Correlation functions of the XXZ open spin chain with unparallel boundary fields

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The open XXZ chain with boundary fields

$$H^{\text{open}}_{\text{XXZ}} = \sum_{m=1}^{N-1} \left\{ \sigma^x_m \sigma^x_{m+1} + \sigma^y_m \sigma^y_{m+1} + \Delta\, \sigma^z_m \sigma^z_{m+1} \right\} + h^x_- \sigma^x_1 + h^y_- \sigma^y_1 + h^z_- \sigma^z_1 + h^x_+ \sigma^x_N + h^y_+ \sigma^y_N + h^z_+ \sigma^z_N$$

- space of states: $\mathcal{H} = \otimes_{n=1}^N \mathcal{H}_n$ with $\mathcal{H}_n \simeq \mathbb{C}^2$
- $\sigma_m^{x,y,z} \in \mathrm{End}(\mathcal{H}_n)$: local spin-1/2 operators (Pauli matrices) at site m
- anisotropy parameter $\Delta = \cosh \eta$
- boundary fields $h_\pm^{x,y,z}$ parametrised in terms of 6 boundary parameters $\varsigma_\pm, \kappa_\pm, \tau_\pm$, or alternatively $\varphi_\pm, \psi_\pm, \tau_\pm$:

$$h^x_\pm = 2\kappa_\pm \sinh\eta \frac{\cosh\tau_\pm}{\sinh\varsigma_\pm}, \quad h^y_\pm = 2i\kappa_\pm \sinh\eta \frac{\sinh\tau_\pm}{\sinh\varsigma_\pm}, \quad h^z_\pm = \sinh\eta \coth\varsigma_\pm$$

$$\sinh\varphi_\pm \cosh\psi_\pm = \frac{\sinh\varsigma_\pm}{2\kappa_\pm}, \quad \cosh\varphi_\pm \sinh\psi_\pm = \frac{\cosh\varsigma_\pm}{2\kappa_\pm}$$

The open XXZ chain with boundary fields

$$H_{\mathrm{XXZ}}^{\mathrm{open}} = \sum_{m=1}^{N-1} \left\{ \sigma_m^x \sigma_{m+1}^x + \sigma_m^y \sigma_{m+1}^y + \Delta\, \sigma_m^z \sigma_{m+1}^z \right\}$$
$$+ h_-^x \sigma_1^x + h_-^y \sigma_1^y + h_-^z \sigma_1^z + h_+^x \sigma_N^x + h_+^y \sigma_N^y + h_+^z \sigma_N^z$$

- space of states: $\mathcal{H} = \otimes_{n=1}^N \mathcal{H}_n$ with $\mathcal{H}_n \simeq \mathbb{C}^2$
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$$h_\pm^x = 2\kappa_\pm \sinh \eta \, \frac{\cosh \tau_\pm}{\sinh \varsigma_\pm}, \quad h_\pm^y = 2i\kappa_\pm \sinh \eta \, \frac{\sinh \tau_\pm}{\sinh \varsigma_\pm}, \quad h_\pm^z = \sinh \eta \, \coth \varsigma_\pm$$
$$\sinh \varphi_\pm \, \cosh \psi_\pm = \frac{\sinh \varsigma_\pm}{2\kappa_\pm}, \quad \cosh \varphi_\pm \, \sinh \psi_\pm = \frac{\cosh \varsigma_\pm}{2\kappa_\pm}$$

Remark: Invariance of the Hamiltonian under the changes

- $\{\eta, \varsigma_\pm\} \to \{-\eta, -\varsigma_\pm\}$
- $\begin{cases} n \to N - n + 1, \quad 1 \le n \le N, \\ \{\varsigma_\pm, \kappa_\pm, \tau_\pm\} \to \{\varsigma_\mp, \kappa_\mp, \tau_\mp\}. \end{cases}$

The open XXZ chain with boundary fields

$$H_{\mathrm{XXZ}}^{\mathrm{open}} = \sum_{m=1}^{N-1} \left\{ \sigma_m^x \sigma_{m+1}^x + \sigma_m^y \sigma_{m+1}^y + \Delta\, \sigma_m^z \sigma_{m+1}^z \right\} + h_-^x \sigma_1^x + h_-^y \sigma_1^y + h_-^z \sigma_1^z + h_+^x \sigma_N^x + h_+^y \sigma_N^y + h_+^z \sigma_N^z$$

- space of states: $\mathcal{H} = \otimes_{n=1}^N \mathcal{H}_n$ with $\mathcal{H}_n \simeq \mathbb{C}^2$
- $\sigma_m^{x,y,z} \in \mathrm{End}(\mathcal{H}_n)$: local spin-1/2 operators (Pauli matrices) at site m
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Question: Correlation functions $\langle \prod_{j=1}^m \sigma_j^{\alpha_j} \rangle = \langle\, G.S. \,|\, \prod_{j=1}^m \sigma_j^{\alpha_j} \,|\, G.S. \,\rangle$ at zero temperature ?

$\rightsquigarrow$ $\exists$ exact formulas for $h_-^{x,y} = h_+^{x,y} = 0$ [Jimbo et al. 95; Kitanine et al. 07] (multiple integral representations in the half-infinite chain limit)

$\rightsquigarrow$ generalize these formulas to a special case of unparallel boundary fields [Niccoli, VT 22] :

- $h_-^{x,y,z}$ arbitrary
- $h_+^{x,y} = 0$ and h_+^z fixed to a specific value

A reminder of the periodic case (1)

- The periodic XXZ chain is solvable in the framework of the Quantum Inverse Scattering Method (QISM) [Faddeev, Sklyanin, Takhtajan, 1979]
  - $\rightsquigarrow$ solution based on the representation theory of the Yang-Baxter algebra:
    - generators: elements of the monodromy matrix $T(\lambda) = \begin{pmatrix} A(\lambda) & B(\lambda) \\ C(\lambda) & D(\lambda) \end{pmatrix}$
    - commutation relations given by the R-matrix of the model:

      $$R(\lambda-\mu)\,\big(T(\lambda)\otimes 1\big)\big(1\otimes T(\mu)\big) = \big(1\otimes T(\mu)\big)\big(T(\lambda)\otimes 1\big)\,R(\lambda-\mu)$$

    - abelian subalgebra generated by the transfer matrix $t(\lambda) = \mathrm{tr}\, T(\lambda)$ such that $[H, t(\lambda)] = 0$

- The eigenstates of the transfer matrix $t(\lambda)$ (and of the Hamiltonian) are constructed by means of ABA as Bethe states:

  $$|\,\{\lambda\}\,\rangle = \prod_{k=1}^{n} B(\lambda_k)|\,0\,\rangle \in \mathcal{H}, \qquad \langle\,\{\lambda\}\,| = \langle\,0\,|\prod_{k=1}^{n} C(\lambda_k) \in \mathcal{H}^*$$

  on a reference state $|\,0\,\rangle \equiv |\uparrow\uparrow\ldots\uparrow\rangle$ such that

  $$C(\lambda)\,|\,0\,\rangle = 0, \quad A(\lambda)\,|\,0\,\rangle = a(\lambda)|\,0\,\rangle, \quad D(\lambda)\,|\,0\,\rangle = d(\lambda)|\,0\,\rangle$$

  - $\rightarrow$ eigenstates ("on-shell" Bethe states) if $\{\lambda\}$ solution of the Bethe equations
  - $\rightarrow$ "off-shell" Bethe states otherwise

A reminder of the periodic case (2)

- Correlation functions can be computed in the ABA framework
  - $\rightarrow$ numerical results [Caux et al. 2005...]
  - $\rightarrow$ analytical derivation of the large distance asymptotic behavior at the thermodynamic limit... [Kitanine, Kozlowski, Maillet, Slavnov, VT 2008, 2011...]

