# Q Operators for Open Quantum Integrable Systems

Robert Weston
Heriot-Watt University, Edinburgh

*GGI, April 2022*

# Plan

- Introduction:origin and properties of Q
- Algebraic picture of Q for closed chains
  - 2 routes to TQ Relations : SES , <u>factorization</u>
- The required representations
  - q-oscillator repns, Verma modules, triangular repn
- $T_v(z,\mu) = \# \, Q(zq^{\mu}) \, \bar{Q}(zq^{-\mu})$ for closed chains
- $T_v(z,\mu) = \# \, Q(zq^{\mu}) \, \bar{Q}(zq^{-\mu})$ for open chains
- Discussion

Based on joint work with Bart Vlaar & Alec Cooper

[J. Phys A / arXiv:2001.10760 , Part II Coming soon]

# Introduction: origin and properties of Q

- Q-op. introduced by Baxter in 72. for 8-vertex model

- Objective : Eigenvectors & eigenvalues of transfer matrix $T^{(1)}(z)$ holomorphic in $z$

- Hence i) $Z = \mathrm{Tr}\,(T^{(1)}(z)^{N})$ partition fn 8-vertex

  ii) Eigenvectors & corr. fns of

$$H_{XYZ} \sim \frac{d \ln T^{(1)}(z)}{dz}\Big|_{z=1}$$

- Problem : no Bethe Ansatz for eigenvectors

Baxter's trick : introduce new operator $Q$

- In $XXZ/sl_2$ case :

$T^{(1)}(z) : M \to M$    i) Diagonalisable & holomorphic

ii) $[T^{(1)}(z), T^{(1)}(z')] = 0$

$Q(z) : M \to M$    i) Diagonalisable & polynomial

ii) $[T^{(1)}(z), Q(z')] = [Q(z), Q(z')] = 0$

key reln $\longrightarrow$ iii) $T^{(1)}(z)\,Q(z) = a(z)\,Q(qz) + b(z)\,Q(q^{-1}z)$

polynomial fns

for eigenvalue $q(z) = \prod_j (z - z_j)$

$$\Rightarrow \quad a(z_i)\prod_j (q z_i - z_j) + b(z_i)\prod_j (q^{-1} z_i - z_j) = 0$$

Bethe Eqns without Bethe Ansatz.

- QISM / Q-group construction of $Q(z)$ emerged slowly.
  - Idea of Q: trace of monodromy matrix over $\infty$-dim aux. space - [Sklyanin? 80s, 90s?]
  - Used in CFT context by [Bazhanov/Lukyanov/Zamolodchikov 96]
  - In spin chain context by [Antonov/Feigin 97, ...]

# Algebraic picture of Q - closed/periodic chains

- We consider 6-Vertex / XXZ case – underlying algebra is $U_q(\hat{sl}_2) = \langle e_0, f_0, e_1, f_1, k_0^{\pm 1}, k_1^{\pm 1} \rangle$

  Also need 2 Borel subalgs

  $U_q(b_+) = \langle e_0, e_1, k_0^{\pm 1}, k_1^{\pm 1} \rangle$, $U_q(b_-) = \langle f_0, f_1, k_0^{\pm 1}, k_1^{\pm 1} \rangle$

- Notation An alg A module is denoted

  $(\alpha, \beta)$ with $\alpha$ = Vector space, $\beta : A \to \mathrm{End}(\alpha)$

Define 4 repns in terms of $W^{(m)} = \bigoplus_{i=0}^{m} \mathbb{C} w_i$, $W = \bigoplus_{i=0}^{\infty} \mathbb{C} w_i$

- $(W^{(m)}, \pi_z^{(m)})$
- $(W, V_{z,\mu})$

$\Big\}$ $U_q(\hat{sl}_2)$ mod.

- $(W, \rho_z)$
- $(W, \bar{\rho}_z)$

$\Big\}$ $U_q(b_+)$ mod.

- $4$ universal transfer matrices $\in U_q(b_-)$
- $\mathcal{R} \in U_q(b_+) \otimes U_q(b_-)$ universal R-matrix

$$T^{(m)}(z) = \mathrm{Tr}_{W^{(m)}}\left((\pi^{(m)}_z \otimes 1)(k_1^{\alpha} \otimes \mathbb{I})\mathcal{R}\right), \quad T_V(z,\mu) = \mathrm{Tr}_W\left((V_{z,\mu} \otimes 1)(k_1^{\alpha} \otimes \mathbb{I})\mathcal{R}\right)$$

$$Q(z) = \mathrm{Tr}_W\left((\rho_z \otimes \mathbb{I})(k_1^{\alpha} \otimes \mathbb{I})\mathcal{R}\right), \quad \bar{Q}(z) = \mathrm{Tr}_W\left((\bar{\rho}_z \otimes \mathbb{I})(k_1^{\alpha} \otimes \mathbb{I})\mathcal{R}\right)$$

to make trace convergent

## - 2 Routes to TQ Relations

I) $U_q(b_+)$ SES (i.e. fuse $\infty$ & finite-dim reps)

$$0 \to (W, \rho_{zq}) \to (W \otimes W^{(1)}, \underbrace{\rho_z}_{\infty\text{-dim}} \otimes \underbrace{\pi_z^{(1)}}_{2\text{ dim}}) \to (W, \rho_{zq^{-1}}) \to 0$$

$$\Rightarrow \quad T^{(1)}(z)\,Q(z) = a(z)\,Q(qz) + b(z)\,Q(q^{-1}z)$$

## II) Factorization

Consider $\mathrm{Tr}_{W\otimes W}\left( \left( \rho_{q^{\mu}z} \otimes \bar{\rho}_{q^{-\mu}z} \otimes \mathbb{I} \right) (\Delta \otimes \mathbb{I}) (k_1^{\alpha} \otimes \mathbb{I}) R \right)$

$$= Q(q^{\mu}z)\, \bar{Q}(q^{-\mu}z)$$

a) Show $\mathbb{T}_V(z,\mu) \propto Q(q^{\mu}z)\, \bar{Q}(q^{-\mu}z)$

$\hookrightarrow$ Verma module

b) $\mathbb{T}^{(m)}(z) = \#\, \mathbb{T}_V(z,\mu) - \#\, \mathbb{T}_V(z,-\mu)$ ; $\mu = -\frac{(m+1)}{2}$

$m \in \mathbb{Z}_{\geq 0}$

$\hookrightarrow$ spin $(m+1)/2$ module

which comes from $U_q(\widehat{sl_2})$ SES

$$0 \to (W, V(z,-\mu)) \to (W, V(z,\mu)) \to (W^{(m)}, \pi_z^{(m)}) \to 0$$

$$\mu = -\frac{(m+1)}{2}$$

$V(z,\mu)$

$V(z,-\mu)$

$\pi_z^{(m)}$

$$T^{(1)}(z) = \# Q(q^{-1}z)\,\bar{Q}(qz) - \# Q(qz)\,\bar{Q}(q^{-1}z) \qquad (1)$$

$$T^{(0)}(z) = \# Q(q^{-1/2}z)\,\bar{Q}(q^{1/2}z) - \# Q(q^{1/2}z)\,\bar{Q}(q^{-1/2}z) \qquad (2)$$

$$(1) + (2) \quad \Rightarrow \quad T^{(1)}(z)\,Q(z) \sim \# Q(qz) + \# Q(q^{-1}z).$$

- Approach I) used by [Antonov/Feigin 97, Rossi/W 02, Korf 03, ...]

