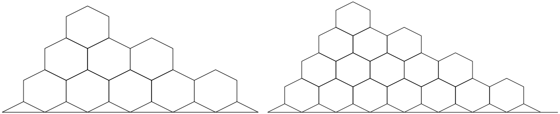
Since TSSCPPs, we only need a twelfth of the hexagon.



On the dual graph, we can map the tiles to edges (dimers). The right boundary is free. For dimers, this turns out to be equivalent to adding a triangle.

Dimers

The dimer graph is now



for size 3 and 4. Each TSSCPP configuration is equivalent to a dimer covering on the dimer-graph, that is subset of edges so that each vertex is covered exactly once by a dimer on the dimer graph.

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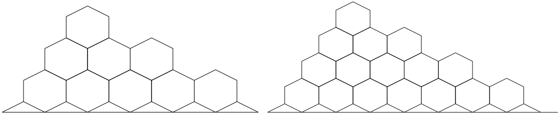

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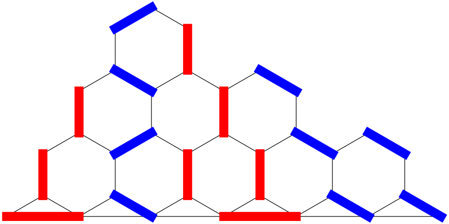

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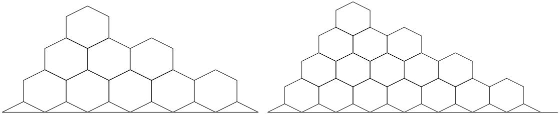

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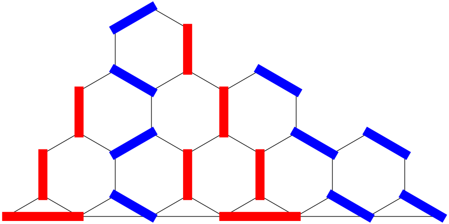

Another interpretation: non-intersecting paths with a free boundary. Mapping between dimers to paths drawn in blue.

Zeilberger's result in our language

Zeilberger found that the number of triangles in bijection with ASMs (Gogs) is equal to the number of triangles in TSSCPPs (Magogs).

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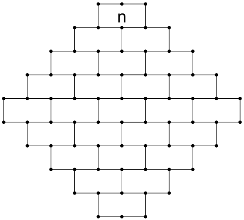

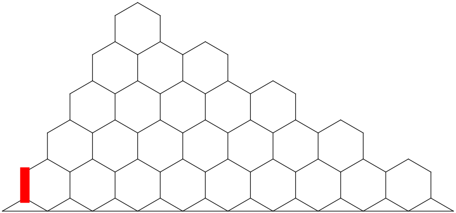

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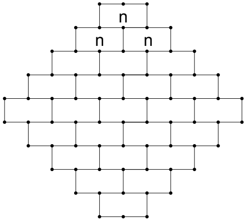

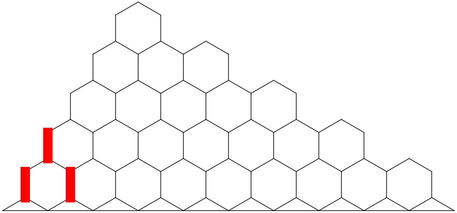

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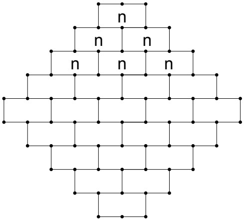

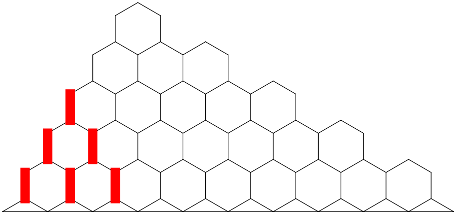

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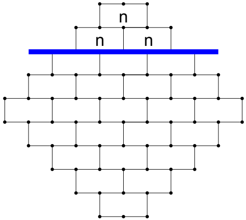

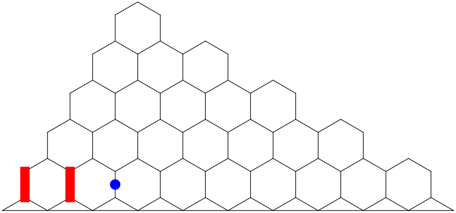

We can relate the maximum of the top path to the position of the leftmost vertical edge not covered by a dimer.

Fluctuations of the leftmost non-vertical dimer

From the previous slide, the number of ASMs where the maximum of the top path is k from the top is the same as the number of the leftmost non-vertical dimer on the bottom row being at site k in the TSSCPPs.

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Thanks for your attention!