Both approaches are based

- on the form factor decomposition of the correlation functions:

$$\langle \psi_g | \sigma_n^\alpha \, \sigma_{n'}^\beta | \psi_g \rangle = \sum_{\substack{\text{eigenstates} \\ |\,\psi_i\,\rangle}} \langle \psi_g | \sigma_n^\alpha | \psi_i \rangle \cdot \langle \psi_i | \sigma_{n'}^\beta | \psi_g \rangle$$

- on the exact determinant representations for the form factors $\langle \psi_i | \sigma_n^\alpha | \psi_j \rangle$ in finite volume [Kitanine, Maillet, VT 1999] , obtained from
  - the action of local operators on Bethe states (using the solution of the quantum inverse problem, e.g. $\sigma_n^- = t(0)^{n-1} \, B(0) \, t(0)^{-n}$)
  - the use of Slavnov's determinant representation for the scalar products of Bethe states [Slavnov 89]

$$\langle \{\mu\}_{\text{off-shell}} | \{\lambda\}_{\text{on-shell}} \rangle \propto \det_{1 \le j,k \le n} \left[\frac{\partial \tau(\mu_j | \{\lambda\})}{\partial \lambda_k} \right]$$

where $t(\mu_j) \, | \{\lambda\} \rangle = \tau(\mu_j | \{\lambda\}) \, | \{\lambda\} \rangle$

The reflection algebra for the XXZ open spin chain

The open spin chains are solvable in the framework of the representation theory of the reflection algebra (or boundary Yang-Baxter algebra) [Sklyanin 88]

- generators $\mathcal{U}_{ij}(\lambda)$, $1 \leq i, j \leq n$ $\leftarrow$ elements of the boundary monodromy matrix $\mathcal{U}(\lambda)$
- commutation relations given by the reflection equation:

$$R_{12}(\lambda-\mu)\,\mathcal{U}_1(\lambda)\,R_{12}(\lambda+\mu-\eta)\,\mathcal{U}_2(\mu) = \mathcal{U}_2(\mu)\,R_{12}(\lambda+\mu-\eta)\,\mathcal{U}_1(\lambda)\,R_{12}(\lambda-\mu)$$

$\hookrightarrow$ most general 2×2 solution of the refl. eq [de Vega, Gonzalez-Ruiz; Ghoshal, Zamolodchikov 93] :

$$K(\lambda;\varsigma,\kappa,\tau) = \frac{1}{\sinh\varsigma}\begin{pmatrix} \sinh(\lambda-\frac{\eta}{2}+\varsigma) & \kappa\, e^{\tau}\sinh(2\lambda-\eta) \\ \kappa\, e^{-\tau}\sinh(2\lambda-\eta) & \sinh(\varsigma-\lambda+\frac{\eta}{2}) \end{pmatrix}$$

$\rightsquigarrow$ boundary matrices $K^-(\lambda) \equiv K(\lambda;\varsigma_+,\kappa_+,\tau_+)$ and $K^+(\lambda) \equiv K(\lambda+\eta;\varsigma_-,\kappa_-,\tau_-)$ describing most general boundary fields in left/right boundaries:

$$h^x_\pm = 2\kappa_\pm \sinh\eta\,\frac{\cosh\tau_\pm}{\sinh\varsigma_\pm}, \quad h^y_\pm = 2i\kappa_\pm\sinh\eta\,\frac{\sinh\tau_\pm}{\sinh\varsigma_\pm}, \quad h^z_\pm = \sinh\eta\,\coth\varsigma_\pm$$

$\rightsquigarrow$ $\mathcal{U}(\lambda) = T(\lambda)\,K^-(\lambda)\,\hat{T}(\lambda) = \begin{pmatrix} \mathcal{A}(\lambda) & \mathcal{B}(\lambda) \\ \mathcal{C}(\lambda) & \mathcal{D}(\lambda) \end{pmatrix}$ with $\hat{T}(\lambda) \propto \sigma^y\, T^t(-\lambda)\,\sigma^y$

$\rightsquigarrow$ transfer matrix: $t(\lambda) = \mathrm{tr}\{K^+(\lambda)\,\mathcal{U}(\lambda)\}$ $\qquad [t(\lambda), t(\mu)] = 0$

$$H^{\text{open}}_{XXZ} \propto \frac{d}{d\lambda}t(\lambda)\Big|_{\lambda=\eta/2}$$

Solution by ABA in the diagonal case

When both boundary matrices $K^{\pm}$ are diagonal ($\kappa_{\pm} = 0$, i.e. boundary fields along σ_1^z and σ_N^z only):

- the state $|\,0\,\rangle$ can still be used as a reference state to construct the eigenstates as Bethe states in the ABA framework [Sklyanin 88]

  $$|\,\{\lambda\}\,\rangle_{\mathcal{B}} = \prod_{k=1}^{n} \mathcal{B}(\lambda_k)|\,0\,\rangle \in \mathcal{H}, \qquad {}_{\mathcal{B}}\langle\,\{\lambda\}\,| = \langle\,0\,|\,\prod_{k=1}^{n} \mathcal{C}(\lambda_k) \in \mathcal{H}^*$$

- $\exists$ generalization of Slavnov's determinant representation for the scalar products of Bethe states $\langle\{\mu\}_{\text{off-shell}}|\{\lambda\}_{\text{on-shell}}\rangle$ [Tsuchiya 98; Wang 02]

- but a simple generalization of the quantum inverse problem to the boundary case (i.e. expressions of σ_n^{α} in terms of elements of the boundary monodromy matrix dressed by the boundary transfer matrix) is missing (except at site 1)

  $\rightsquigarrow$ no simple closed formula for the form factors

- correlation functions for the half-infinite chain can be computed as multiple integrals [Kitanine et al. 07] (recovering the results of [Jimbo et al. 95] from q-vertex operators):

  - decompose boundary Bethe states into bulk Bethe states
  - use the bulk inverse problem to compute the action of local operators
  - reconstruct the result in terms of boundary Bethe states

  $\rightsquigarrow$ multiple sums over scalar products. . .

The non-diagonal case ?