  Extended ~~to~~ to open chains by [Frassek/Szécsényi (XXX) 15, Baseilhac/Tsuboi (XXZ & Q) 18, Vlaar/Weston 20 (XXZ, TQ)]

- – Approach II) superficially more complicated, but was the one taken by BLZ (App D of paper III, 98)

  See also Boos/Jimbo/Miwa/Smirnov/Takeyama II, 08, App C for algebraic description.

- Related to 'factorization' of Verma R-matrix [Derkachov/collaborators 05...] in turn inspired by [Bazhanov/Stroganov 90].

- Factorization property in closed case used by [Bazhanov, Lukowski, Meneghelli, Staudacher 10, ..., Miao, Lamers, Pasquier 21]

- Described most algebraically by [Khoroshkin/Tsuboi 14] ⟵ XXZ ($q \neq 1$)

We use and develop this framework and extend to open chains.
[See also Lazarescu & Pasquier 14]

# The required representations

- We construct all of our $U_q(\hat{sl}_2)$, $U_q(b_\pm)$ repns using 2 q-oscillator algebras $A$, $\bar{A}$ $(|q|<1;\ q\neq 0)$

- $A = \langle a, a^+, f(D)\rangle$; $\bar{A} = \langle \bar{a}, \bar{a}^+, f(\bar{D})\rangle$ ; $f: \mathbb{Z} \to \mathbb{C}$

- $a^+ a = 1 - q^{2D}$ ; $a a^+ = 1 - q^{2(D+1)}$ ; $a f(D) = f(D+1)\, a$, $a^+ f(D) = f(D-1)\, a$

- $\bar{a}^+ \bar{a} = 1 - q^{-2\bar{D}}$ ; $\bar{a}\bar{a}^+ = 1 - q^{-2(\bar{D}+1)}$ ; $\bar{a} f(\bar{D}) = f(\bar{D}+1)\, \bar{a}$, $\bar{a}^+ f(\bar{D}) = f(\bar{D}-1)\, \bar{a}$

- We use $W = \bigoplus_{j=0}^{\infty} \mathbb{C}\, w_j$ given faithful $A$ & $\bar{A}$ module structure:

- $a\, w_{j+1} = w_j$ ; $a^+ w_j = (1 - q^{2(j+1)})\, w_{j+1}$ ; $D\, w_j = j\, w_j$

- $\bar{a}\, w_{j+1} = w_j$ ; $\bar{a}^+ w_j = (1 - q^{-2(j+1)})\, w_{j+1}$ ; $\bar{D}\, w_j = j\, w_j$

i.e. identify $A, \bar{A}$ with $\mathrm{End}(W)$.

- $\rho_z : U_q(b_+) \to A \quad (\simeq \mathrm{End}\, W) \qquad (\lambda = q - q^{-1}$
  $|q|<1, \neq 0$.

$$\rho_z(e_0) = \frac{q^{-1} z a^+}{\lambda} ; \quad \rho_z(e_1) = \frac{q z a}{\lambda} ; \quad \rho_z(k_1) = q^{-2D}$$

$k_0 \sim k_1^{-1}$

all repns are level 0.

- $\bar{\rho}_z : U_q(b_+) \to \bar{A}$

$$\bar{\rho}_z(e_0) = \frac{q z \bar{a}^+}{\lambda}, \quad \bar{\rho}_z(e_1) = \frac{q^{-1} z \bar{a}}{\lambda} ; \quad \bar{\rho}_z(k_1) = q^{-2(\bar{D}+1)}$$

- $V_{z,\mu} : U_q(\hat{sl}_2) \to A \qquad (\mu \in \mathbb{C})$

$$V_{z,\mu}(e_0) = \frac{q^{-1} z a^+}{\lambda} ; \quad V_{z,\mu}(e_1) = \frac{q z}{\lambda} a (q^{2\mu} - q^{-2(\mu+D)})$$

$$V_{z,\mu}(f_0) = \frac{q z^{-1}}{\lambda} a (q^{2\mu} - q^{-2(\mu+D)}) ; \quad V_{z,\mu}(f_1) = \frac{q^{-1} z^{-1} a^+}{\lambda}$$

$$V_{z,\mu}(k_1) = q^{-2\mu-1-D}$$

- $\pi_z^{(m)} = V_{z, -\frac{(m+1)}{2}} \Big|_{W^{(m)} = \bigoplus_{j=0}^{m} \mathbb{C} w_j}$ ; spin$\left(\frac{m+1}{2}\right)$ evaluation repn.

- We need one more $U_q(b_+)$ repn $(\omega, C_{z,\mu})$ triangular repn.

$$C_{z,\mu}(e_0) = 0 \quad ; \quad C_{z,\mu}(e_1) = \frac{z\bar{a}}{\lambda}$$

$$C_{z,\mu}(k_1) = q^{2\mu-1-2\bar{\jmath}} \quad ; \quad k_0 \sim k_1^{-1}$$

$T_V(z,\mu) \propto Q(zq^{\mu})\,\bar{Q}(zq^{-\mu})$ for closed chains

**Theorem** There is a $U_q(b_+)$ module isomorphism

$$\Theta_\mu^+ : \quad (W\otimes W,\ \rho_{zq^{\mu}}\otimes\bar{\rho}_{zq^{-\mu}}) \to (W\otimes W,\ V_{z,\mu}\otimes C_{z,\mu})$$

with

$$\Theta_\mu^+ = (q^2 a\otimes \bar{a}^+;\ q^2)_\infty\ q^{\mu(1\otimes\bar{D} - D\otimes 1)}$$

with $(q^2 x; q^2)_\infty = \sum_{k=0}^{\infty} \frac{q^{k(k+1)}(-x)^k}{(q^2;q^2)_k} = e_{q^2}^{-1}(q^2 x)$

**Proof** check

$$\Theta_\mu^+ \circ \left(\rho_{zq^{\mu}}\otimes\bar{\rho}_{zq^{-\mu}}\,\Delta(x)\right)$$

$$= \left(V_{z,\mu}\otimes C_{z,\mu}\,\Delta(x)\right)\circ\Theta_\mu^+ \qquad \in A\otimes\bar{A}$$

for $x \in U_q(b_+)$

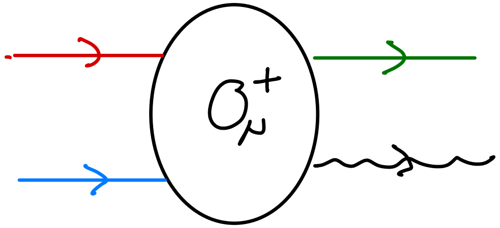
# Pictures!