- It is possible to generalize usual Bethe ansatz equations to the case of non-longitudinal boundary fields with one constraint on the boundary parameters $\varphi_\pm, \psi_\pm, \tau_\pm$ [Nepomechie 03] :

$$\cosh(\tau_+ - \tau_-) = \epsilon_{\varphi_+}\epsilon_{\varphi_-} \cosh(\epsilon_{\varphi_+}\varphi_+ + \epsilon_{\varphi_-}\varphi_- + \epsilon_{\psi_+}\psi_+ - \epsilon_{\psi_-}\psi_- + (N-1-2M)\eta)$$

with $M \in \mathbb{N}$ (numbers of Bethe roots), $\epsilon_{\varphi_\pm}, \epsilon_{\psi_\pm} \in \{+,-\}$

$\rightsquigarrow$ incomplete in general (except for $M = N$)

+ construction of the Bethe states by means of a Vertex-IRF transformation [Cao et al 03; Yang, Zhang 07; Filali, Kitanine 11] (cf. the solution of the 8-vertex model by [Baxter 73; Faddeev, Takhtajan 79]) but problems in the ABA construction of a complete set of Bethe states both in $\mathcal{H}$ and $\mathcal{H}^*$

  $\rightsquigarrow$ scalar products and correlation functions could not be computed

- most general boundaries ? a usual ABA solution is missing...

  Alternative proposals:

  - Off-diagonal Bethe Ansatz [Cao et al 13...]
  - Modified Bethe Ansatz [Belliard et al 13...]
  - Separation of Variables [Frahm et al 10, Niccoli 12, Faldella et al 13...]

Solution by SOV in the general case

Goal: identity a basis of the space of state which "separates the variables" for the transfer matrix spectral problem

Sklyanin's method [Sklyanin 85,90] : construct this basis by means of the "operator roots" X_j of a one-parameter family of commuting operators $\mathbb{X}(\lambda)$

- $\mathbb{X}(\lambda)$ should be diagonalizable with simple spectrum

  $\rightsquigarrow$ the N commuting "operators roots" X_j (with $S_j \cap S_k = \emptyset$ if $j \neq k$, $S_j \equiv \mathrm{Spec}(X_j)$) can be used to define a basis of the space of states $\mathcal{H}$:

$$X_n \,|\, x_1, \ldots, x_N \,\rangle = x_n \,|\, x_1, \ldots, x_N \,\rangle, \quad (x_1, \ldots, x_N) \in S_1 \times \cdots \times S_N$$

- such that the transfer matrix $t(\lambda)$ at $\lambda = X_n$ acts as simple shifts on the basis elements:

$$\begin{aligned} t(X_n) \,|\, x_1, \ldots, x_n, \ldots, x_N \,\rangle = \Delta_+(x_n) \,|\, x_1, \ldots, x_n + \eta, \ldots, x_N \,\rangle \\ + \Delta_-(x_n) \,|\, x_1, \ldots, x_n - \eta, \ldots, x_N \,\rangle \end{aligned}$$

$\rightsquigarrow$ For the XXZ chain with non-diagonal b.c., such an operator $\mathbb{X}(\lambda)$ can be obtained as an entry of the monodromy matrix of a generalized gauge transformed model with inhomogeneities $\xi_1, \ldots, \xi_N$

Generalized method [Maillet, Niccoli 19] : use the multiple action of the transfer matrix $t(\lambda)$ itself, evaluated in distinguished points related to the inhomogeneities ξ_n, on a generically chosen vector

Solution by Sklyanin's SOV approach: more details

1. simplify the expression of $t(\lambda) = \mathrm{tr}\{K^+(\lambda)\mathcal{U}(\lambda)\}$: use (a trigonometric version of) Baxter's Vertex-IRF tranformation to diagonalize K^+

$$R_{12}(\lambda-\mu)\, S_1(\lambda|\alpha,\beta)\, S_2(\mu|\alpha,\beta+\sigma_1^z) = S_2(\mu|\alpha,\beta)\, S_1(\lambda|\alpha,\beta+\sigma_2^z)\, R_{12}^{\mathrm{SOS}}(\lambda-\mu|\beta)$$

with $S(\lambda|\alpha,\beta) = \begin{pmatrix} e^{\lambda-\eta(\beta+\alpha)} & e^{\lambda+\eta(\beta-\alpha)} \\ 1 & 1 \end{pmatrix}$ $\begin{cases} \beta : \text{dynamical parameter} \\ \alpha : \text{arbitrary shift} \end{cases}$

$\rightsquigarrow$ gauged transformed boundary/bulk monodromy matrices and boundary $K^\pm$ matrices:

$$\begin{aligned}\mathcal{U}(\lambda|\alpha,\beta) &= S^{-1}(\eta/2-\lambda|\alpha,\beta)\,\mathcal{U}(\lambda)\, S(\lambda-\eta/2|\alpha,\beta) \\ &= T(\lambda|(\alpha,\beta),(\gamma,\delta))\, K_-(\lambda|(\gamma,\delta),(\gamma',\delta'))\, \hat{T}(\lambda|(\gamma',\delta'),(\alpha,\beta)) \\ &= \begin{pmatrix} \mathcal{A}(\lambda|\alpha,\beta) & \mathcal{B}(\lambda|\alpha,\beta) \\ \mathcal{C}(\lambda|\alpha,\beta) & \mathcal{D}(\lambda|\alpha,\beta) \end{pmatrix}\end{aligned}$$

$\rightsquigarrow$ choice of α, β such that

$$t(\lambda) = \bar{\mathsf{a}}_+(\lambda)\, \mathcal{A}(\lambda|\alpha,\beta-1) + \bar{\mathsf{a}}_+(-\lambda)\, \mathcal{A}(-\lambda|\alpha,\beta-1)$$

2 construct a SOV basis which quasi-diagonalises $\mathcal{B}(\lambda|\alpha,\beta)$:

$$|\,\mathbf{h},\alpha,\beta+1\,\rangle_{\mathrm{Sk}} \propto \prod_{j=1}^{N} \mathcal{D}(\xi_j+\eta/2|\alpha,\beta+1)^{h_j}\, S_{1\dots N}(\{\xi\}|\alpha,\beta)\,|\,\underline{0}\,\rangle$$

$$_{\mathrm{Sk}}\langle\,\alpha,\beta-1,\mathbf{h}\,| \propto \langle\,0\,|\,S_{1\dots N}(\{\xi\}|\alpha,\beta)^{-1}\prod_{j=1}^{N}\mathcal{A}(\eta/2-\xi_j|\alpha,\beta-1)^{1-h_j}$$

for $\mathbf{h}\equiv(h_1,\dots,h_N)\in\{0,1\}^N$, $\langle\,0\,|=\otimes_{n=1}^{\mathrm{N}}(1,0)_n$, $|\,\underline{0}\,\rangle=\otimes_{n=1}^{\mathrm{N}}\begin{pmatrix}0\\1\end{pmatrix}_n$ and

$$S_{1\dots N}(\{\xi\}|\alpha,\beta)=\prod_{n=1\to N} S_n\left(-\xi_n\,\middle|\,\alpha,\beta+\sum_{j=1}^{n-1}\sigma_j^z\right)$$

such that

$$\mathcal{B}(\lambda|\alpha,\beta-1)\,|\,\mathbf{h},\alpha,\beta-1\,\rangle_{\mathrm{Sk}}=\mathsf{b}_R(\lambda|\alpha,\beta)\,a_{\mathbf{h}}(\lambda)\,a_{\mathbf{h}}(-\lambda)\,|\,\mathbf{h},\alpha,\beta+1\,\rangle_{\mathrm{Sk}},$$
$$_{\mathrm{Sk}}\langle\,\alpha,\beta+1,\mathbf{h}\,|\,\mathcal{B}(\lambda|\alpha,\beta+1)=\mathsf{b}_L(\lambda|\alpha,\beta)\,a_{\mathbf{h}}(\lambda)\,a_{\mathbf{h}}(-\lambda)\,{}_{\mathrm{Sk}}\langle\,\alpha,\beta-1,\mathbf{h}\,|,$$

where $a_{\mathbf{h}}(\lambda)=\prod_{n=1}^{N}\sinh(\lambda-\xi_n^{(h_n)})$ with $\xi_n^{(h_n)}=\xi_n+\eta/2-h_n\eta$