$\rho_z \sim z \longrightarrow$ , $\bar{\rho}_z \sim z \longrightarrow$

$V_{z,\mu} \sim z,\mu \rightsquigarrow$ , $C_{z,\mu} \sim z,\mu \longrightarrow$ , $\pi^{(1)}_z \sim z \longrightarrow$

Then $O^+_\mu : (\omega \otimes \omega,\ \rho_{zq^{\mu}} \otimes \bar{\rho}_{zq^{-\mu}}) \to (\omega \otimes \omega,\ V_{z,\mu} \otimes C_{z,\mu})$

is




# L operators

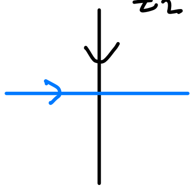
Consider $\mathcal{L}(z_1/z_2) = (M_{z_1} \otimes \pi^{(1)}_{z_2}) R$

anyone of above 5 repns

$\in (A \text{ or } \bar{A}) \otimes \mathrm{End}(\mathbb{C}^2)$

e.g. $\mathcal{L}(z) = (\rho_{z_1} \otimes \pi^{(1)}_{z_2}) R \propto \begin{pmatrix} q^{D} & -a^{+} q^{-D-1} z \\ -a\, q^{D+1} z & q^{-D} - q^{D+2} z^2 \end{pmatrix}$

$\sim$



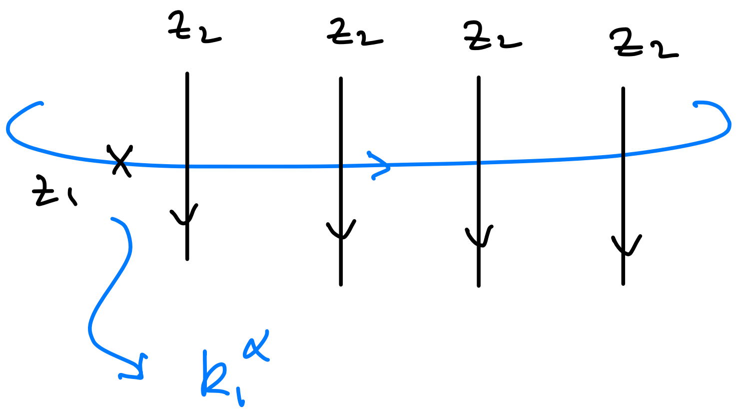

$z = z_1/z_2$

# Transfer matrices

$$T(z) \propto \mathrm{Tr}_W \left( M_{z_1} \otimes \underbrace{\pi_{z_2}^{(1)} \otimes \cdots \otimes \pi_{z_2}^{(1)}}_{N} \, (\mathbb{1} \otimes \Delta^{N-1}) \, (k_1^{\alpha} \otimes \mathbb{1}) \, R \right)$$

$\hookrightarrow \Delta' = \Delta$

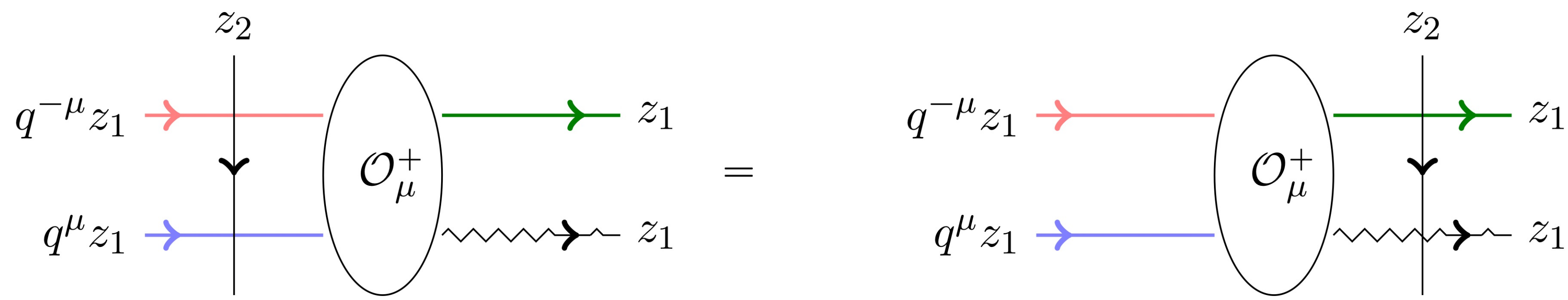
$=$




when $M(z) = \rho(z)$

| $T(z)$ | $M(z)$ |
|---|---|
| $Q(z)$ | $\rho(z)$ |
| $\bar{Q}(z)$ | $\bar{\rho}(z)$ |
| $T_V(z,\mu)$ | $V(z,\mu)$ |
| $T_C(z,\mu)$ | $C(z,\mu)$ |

- **Proposition**




i.e.

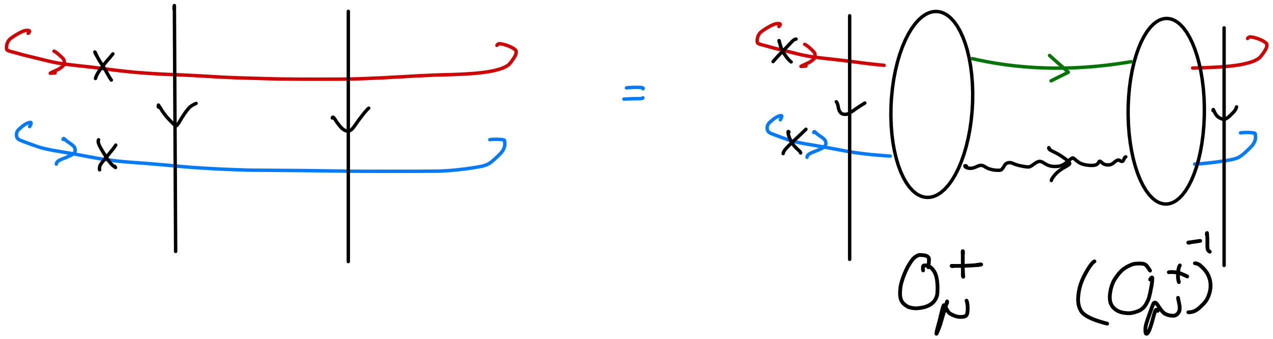
$$\boxed{\mathcal{O}_\mu^+ \, \mathcal{L}(zq^{\mu}) \, \bar{\mathcal{L}}(zq^{-\mu}) = \mathcal{L}_V(z,\mu) \, \mathcal{L}_c(z,\mu) \, \mathcal{O}_\mu^+} \qquad *$$

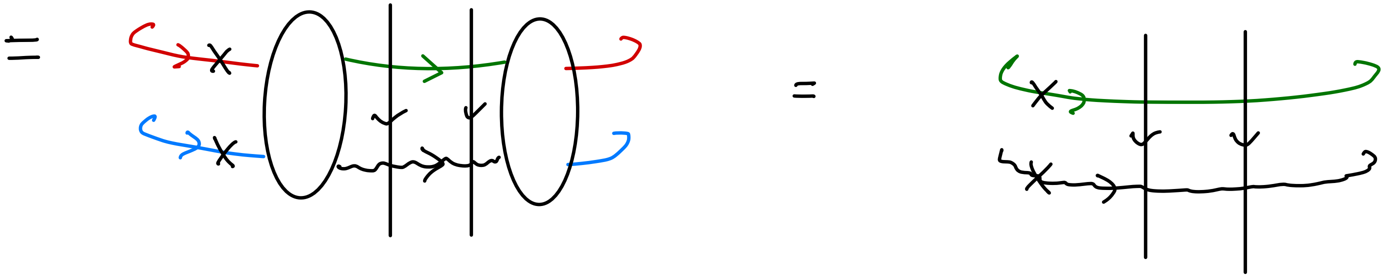
- **Note** $\quad \mathcal{L}_c(z,\mu) = \begin{pmatrix} q^{\bar{D}+1} & 0 \\ -\bar{a} q^{\bar{D}+1} z & q^{-\bar{D}+2\mu} \end{pmatrix}$ triangular.

* behind Derkachov's R-matrix factorization.

- Immediately implies: $Q(zq^{\mu})\,\bar{Q}(zq^{-\mu}) = \overline{T_V(z,\mu)}\,\overline{T_C(z,\mu)}$

since





- But $\mathcal{L}_C$ form $\Rightarrow$ $\overline{T_C(z,\mu)} = \sum_{j=0}^{\infty} q^{(j+1)N_0 + (-j+2\mu)N_1}\,\alpha^j$

$$= \frac{q^{(N_0+2\mu N_1)}}{(1-\alpha\, q^{(N_0-N_1)})}$$

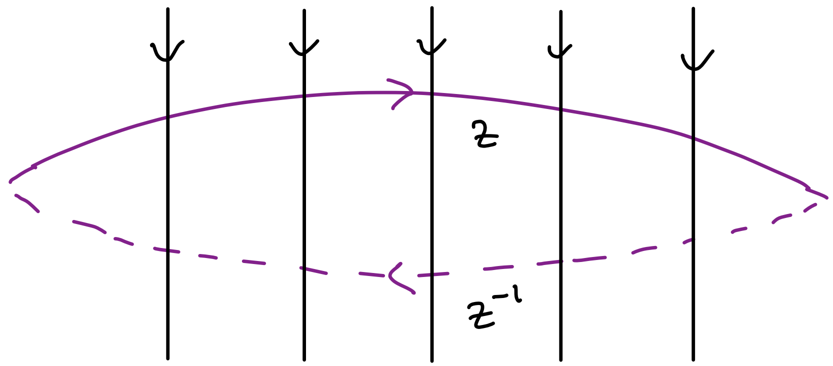
$N_0 = \#\, 0\text{'s}$, $N_1 = \#\, 1\text{'s}$.

So $\overline{T_V(z,\mu)} = Q(zq^{\mu})\,\bar{Q}(zq^{-\mu}) \times (\text{above})^{-1}$

$T_V(z,\mu) \propto Q(zq^{\mu})\,\bar{Q}(zq^{-\mu})$ for open chains

- The double row transfer matrix

Recall: We are interested in transfer matrices of the form [Sklyanin 88]

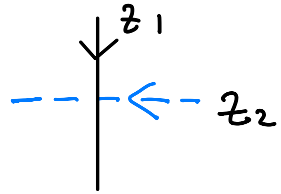



- To construct lower $\mathcal{L}$-ops for our different aux spaces, we use $U_q(b_-)$ repns given by $U_q(\widehat{sl_2})$ alg. automorphism:

$\psi(e_i) = f_{1-i}$, $\psi(f_i) = e_{1-i}$, $\psi(k_i) = k_{1-i}^{-1}$, from which

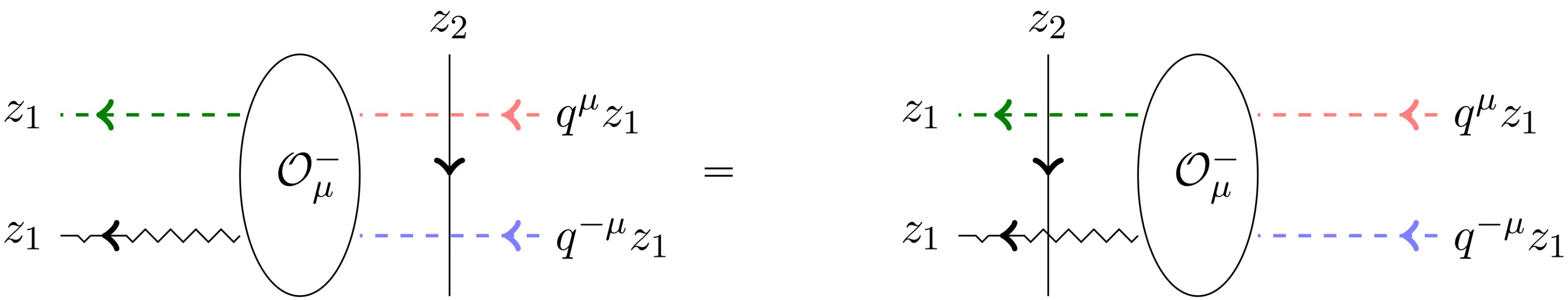
$$\rho^-_z = \rho_{z^{-1}} \circ \psi, \quad \bar{\rho}^-_z = \bar{\rho}_{z^{-1}} \circ \psi, \quad C^-_{z,\mu} = C_{z^{-1},\mu} \circ \psi$$

Then $\mathcal{L}^-(z) \propto (\pi^{(1)}_{z_1} \otimes \rho^-_{z_2})\, R$ =




$= P\, \mathcal{L}(z)\, P$ etc.