2. construct a SOV basis which quasi-diagonalises $\mathcal{B}(\lambda|\alpha,\beta)$:

$|\,\mathbf{h},\alpha,\beta+1\,\rangle_{\mathrm{Sk}}$ and ${}_{\mathrm{Sk}}\langle\,\alpha,\beta-1,\mathbf{h}\,|$ for $\mathbf{h}\equiv(h_1,\ldots,h_N)\in\{0,1\}^N$

such that

$$\mathcal{B}(\lambda|\alpha,\beta-1)\,|\,\mathbf{h},\alpha,\beta-1\,\rangle_{\mathrm{Sk}}=\mathrm{b}_R(\lambda|\alpha,\beta)\,a_{\mathbf{h}}(\lambda)\,a_{\mathbf{h}}(-\lambda)\,|\,\mathbf{h},\alpha,\beta+1\,\rangle_{\mathrm{Sk}},$$
$${}_{\mathrm{Sk}}\langle\,\alpha,\beta+1,\mathbf{h}\,|\,\mathcal{B}(\lambda|\alpha,\beta+1)=\mathrm{b}_L(\lambda|\alpha,\beta)\,a_{\mathbf{h}}(\lambda)\,a_{\mathbf{h}}(-\lambda)\,{}_{\mathrm{Sk}}\langle\,\alpha,\beta-1,\mathbf{h}\,|,$$

where $a_{\mathbf{h}}(\lambda)=\prod_{n=1}^{N}\sinh(\lambda-\xi_n^{(h_n)})$ with $\xi_n^{(h_n)}=\xi_n+\eta/2-h_n\eta$

$+$ orthogonality conditions:

$${}_{\mathrm{Sk}}\langle\,\alpha,\beta-1,\mathbf{h}\,|\,\mathbf{k},\alpha,\beta+1\,\rangle_{\mathrm{Sk}}\propto\delta_{\mathbf{h},\mathbf{k}}\,\frac{e^{2\sum_{j=1}^{N}h_j\xi_j}}{V_{\mathbf{h}}(\boldsymbol{\xi})}$$

with $V_{\mathbf{h}}(\boldsymbol{\xi})=V(\xi_1^{(h_1)},\ldots,\xi_N^{(h_N)})=\det_N\left[\sinh^{2(j-1)}(\xi_i^{(h_i)})\right]$

Remarks: This construction

→ works only on an inhomogeneous deformation of the model:

$$T(\lambda)\longrightarrow T(\lambda;\xi_1,\ldots,\xi_N)$$

such that $\xi_i\neq\xi_k\pm\eta \mod i\pi$ if $i\neq k$

→ needs $[K^-(\lambda|\alpha,\beta)]_{12}\neq 0$

The new SOV approach [Maillet, Niccoli 19]

Under the hypothesis that

- $\xi_i \neq \xi_k \pm \eta \mod i\pi$ if $i \neq k$
- the two boundary matrices $K^{\pm}$ are not both proportional to the identity

one can construct, for almost any choice of the co-vector $\langle S \,|$, the following SOV basis:

$$_S\langle \mathbf{h} \,| \propto \langle S \,| \prod_{n=1}^{N} t(\xi_n - \eta/2)^{1-h_n}, \qquad \mathbf{h} \equiv (h_1, \ldots, h_N) \in \{0,1\}^N$$

$$|\, \mathbf{h} \rangle_S \propto \prod_{n=1}^{N} t(\xi_n + \eta/2)^{h_n} |\, R \,\rangle, \qquad \mathbf{h} \in \{0,1\}^N$$

where $|\, R \,\rangle$ is uniquely fixed by adequate orthogonality conditions:

$$_S\langle \mathbf{h} \,|\, R \,\rangle = N(\{\xi\})\, \delta_{\mathbf{h},\mathbf{0}}$$

They satisfy the following orthogonality conditions (same as previous basis):

$$_S\langle \mathbf{h} \,|\, \mathbf{h}' \,\rangle_S \propto \delta_{\mathbf{h},\mathbf{h}'} \frac{e^{2\sum_{j=1}^{N} h_j \xi_j}}{V_{\mathbf{h}}(\boldsymbol{\xi})}$$

Spectrum and eigenstates by SOV

In both types of SOV basis ($|\,\mathbf{h}\,\rangle \equiv |\,\mathbf{h},\alpha,\beta+1\,\rangle_{\mathrm{Sk}}$ or $|\,\mathbf{h}\,\rangle_S$):

- the multi-dimensional spectral problem for the transfer matrix $t(\lambda)$ can be reduced to a set of N one-dimensional ones:

$$t(\lambda)\,|\,\Psi_\tau\,\rangle = \tau(\lambda)\,|\,\Psi_\tau\,\rangle \quad \text{with} \quad |\,\Psi_\tau\,\rangle = \sum_{\mathbf{h}\in\{0,1\}^N} \psi_\tau(\mathbf{h})\,|\,\mathbf{h}\,\rangle,$$

is solved by

$$\psi_\tau(\mathbf{h}) = \prod_{n=1}^{N} Q_\tau(\xi_n^{(h_n)}) \cdot V_{\mathbf{h}}(\boldsymbol{\xi})$$

where Q_τ and τ are solution of a discrete version of Baxter's T-Q equation:

$$\tau(x)\,Q_\tau(x) = \mathbf{A}(x)\,Q(x+\eta) + \mathbf{A}(-x)\,Q_\tau(x-\eta), \quad x \in \cup_{n=1}^{N}\{\xi_n^{(0)},\xi_n^{(1)}\}$$