We also need




which represents

$$O_\mu^-\, \mathcal{L}^-(q^\mu z)\, \bar{\mathcal{L}}^-(q^{-\mu} z_1) = \mathcal{L}_V^-(z,\mu)\, \mathcal{L}_C^-(z,\mu)\, O_\mu^-$$

works for $O_\mu^- = P\, O_\mu\, P$.

- K matrices

  - We need right & left K-matrices for our 4 infinite dim modules, and $\pi_z^{(1)}$ that satisfy REs.

Step 1 : choose a left coideal subalg : in this case

$$B = \langle e_0 - q^{-1}\xi^{-1} k_0 f_1,\ e_1 - q^{-1}\xi k_1 f_0,\ k_0 k_1^{-1},\ k_1 k_0^{-1} \rangle \subset U_q(\hat{sl}_2)$$

$$\Delta(B) \subset U_q(\hat{sl}_2) \otimes B$$

Step 2 : For $U_q(\hat{sl}_2)$ repns (which are thus also $B$ repns)

– Solve $\pi^{(1)}_{z^{-1}}(x) \circ K^{(1)}(z) = K^{(1)}(z) \circ \pi^{(1)}_{z}(x) \qquad x \in B$

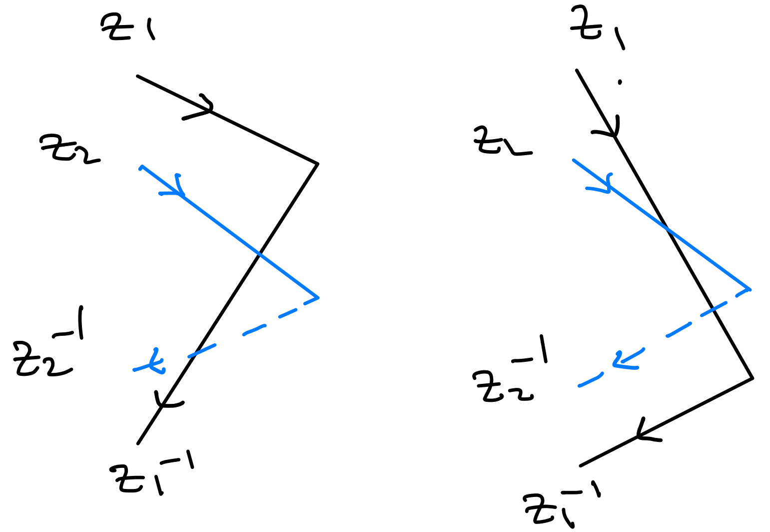
gives $$K^{(1)}(z) = \begin{pmatrix} \xi z^2 - 1 & 0 \\ 0 & (\xi - z^2) \end{pmatrix}$$

– solve analagous reln for Verma module

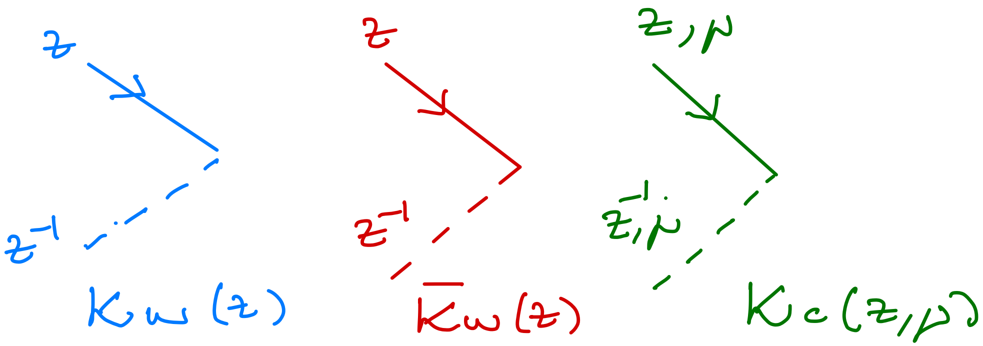
gives $\quad K(z,\mu) = z^{-2D} \dfrac{(\xi^{-1} q^{2\mu+2} z^{2}; q^{2})_D}{(\xi^{-1} q^{2\mu+2} z^{-2}; q^{2})_D}$

Step 3 : $\quad \rho_z$, $\bar{\rho}_z$, $C_{z,\mu}$ are $U_q(b_+)$ modules which cannot be lifted to $U_q(\hat{sl}_2)$ & then restricted to B. So

solve RE [Cherednik]



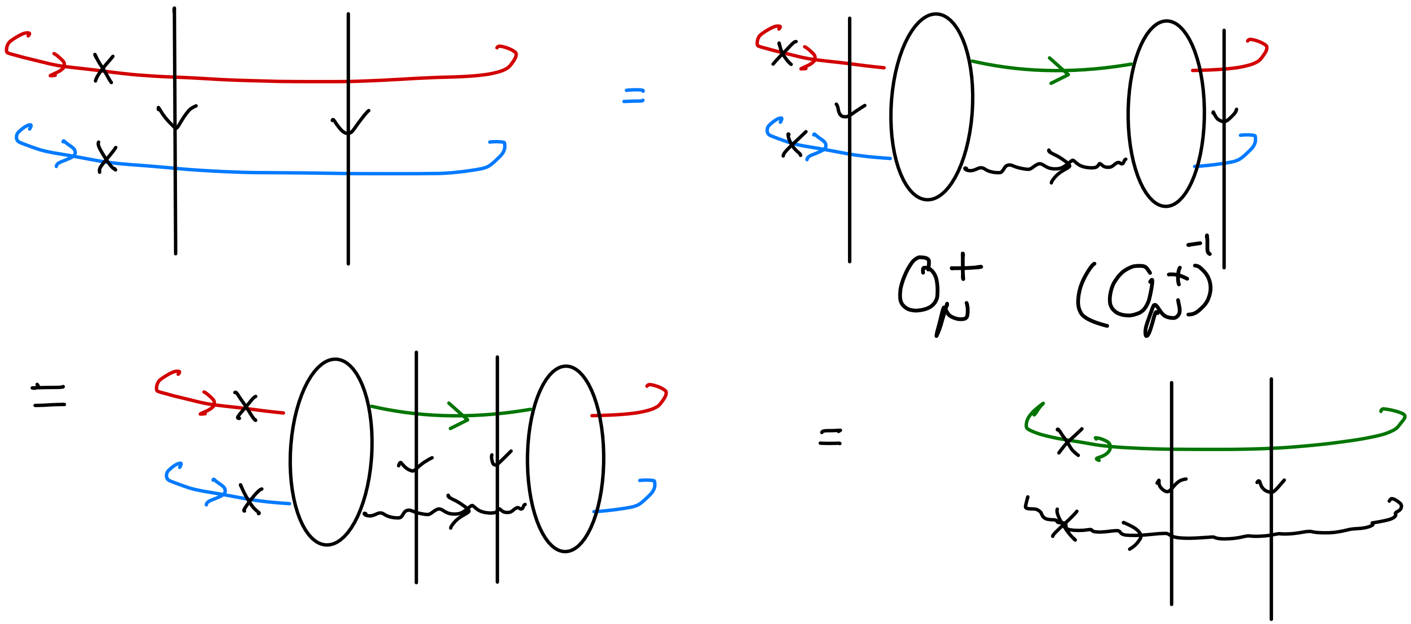
to find



; obtain left counterparts using transformation of right; label $\tilde{K}$ on left; dep on $\tilde{\xi}$.

- $T_V(z,\mu) \propto Q(zq^{\mu})\,\bar{Q}(zq^{-\mu})$ derivation

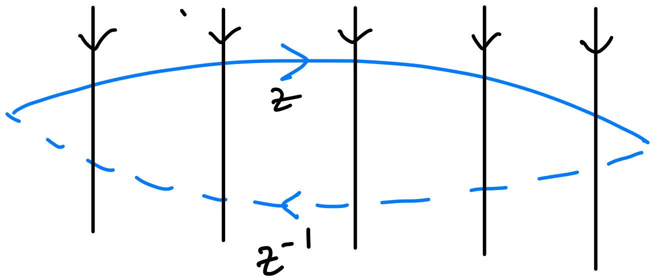
We want open analogue of




$$\Rightarrow \quad Q(zq^{\mu})\,\bar{Q}(zq^{-\mu}) = T_V(z,\mu)\,T_C(z,\mu)$$

Recall $\quad Q(z) = \mathrm{Tr}_W\left(\tilde{K}_W(z)\, M_W(z)\right)$

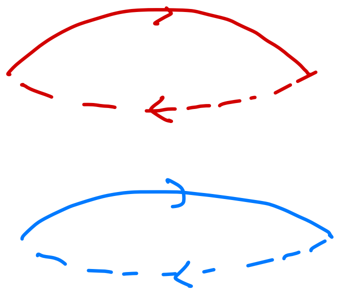
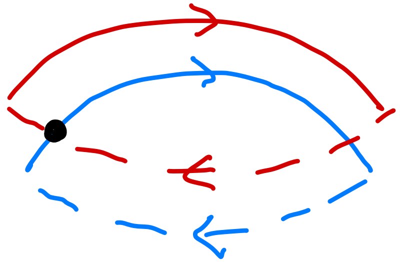
$$M(z) = \mathcal{L}_1^-(z)\cdots\mathcal{L}_N^-(z)\, K_W(z)\, \mathcal{L}_N(z)\cdots\mathcal{L}_1(z) \qquad \text{etc}$$

$=$ 


Note – no twist required for convergence now

- **Step 1** (missing out $\dagger$ for clarity)

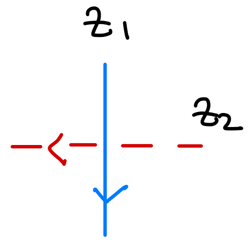
$Q(zq^{\mu})\,\bar{Q}(zq^{-\mu})$

$=$  $=$ 

usually the 1st step in showing commutativity of transfer matrices

## 2 Complications :

1) $\bar{R}(z_1|z_2) \sim$



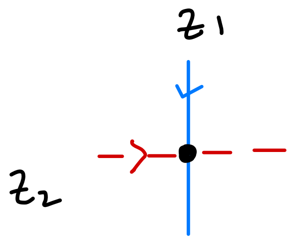

$\sim (\rho_{z_1} \otimes \bar{\rho}_{\bar{z}_2}) R \in A \otimes \bar{A}$

This is surprisingly simple :

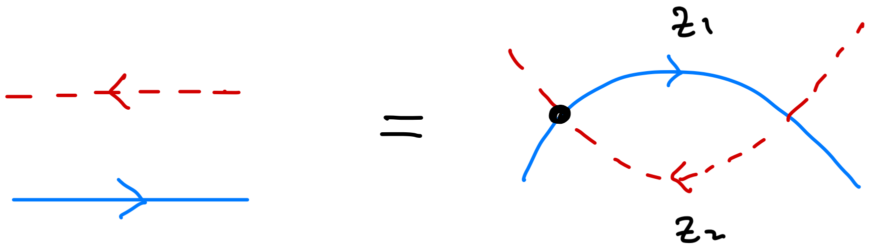
$$\bar{R}(z) = e_{q^2}(q z \, a \otimes \bar{a}^+) \, e_{q^2}(q^{-3} z \, a^+ \otimes \bar{a}) \, q^{-2D(\bar{D}+1)}$$

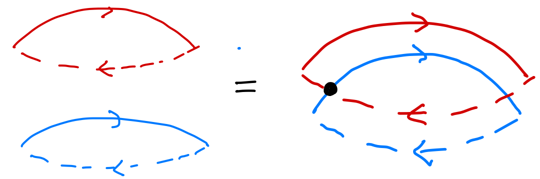
$$e_q(x) := \frac{1}{(qx;q)_\infty} = \sum_{k=0}^{\infty} \frac{q^k x^k}{(q;q)_k}$$

nicely convergent.

2) The $\tilde{R}(z) =$ 

matrix is defined s.t.