- The scalar products of separate states can be expressed as determinants:

$$\langle\,P\,| = \sum_{\mathbf{h}} \prod_{n=1}^{N} [v_n^{h_n} P(\xi_n^{(h_n)})]\, V_{1-\mathbf{h}}(\boldsymbol{\xi})\langle\,\mathbf{h}\,|, \quad |\,Q\,\rangle = \sum_{\mathbf{h}} \prod_{n=1}^{N} Q(\xi_n^{(h_n)})\, V_{\mathbf{h}}(\boldsymbol{\xi})|\,\mathbf{h}\,\rangle$$

where P and Q are arbitrary and

$$\langle\mathbf{h}\,|\,\mathbf{k}\rangle \propto \frac{\delta_{\mathbf{h},\mathbf{k}}}{V_{\mathbf{h}}(\boldsymbol{\xi})} \quad \text{with} \quad V_{\mathbf{h}}(\boldsymbol{\xi}) = \det_N\left[\sinh^{2(j-1)}(\xi_i^{(h_i)})\right]$$

$$\rightsquigarrow \;\langle\,P\,|\,Q\,\rangle = \det_{1\le i,j\le N}\left[\sum_{h\in\{0,1\}} f(\xi_i^{(h_i)})P(\xi_i^{(h_i)})Q(\xi_i^{(h_i)})\,\sinh^{2(j-1)}(\xi_i^{(1-h_i)})\right]$$

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  is solved by

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  where P and Q are arbitrary and

  $$\langle \mathbf{h}\,|\,\mathbf{k}\rangle \propto \frac{\delta_{\mathbf{h},\mathbf{k}}}{V_{\mathbf{h}}(\boldsymbol{\xi})} \quad \text{with} \quad V_{\mathbf{h}}(\boldsymbol{\xi}) = \det_N\left[\sinh^{2(j-1)}(\xi_i^{(h_i)})\right]$$

  $$\rightsquigarrow \; \langle\,P\,|\,Q\,\rangle = \det_{1\le i,j\le N}\left[\sum_{h\in\{0,1\}} f(\xi_i^{(h_i)})P(\xi_i^{(h_i)})Q(\xi_i^{(h_i)})\,\sinh^{2(j-1)}(\xi_i^{(1-h_i)})\right]$$

From discrete to continuous T-Q equations

Question: Can we characterize a class of (entire ?) functions Σ_Q such that

$$\tau(\lambda) \text{ eigenvalue of } t(\lambda) \text{ (+ simple conditions on } \tau(\lambda) \text{ ?)}$$
$$\Updownarrow$$
$$\exists! Q \in \Sigma_Q \quad \text{s.t.} \quad \tau(\lambda)\, Q(\lambda) = \mathbf{A}(\lambda)\, Q(\lambda+\eta) + \mathbf{A}(-\lambda)\, Q_\tau(\lambda-\eta)$$

$\rightarrow$ not known in general

but this SOV characterisation of the spectrum can be equivalently reformulated in terms of polynomials (in $\cosh(2\lambda)$) Q-solutions of a functional T-Q equation with an inhomogeneous term [Kitanine et al 13] , (see also [Cao et al. 13; Belliard, Crampé 13. . .]):

An entire function $\tau(\lambda)$ is an eigenvalue of the antiperiodic transfer matrix iff there exists a unique function $Q(\lambda) \in \Sigma_Q$ such that

$$\tau(\lambda)\, Q(\lambda) = \mathbf{A}(\lambda)\, Q(\lambda-\eta) + \mathbf{A}(-\lambda)\, Q(\lambda+\eta) + \mathbf{F}(\lambda),$$

where $\mathbf{A}(\lambda) \equiv \mathbf{A}_{\zeta_\pm,\kappa_\pm}(\lambda)$ and $\mathbf{F}(\lambda) \equiv \mathbf{F}_{\zeta_\pm,\kappa_\pm,\tau_\pm}(\lambda)$ depend on the boundary parameters, with $\mathbf{F}(\xi_n^{(0)}) = \mathbf{F}(\xi_n^{(1)}) = 0$, $n = 1, \dots, N$.

$\mathbf{F} = 0$ identically $\Longleftrightarrow$ constraint on the boundary param. (cf [Nepomechie 03])

More precisions on the spectrum

$$\mathbf{F}_{\varepsilon}(\lambda) = \frac{2\kappa_+\kappa_-}{\sinh\varsigma_+\ \sinh\varsigma_-}\ \mathfrak{g}_{\varepsilon}^{(N)}\ a(\lambda)\, a(-\lambda)\, d(\lambda)\, d(-\lambda)\, [\cosh^2(2\lambda) - \cosh^2\eta]$$

with, for $M \in \mathbb{N}$ and $\varepsilon \in \{+,-\}^4$,

$$\mathfrak{g}_{\varepsilon}^{(M)} \equiv \mathfrak{g}_{\varepsilon,\tau_\pm,\varphi_\pm,\psi_\pm}^{(M)} = \cosh(\tau_+ - \tau_-) \\ -\epsilon_{\varphi_+}\epsilon_{\varphi_-}\ \cosh(\epsilon_{\varphi_+}\varphi_+ + \epsilon_{\varphi_-}\varphi_- + \epsilon_{\psi_+}\psi_+ - \epsilon_{\psi_-}\psi_- + (N-1-2M)\eta)$$

and set

$$\Sigma_Q^M = \left\{ Q(\lambda) = \prod_{j=1}^{M} \frac{\cosh(2\lambda)-\cosh(2\lambda_j)}{2} \,\middle|\, \cosh(2\lambda_j) \neq \cosh(2\xi_n^{(h)}), \quad \forall\,(j,n,h) \right\}$$

1. complete description of the spectrum in terms of $Q(\lambda) \in \Sigma_Q^N$ solution of

$$\tau(\lambda)\, Q(\lambda) = \mathbf{A}_{\varepsilon}(\lambda)\, Q(\lambda-\eta) + \mathbf{A}_{\varepsilon}(-\lambda)\, Q(\lambda+\eta) + \mathbf{F}_{\varepsilon}(\lambda),$$

if $\frac{2\kappa_+\kappa_-}{\sinh\varsigma_+\ \sinh\varsigma_-} \neq 0$ and $\mathfrak{g}_{\varepsilon}^{(M)} \neq 0\ \forall M \in \{0,\ldots,N-1\}$

2. incomplete description of the spectrum in terms of $Q(\lambda) \in \Sigma_Q^M$ solution of

$$\tau(\lambda)\, Q(\lambda) = \mathbf{A}_{\varepsilon}(\lambda)\, Q(\lambda-\eta) + \mathbf{A}_{\varepsilon}(-\lambda)\, Q(\lambda+\eta) \qquad (1)$$

if $\frac{2\kappa_+\kappa_-}{\sinh\varsigma_+\ \sinh\varsigma_-} \neq 0$ and $\mathfrak{g}_{\varepsilon}^{(M)} = 0$ for some $M \in \{0,\ldots,N-1\}$

3. complete description of the spectrum in terms of $Q(\lambda) \in \Sigma_Q = \cup_{M=0}^{N}\Sigma_Q^M$ solution of (1) if $\frac{2\kappa_+\kappa_-}{\sinh\varsigma_+\ \sinh\varsigma_-} = 0$. This is the case for our special boundary conditions which can be reached as $\varsigma_+ \to +\infty$