& hence 

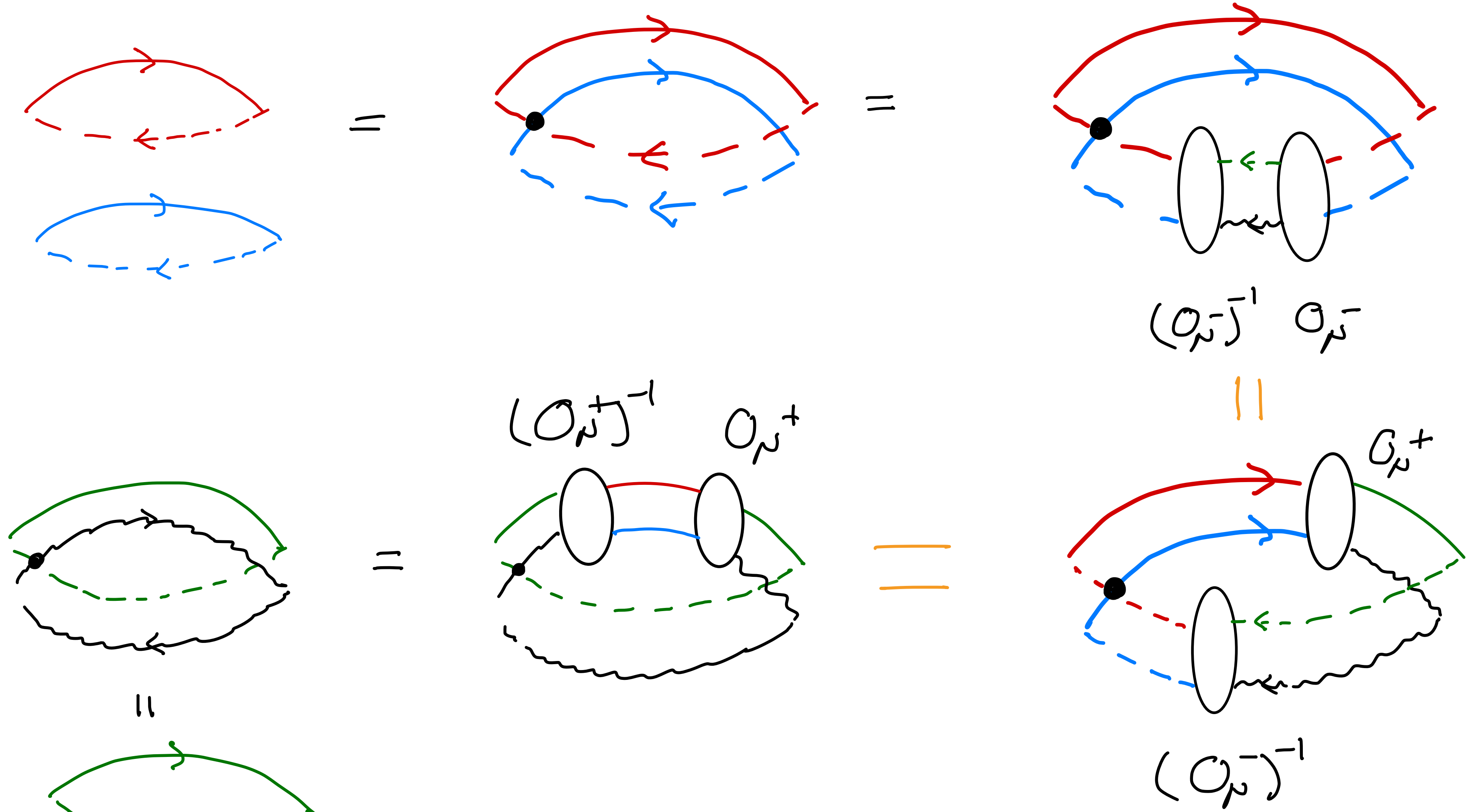
i.e. $\mathbb{I} = \tilde{R}(z)^{t_1} \, \bar{R}(z)^{t_1}$

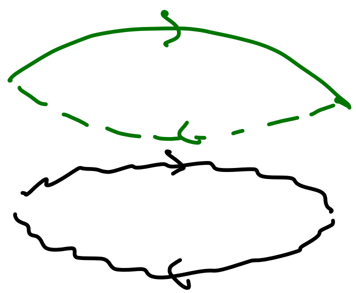
It is $\tilde{R}(z) := \left( \left( \bar{R}(z)^{t_1} \right)^{-1} \right)^{t_1}$

[ See Vlaar 15, 'Boundary transfer matrices ...' ]

- You need to be careful in taking partial trace — we are.

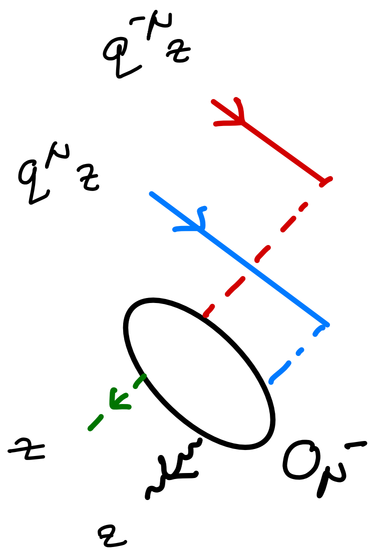
# Remaining Steps:



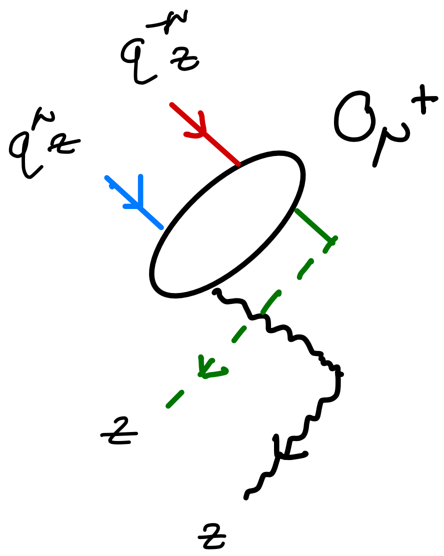



$$\Rightarrow \quad Q(zq^{\mu})\,\bar{Q}(zq^{-\mu}) = \bar{T}_V(z,\mu)\,\bar{T}_C(z,\mu)$$

We need 2 boundary factorization relns. (which hold)