More precisions on the spectrum

$$\mathbf{F}_{\varepsilon}(\lambda) = \frac{2\kappa_+\kappa_-}{\sinh\varsigma_+\,\sinh\varsigma_-}\,\mathfrak{g}_{\varepsilon}^{(N)}\,a(\lambda)\,a(-\lambda)\,d(\lambda)\,d(-\lambda)\,[\cosh^2(2\lambda)-\cosh^2\eta]$$

with, for $M \in \mathbb{N}$ and $\varepsilon \in \{+,-\}^4$,

$$\mathfrak{g}_{\varepsilon}^{(M)} \equiv \mathfrak{g}_{\varepsilon,\tau_\pm,\varphi_\pm,\psi_\pm}^{(M)} = \cosh(\tau_+-\tau_-) \\ -\epsilon_{\varphi_+}\epsilon_{\varphi_-}\cosh(\epsilon_{\varphi_+}\varphi_+ + \epsilon_{\varphi_-}\varphi_- + \epsilon_{\psi_+}\psi_+ - \epsilon_{\psi_-}\psi_- + (N-1-2M)\eta)$$

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More precisions on the spectrum

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$$\Sigma_Q^M = \left\{ Q(\lambda) = \prod_{j=1}^{M} \frac{\cosh(2\lambda)-\cosh(2\lambda_j)}{2} \,\middle|\, \cosh(2\lambda_j) \neq \cosh(2\xi_n^{(h)}), \quad \forall\,(j,n,h) \right\}$$

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$$\tau(\lambda)\,Q(\lambda) = \mathbf{A}_\varepsilon(\lambda)\,Q(\lambda-\eta) + \mathbf{A}_\varepsilon(-\lambda)\,Q(\lambda+\eta) \qquad (1)$$

if $\frac{2\kappa_+\kappa_-}{\sinh\varsigma_+\,\sinh\varsigma_-} \neq 0$ and $\mathfrak{g}_\varepsilon^{(M)} = 0$ for some $M \in \{0,\ldots,N-1\}$
(Nepomechie's constraint)

3. complete description of the spectrum in terms of $Q(\lambda) \in \Sigma_Q = \cup_{M=0}^{N}\Sigma_Q^M$ solution of (1) if $\frac{2\kappa_+\kappa_-}{\sinh\varsigma_+\,\sinh\varsigma_-} = 0$. This is the case for our special boundary conditions which can be reached as $\varsigma_+ \to +\infty$

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$$\mathbf{F}_\varepsilon(\lambda) = \frac{2\kappa_+\kappa_-}{\sinh\varsigma_+ \sinh\varsigma_-}\, \mathfrak{g}_\varepsilon^{(N)}\, a(\lambda)\, a(-\lambda)\, d(\lambda)\, d(-\lambda)\, [\cosh^2(2\lambda) - \cosh^2\eta]$$

with, for $M \in \mathbb{N}$ and $\varepsilon \in \{+,-\}^4$,

$$\mathfrak{g}_\varepsilon^{(M)} \equiv \mathfrak{g}^{(M)}_{\varepsilon,\tau_\pm,\varphi_\pm,\psi_\pm} = \cosh(\tau_+ - \tau_-) \\ -\epsilon_{\varphi_+}\epsilon_{\varphi_-} \cosh(\epsilon_{\varphi_+}\varphi_+ + \epsilon_{\varphi_-}\varphi_- + \epsilon_{\psi_+}\psi_+ - \epsilon_{\psi_-}\psi_- + (N-1-2M)\eta)$$

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Eigenstates as generalised Bethe states

- In the range of Sklyanin's approach, separate states can be reformulated as generalised Bethe states:

$$| Q \rangle_{\mathrm{Sk}} \propto \prod_{j=1\to M} \mathcal{B}(\lambda_j|\alpha, \beta - 2j + 1) \, | \Omega_{\alpha,\beta+1-2M} \rangle_{\mathrm{Sk}}$$

$$_{\mathrm{Sk}}\langle Q | \propto {}_{\mathrm{Sk}}\langle \Omega_{\alpha,\beta-1+2M} | \prod_{j=1\to M} \mathcal{B}(\lambda_j|\alpha, \beta + 2M - 2j + 1)$$

  for any $Q(\lambda) = \prod_{j=1}^{M} \frac{\cosh(2\lambda)-\cosh(2\lambda_j)}{2}$

  with $| \Omega_{\alpha,\beta+1-2M} \rangle_{\mathrm{Sk}}$ and $_{\mathrm{Sk}}\langle \Omega_{\alpha,\beta-1+2M} |$ special separate states

  Remark: if $| Q \rangle$ and $\langle Q |$ are eigenstates obtained via the new SOV approach, we have also $| Q \rangle_{\mathrm{Sk}} = \mathrm{c}_Q^{\mathrm{Sk}} \, | Q \rangle$, $_{\mathrm{Sk}}\langle Q | = \langle Q | / \mathrm{c}_Q^{\mathrm{Sk}}$

- With the special choice of α, β diagonalising K^+, and under the constraint

$$[K^-(\lambda|(\alpha, \beta + N - 1 - 2M), (\alpha, \beta + N - 1 - 2M)]_{21} = 0$$

  (which implies Nepomechie's constraint $\mathfrak{g}_\varepsilon^{(M)} = 0$), the reference state $| \Omega_{\alpha,\beta+1-2M} \rangle$ can be identified as (cf. [Cao et al 03])

$$| \eta, \alpha + \beta + N - 1 - 2M \rangle \equiv \prod_{n=1}^{N} S_n(-\xi_n|\alpha, \beta + n - 1 - 2M) \, | 0 \rangle$$

  up to a proportionality coefficient which only depends on M

Spectrum and eigenstates in the limit $\varsigma_+ \to +\infty$

$K^-(\lambda;\varsigma_+ = -\infty, \kappa_+, \tau_+) = e^{(\eta/2-\lambda)\sigma^z}$

- out of the range of Sklyanin's SOV approach but still in the range of the new SOV approach

  $\rightsquigarrow$ the transfer matrix is diagonalizable with simple spectrum and the complete set of eigenstates is given by the separate states $|\,Q\,\rangle$ and $\langle\,Q\,|$ with

  $$Q(\lambda) = \prod_{j=1}^{M} \frac{\cosh(2\lambda)-\cosh(2\lambda_j)}{2} \quad (1 \le M \le N)$$

  solution with the corresponding eigenvalue $\tau(\lambda)$ of the homogeneous TQ-equation

- with the special choice of α, β diagonalizing K^+, it can be shown by direct computation that the Bethe state

  $$\prod_{j=1\to M} \mathcal{B}(\lambda_j|\alpha, \beta-2j+1)\,|\,\eta, \alpha+\beta+N-1-2M\,\rangle$$

  is an eigenstate of $t(\lambda)$ with eigenvalue $\tau(\lambda)$ (cf. [Cao et al 03]), and hence should be proportional to $|\,Q\,\rangle$

- the transfer matrix is isospectral to the transfer matrix of an open spin chain with diagonal boundary conditions with boundary parameters $\varsigma_\pm^{(D)}$:

  $$\varsigma_\epsilon^{(D)} = \epsilon_{\varphi_-}\varphi_-, \quad \varsigma_{-\epsilon}^{(D)} = -\epsilon_{\varphi_-}\psi_- + i\pi/2, \qquad \text{for } \epsilon_{\varphi_-} = 1 \text{ or } -1.$$

Computation of the scalar products [Kitanine, Maillet, Niccoli, VT 18]

$$\langle P \mid Q \rangle \propto \det_{1\le i,j\le N}\left[\sum_{\epsilon=\pm} f_{\{a\}}(\epsilon\xi_i)\, P\Big(\xi_i-\epsilon\frac{\eta}{2}\Big)\, Q\Big(\xi_i-\epsilon\frac{\eta}{2}\Big)\, \cosh^{j-1}(2\xi_i+\epsilon\eta)\right]$$

with arbitrary $P(\lambda)=\prod_{j=1}^{p}(\cosh 2\lambda-\cosh 2p_j)$, $Q(\lambda)=\prod_{j=1}^{q}(\cosh 2\lambda-\cosh 2q_j)$,

where $f_{\{a\}}(\lambda)$ depends on combinations $\{a\}$ of the $\pm$ boundary parameters $\zeta_\pm, \kappa_\pm$

$\rightsquigarrow$ not convenient for the consideration of the homogeneous/thermodynamic limit

- When $p+q=N$, can be transformed into a new determinant in which the role of the set of variables $\{\xi_j\}$ and $\{\gamma_j\}\equiv\{p_j\}\cup\{q_j\}$ has been exchanged at the price of modifying the last column:

$$\langle P \mid Q \rangle \propto \det_{1\le i,j\le p+q}\left[\sum_{\epsilon=\pm} f_{\{\frac{\eta}{2}-a\}}(\epsilon\gamma_i)\prod_{\ell=1}^{L}\Big(\cosh(2\gamma_i-\epsilon\eta)-\cosh 2\xi_\ell\Big)\, \cosh^{j-1}(2\gamma_i+\epsilon\eta) + \delta_{j,L}\, g_{\{a\}}^{(p+q)}(\gamma_i)\right]$$

- Generalization to $p+q\neq N$ by considering limits of the previous result
- In its turn, this new determinant can be transformed into a generalized (and much more complicated !) version of Slavnov's determinant
- In the case with a constraint, the determinant simplifies drastically if one of the state is an eigenstate thanks to Bethe equations

  $\rightsquigarrow$ usual Slavnov formula if $p=q$!

Generalized Slavnov determinant for open XXZ

Example: the case $p = q$

$$\langle\, P \,|\, Q \,\rangle \propto \det_p \mathcal{S}$$

$$\mathcal{S}_{i,k} = \sum_{\epsilon\in\{+,-\}} f(\epsilon q_i)\, P(q_i{+}\epsilon\eta) \left[\frac{f(-p_k)}{\varsigma(q_i+\epsilon\frac{\eta}{2})-\varsigma(p_k+\frac{\eta}{2})} - \frac{f(p_k)\,\varphi(p_k)}{\varsigma(q_i+\epsilon\frac{\eta}{2})-\varsigma(p_k-\frac{\eta}{2})} \right.$$

$$\left. + \frac{f(-p_k)-f(p_k)\,\varphi(p_k)}{1+\sum_{\ell=1}^{p} P^g_{f,\ell}} \sum_{j=1}^{p} \frac{P^g_{f,j}}{\varsigma(q_i+\epsilon\frac{\eta}{2})-\varsigma(p_j-\frac{\eta}{2})} \right] + \frac{g(q_i)}{P(q_i)}\, \frac{f(-p_k)-f(p_k)\,\varphi(p_k)}{1+\sum_{\ell=1}^{p} P^g_{f,\ell}}.$$

with $\qquad \varsigma(\lambda) = \dfrac{\cosh(2\lambda)}{2}$

and $\qquad P^g_{f,k} = \dfrac{g(p_k)\,\sinh(2p_k-\eta)}{f(-\alpha_k)\,P'(p_k)\,P(p_k-\eta)}, \quad \varphi(\lambda) = \dfrac{\sinh(2\lambda-\eta)}{\sinh(2\lambda+\eta)}\,\dfrac{P(\lambda+\eta)}{P(\lambda-\eta)}.$

The functions f and g depend on the boundary parameters.

Generalized Slavnov determinant for open XXZ

Example: the case $p = q$

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$$\mathcal{S}_{i,k} = \sum_{\epsilon\in\{+,-\}} f(\epsilon q_i)\, P(q_i{+}\epsilon\eta) \left[\frac{f(-p_k)}{\varsigma(q_i+\epsilon\frac{\eta}{2}) - \varsigma(p_k+\frac{\eta}{2})} - \frac{f(p_k)\,\varphi(p_k)}{\varsigma(q_i+\epsilon\frac{\eta}{2}) - \varsigma(p_k-\frac{\eta}{2})} \right.$$

$$\left. + \frac{f(-p_k) - f(p_k)\,\varphi(p_k)}{1+\sum_{\ell=1}^{p} P^g_{f,\ell}} \sum_{j=1}^{p} \frac{P^g_{f,j}}{\varsigma(q_i+\epsilon\frac{\eta}{2}) - \varsigma(p_j-\frac{\eta}{2})} \right] + \frac{g(q_i)}{P(q_i)}\, \frac{f(-p_k) - f(p_k)\,\varphi(p_k)}{1+\sum_{\ell=1}^{p} P^g_{f,\ell}}.$$

In the case with a constraint, the Bethe equations are

$$f(-p_k) - f(p_k)\,\varphi(p_k) = 0, \;\; k = 1, \ldots p$$

$\rightsquigarrow$ if $|\, P \,\rangle$ is an eigenstate the determinant simplifies into

$$\mathcal{S}_{i,k} = \sum_{\epsilon\in\{+,-\}} f(\epsilon q_i)\;\; P(q_i+\epsilon\eta) \left[\frac{f(-p_k)}{\varsigma(q_i+\epsilon\frac{\eta}{2}) - \varsigma(p_k+\frac{\eta}{2})} - \frac{f(q_k)\,\varphi(q_k)}{\varsigma(p_i+\epsilon\frac{\eta}{2}) - \varsigma(p_k-\frac{\eta}{2})} \right]$$

$$\propto \frac{\partial \tau(q_j|\{p\})}{\partial p_k}$$

Computation of correlation functions: general strategy

Compute $\langle O_{1\to m}\rangle \equiv \dfrac{\langle\, Q \,|\, O_{1\to m} |\, Q\,\rangle}{\langle Q \,|\, Q\rangle}$ for $|\, Q\,\rangle =$ ground state and $O_{1\to m} \in \mathrm{End}(\otimes_{n=1}^{m} \mathcal{H}_n)$ acts on sites 1 to m?

1. rewrite $|\, Q\,\rangle$ as a generalized Bethe state
$$\prod_{j=1\to M} \mathcal{B}(\lambda_j|\alpha, \beta - 2j + 1)\,|\, \eta, \alpha + \beta + N - 1 - 2M\,\rangle$$
2. use a similar strategy as in the diagonal case [Kitanine et al. 07] to act with $O_{1\to m}$ on this Bethe state, i.e.
    - decompose the boundary Bethe state as a sum of bulk Bethe states
    - use the solution of the bulk inverse problem to act with local operators on bulk Bethe states
    - reconstruct the result of this action as sums over boundary Bethe states, and hence as a sum over separate states
3. compute the resulting scalar products using the determinant representation for the scalar products of separate states issued from SOV

but difficulties due to the use in all the steps of 2 of a gauged transformed boundary/bulk YB algebra !