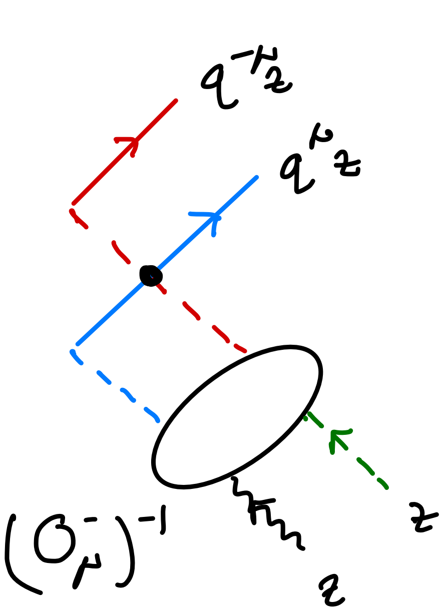

$=$



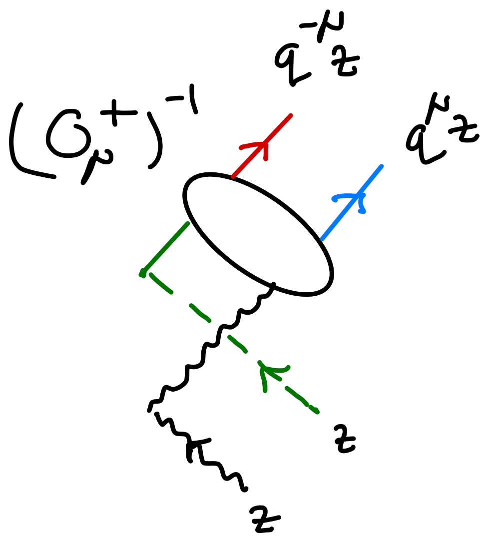

$$(O_\mu^-)_{21}\, K_w(q^\mu z)\, \bar{R}(z^2)\, \bar{K}_w(q^{-\mu} z)$$

$$\|$$

$$K_V(z,\mu)\, R_{V,\bar{c}}(z^2,\mu)\, K_c(z,\mu)\, O_\mu^+$$




$=$




$$\tilde{\bar{K}}_w(q^{-\mu}z)\, \tilde{R}(z^2)\, \tilde{K}_w(q^\mu z)\, (O_\mu^-)^{-1}_{21}$$

$$\|$$

$$(O_\mu^+)^{-1}\, \tilde{K}_c(z)\, \tilde{R}_{V\bar{c}}(z^2)\, \tilde{K}_V(z,\mu)$$

- In this way, we obtain

$$Q(zq^{\mu})\,\bar{Q}(zq^{-\mu})\,\overline{T}_c(z,\mu)^{-1} = \overline{T}_V(z,\mu)$$

- Again $\overline{T}_c(z,\mu) = \dfrac{q^{2(N_0+2\mu N_1)}}{1-\xi\tilde{\xi}\,q^{2(N_0-N_1)}}$ simple.

- Hence $\overline{T}_V(z,\mu)$ &

$$0 \to (\omega, V(z,-\mu)) \to (\omega, V(z,\mu)) \to (\omega^{(m)}, \pi_z^{(m)}) \to 0$$

$\mu = -\frac{(m+1)}{2}$

$$\Rightarrow \quad T^{(1)}(z) = \#\,\overline{T}_V(z,-1) - \#\,\overline{T}_V(z,1)$$

Q determinant condition

$$\Rightarrow \quad T^{(0)}(z) = \#\,\overline{T}_V(z,-1/2) - \#\,\overline{T}_V(z,1/2) = \text{const}$$

$$\Rightarrow \quad T^{(1)}(z)\,Q(z) = \alpha(z)\,Q(zq) + \beta(z)\,Q(zq^{-1})$$

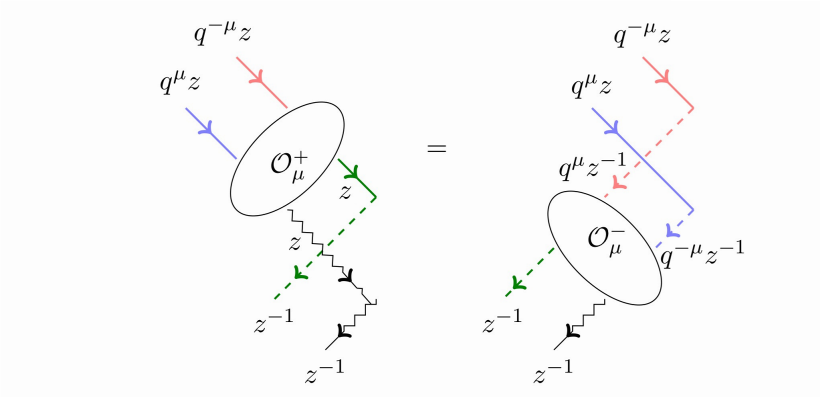
- From this we reproduce standard open Bethe eqns [Sklyanin'88] (paper 1).

- Trace involved in $Q$ converges. (no twist required)

- Diag. elements of $Q$ polynomial. ($\nexists$ all elem polynomial for $N=2$)

# Discussion

- We are used to understanding relns like

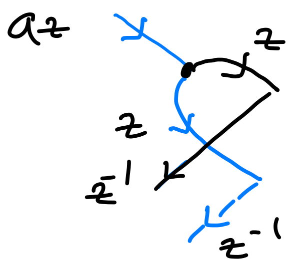



$$K_V(z,\mu)\, R_{V,\bar{C}}(z^2,\mu)\, K_C(z,\mu)\, \mathcal{O}_\mu^+ = (\mathcal{O}_\mu^-)_{21}\, K_W(q^\mu z)\, \bar{R}(z^2)\, K_W(q^{-\mu} z)$$

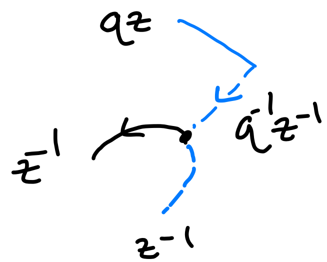
as coming from Schur's lemma. But these maps are not $U_q(\widehat{sl}_2)$, $B$, $U_q(b_\pm)$ homomorphisms. Purely lin. alg. identities as yet.

- Also true for SES, approach, which relies on

$$K_W(z)\, L(z^2)\, K^{(1)}(z)\, \iota \;\propto\; \iota\, K_W(qz)$$




$\propto$




boundary fusion

- We hope to generalise one or other approach to other $U_q(\mathfrak{g})$ using asymptotic / fundamental repns.

[Hernandez / Jimbo 11, Frenkel / Hernandez 14, Baseilhac / Tsuboi 17]

Indeed answer to above query may be to lift from $U_q(b_+)$ repns to repns of asymptotic alg - big enough to include coideal subalg?

TQ relns appear in Grothendieck ring of $U_q(b_+)$ repns

$\chi_q$ : G ring $\to$ transfer matrix.

- Does this work for non-diag $K^{(1)}(z)$ ?

Do not know yet.

Non-diag. RE difficult to solve and boundary factorization difficult to prove.