Difficulties due to use of the gauged algebra

- the action of the usual basis of local operators given by $E_n^{i,j} \in \mathrm{End}(\mathcal{H}_n)$ (such that $(E^{i,j})_{k,\ell} = \delta_{i,k}\,\delta_{j,\ell}$) is very intricate on the gauged bulk Bethe states

  $\rightsquigarrow$ identification of a basis of $\mathrm{End}(\otimes_{n=1}^m \mathcal{H}_n)$ whose action is simpler to compute:

$$\mathbb{E}_m(\alpha,\beta) = \left\{ \prod_{n=1}^{m} E_n^{\epsilon'_n,\epsilon_n}(\xi_n|(a_n,b_n),(\bar{a}_n,\bar{b}_n)) \;\Big|\; \boldsymbol{\epsilon},\boldsymbol{\epsilon}' \in \{1,2\}^m \right\},$$

where $E_n^{\epsilon'_n,\epsilon_n}(\lambda|(a_n,b_n),(\bar{a}_n,\bar{b}_n))) = S_n(-\lambda|\bar{a}_n,\bar{b}_n)\,E_n^{\epsilon'_n,\epsilon_n}\,S_n^{-1}(-\lambda|a_n,b_n)$ and the gauge parameters $a_n, \bar{a}_n, b_n, \bar{b}_n$, $1 \le n \le m$, are fixed in terms of α, β and of the m-tuples $\boldsymbol{\epsilon} \equiv (\epsilon_1,\ldots,\epsilon_m)$ and $\boldsymbol{\epsilon}' \equiv (\epsilon'_1,\ldots,\epsilon'_m)$ as

$$a_n = \alpha + 1, \qquad b_n = \beta - \sum_{r=1}^{n} (-1)^{\epsilon_r},$$

$$\bar{a}_n = \alpha - 1, \qquad \bar{b}_n = \beta + \sum_{r=n+1}^{m} (-1)^{\epsilon'_r} - \sum_{r=1}^{m} (-1)^{\epsilon_r} = b_n + 2\tilde{m}_{n+1},$$

with $\tilde{m}_n = \sum_{r=n}^{m} (\epsilon'_r - \epsilon_r)$.

$\rightsquigarrow$ compute "elementary building blocks" $\langle \prod_{n=1}^{m} E_n^{\epsilon'_n,\epsilon_n}(\xi_n|(a_n,b_n),(\bar{a}_n,\bar{b}_n)) \rangle$

- the action of $\prod_{n=1}^{m} E_n^{\epsilon'_n,\epsilon_n}(\xi_n|(a_n,b_n),(\bar{a}_n,\bar{b}_n))$ for

$$\sum_{r=1}^{m}(\epsilon'_r - \epsilon_r) \neq 0$$

on the Bethe state

$$\prod_{j=1\to M} \mathcal{B}(\lambda_j|\alpha,\beta-2j+1)\,|\,\eta,\alpha+\beta+N-1-2M\,\rangle$$

produces a state written on a SOV basis with shifted gauge parameters β

$\rightsquigarrow$ the expression of the resulting scalar product is not known in that case

$\rightsquigarrow$ we had to restrict our study to the computation of "elementary blocks" $\langle\prod_{n=1}^{m} E_n^{\epsilon'_n,\epsilon_n}(\xi_n|(a_n,b_n),(\bar{a}_n,\bar{b}_n))\rangle$ for which

$$\sum_{r=1}^{m}(\epsilon'_r - \epsilon_r) = 0$$

Result

As in the diagonal case, the result is given as a multiple sum over scalar products, which turn in the half-infinite chain limit into multiple integrals over the Fermi zone $[-\Lambda, \Lambda]$ on which the Bethe roots condensate with density $\rho(\lambda)$ + possible contribution of two (instead of one in the diagonal case) isolated complex roots (the boundary roots $\check{\lambda}_\pm$ converging towards $\eta/2 - \varsigma_\pm^{(D)}$):

$$\langle \prod_{n=1}^{m} E_n^{\epsilon'_n,\epsilon_n}(\xi_n|(a_n,b_n),(\bar{a}_n,\bar{b}_n))\rangle = \prod_{n=1}^{m} \frac{e^\eta}{\sinh(\eta b_n)} \frac{(-1)^s}{\prod_{j<i}\sinh(\xi_i-\xi_j)\prod_{i\leq j}\sinh(\xi_i+\xi_j)}$$

$$\times \int_{\mathcal{C}} \prod_{j=1}^{s} d\lambda_j \int_{\mathcal{C}_{\boldsymbol{\xi}}} \prod_{j=s+1}^{m} d\lambda_j \quad \underbrace{H_m(\{\lambda_j\}_{j=1}^M; \{\xi_k\}_{k=1}^m)}_{\substack{\text{similar to the diagonal case} \\ \text{except that it depends on both parameters } \bar{\varsigma}_\pm^{(D)}}} \quad \underbrace{\det_{1\leq j,k\leq m}\left[\Phi(\lambda_j,\xi_k)\right]}_{\text{determinant of densities}},$$

The contours $\mathcal{C}$ and $\mathcal{C}_{\boldsymbol{\xi}}$ are defined as

$$\mathcal{C} = \begin{cases} [-\Lambda,\Lambda] & \text{if the GS has no boundary roots} \\ [-\Lambda,\Lambda]\cup\Gamma(\bar{\varsigma}_\sigma^{(D)}-\eta/2) & \text{if the GS contains the b.r. } \check{\lambda}_\sigma \end{cases}$$

$$\mathcal{C}_{\boldsymbol{\xi}} = \mathcal{C}\cup\Gamma(\{\xi_k^{(1)}\}_{k=1}^m)$$

where $\Gamma(\bar{\varsigma}_\sigma^{(D)}-\eta/2)$ (respectively $\Gamma(\{\xi_k^{(1)}\}_{k=1}^m)$) surrounds the point $\bar{\varsigma}_\sigma^{(D)}-\eta/2$ (respectively the points $\xi_1^{(1)},\ldots,\xi_m^{(1)}$) with index 1, all other poles being outside.

Perspectives and open problems

- generalize this study to a general boundary field on site N (case with a constraint)
- generalize this study to (some particular case of) the open XYZ chain ?
- compute more general matrix elements with $\sum_{r=1}^{m}(\epsilon'_r - \epsilon_r) \neq 0$?
- case without constraint ?
  - form of the (homogeneous) functional T-Q equation for the general open chain ($\rightsquigarrow$ Q not a polynomial) ?
  - transformation of the determinant of the scalar product in the non-polynomial case (cf antiperiodic XXZ $\rightsquigarrow$ difficult) ?
- Form factor of a local operator at distance m from the boundary (even in the diagonal case) ?
- Temperature case ?