Permutations, moments, measures

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joint work with Einar Steingrímsson (Strathclyde)

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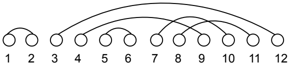
Combinatorics $\longleftrightarrow$ Probability

Moments

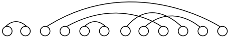
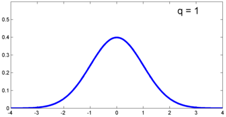
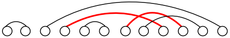
$$\int_{-\infty}^{\infty} x^{2n} \frac{exp(-x^2/2)}{\sqrt{2\pi}}\, dx \quad = \quad \#\text{ perfect matchings on } [2n]$$



$$\int_{-2}^{2} x^{2n} \frac{\sqrt{4-x^2}}{2\pi}\, dx \quad = \quad \#\text{ non-crossing perfect matchings on } [2n]$$

$$\sum_{k\geq 0} k^n \frac{e^{-1}}{k!} \quad = \quad \#\text{ set partitions on } [n]$$

$$\int_{0}^{\infty} x^n exp(-x)\, dx \quad = \quad \#\text{ permutations on } [n]$$

Comm. rel.	**Law of $a_i + a_i^*$ on $\bigoplus_n H^{\otimes n}$**	**CLT & Moments**
$a_i a_j^* - a_j^* a_i = \delta_{i,j}$ Fock 1932, Segal 1956	$\mathcal{N}(0,1)$	Classical CLT 
$a_i a_j^* - q a_j^* a_i = \delta_{i,j}$ Physics '50s/'70s Maths '90s Bożejko & Speicher 1991	q-Gaussian $q \in [-1,1]$  	$X_1, X_2, \ldots$ "iid" $\mathbb{E}(X_i) = 0, \mathbb{E}(X^2) = 1$ $X_i X_j = s(j,i) X_j X_i$ $s(j,i) \in \{-1,1\}$ $\frac{X_1+\ldots+X_N}{\sqrt{N}} \Longrightarrow a + a^*$ Speicher 1992  $q^{\#\text{crossings}}$

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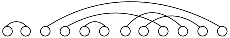
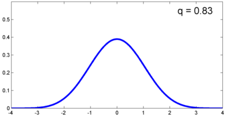
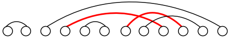
$a_i a_j^* - a_j^* a_i = \delta_{i,j}$ Fock 1932, Segal 1956	$\mathcal{N}(0,1)$	Classical CLT 
$a_i a_j^* - q a_j^* a_i = \delta_{i,j}$ Physics '50s/'70s Maths '90s Bożejko & Speicher 1991	q-Gaussian $q \in [-1,1]$ 	$X_1, X_2, \ldots$ "iid" $\mathbb{E}(X_i) = 0, \mathbb{E}(X^2) = 1$ $X_i X_j = s(j,i) X_j X_i$ $s(j,i) \in \{-1, 1\}$ $\frac{X_1 + \ldots + X_N}{\sqrt{N}} \implies a + a^*$ Speicher 1992  $q^{\#\text{crossings}}$

Comm. rel.	Law of $a_i + a_i^*$ on $\bigoplus_n H^{\otimes n}$	CLT & Moments

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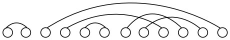
Fock 1932, Segal 1956

$$\mathcal{N}(0,1)$$

Classical CLT



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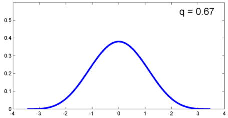
Physics ‘50s/‘70s

Maths ‘90s

Bożejko & Speicher 1991

q-Gaussian

$q \in [-1,1]$



$X_1, X_2, \ldots$ “iid”

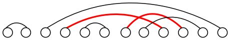
$\mathbb{E}(X_i) = 0, \mathbb{E}(X^2) = 1$

$X_i X_j = s(j,i) X_j X_i$

$s(j,i) \in \{-1, 1\}$

$$\frac{X_1 + \ldots + X_N}{\sqrt{N}} \implies a + a^*$$

Speicher 1992



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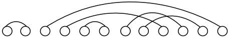
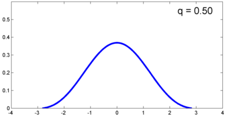
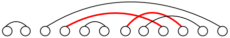
$a_i a_j^* - a_j^* a_i = \delta_{i,j}$ Fock 1932, Segal 1956	$\mathcal{N}(0,1)$	Classical CLT 
$a_i a_j^* - q a_j^* a_i = \delta_{i,j}$ Physics ‘50s/‘70s Maths ‘90s Bożejko & Speicher 1991	q-Gaussian $q \in [-1,1]$  	$X_1, X_2, \ldots$ “iid” $\mathbb{E}(X_i) = 0, \mathbb{E}(X^2) = 1$ $X_i X_j = s(j,i) X_j X_i$ $s(j,i) \in \{-1,1\}$ $\frac{X_1 + \ldots + X_N}{\sqrt{N}} \implies a + a^*$ Speicher 1992  $q^{\#\text{crossings}}$

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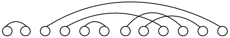
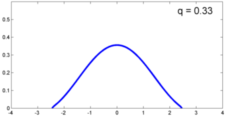
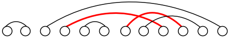
$a_i a_j^* - a_j^* a_i = \delta_{i,j}$ Fock 1932, Segal 1956	$\mathcal{N}(0,1)$	Classical CLT 
$a_i a_j^* - q a_j^* a_i = \delta_{i,j}$ Physics '50s/'70s Maths '90s Bożejko & Speicher 1991	q-Gaussian $q \in [-1,1]$  	$X_1, X_2, \ldots$ "iid" $\mathbb{E}(X_i) = 0, \mathbb{E}(X^2) = 1$ $X_i X_j = s(j,i) X_j X_i$ $s(j,i) \in \{-1,1\}$ $\frac{X_1 + \ldots + X_N}{\sqrt{N}} \implies a + a^*$ Speicher 1992  $q^{\#\text{crossings}}$

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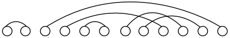
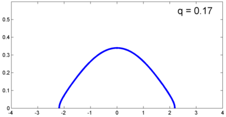
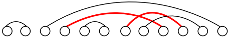
$a_i a_j^* - a_j^* a_i = \delta_{i,j}$ Fock 1932, Segal 1956	$\mathcal{N}(0,1)$	Classical CLT 
$a_i a_j^* - q a_j^* a_i = \delta_{i,j}$ Physics '50s/'70s Maths '90s Bożejko & Speicher 1991	q-Gaussian $q \in [-1,1]$ 	$X_1, X_2, \ldots$ "iid" $\mathbb{E}(X_i) = 0, \mathbb{E}(X^2) = 1$ $X_i X_j = s(j,i) X_j X_i$ $s(j,i) \in \{-1,1\}$ $\frac{X_1 + \ldots + X_N}{\sqrt{N}} \implies a + a^*$ Speicher 1992  $q^{\#\text{crossings}}$

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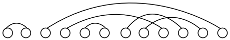
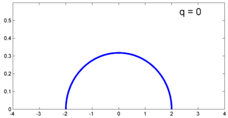
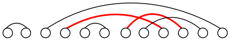
$a_i a_j^* - a_j^* a_i = \delta_{i,j}$ Fock 1932, Segal 1956	$\mathcal{N}(0,1)$	Classical CLT 
$a_i a_j^* - q a_j^* a_i = \delta_{i,j}$ Physics '50s/'70s Maths '90s Bożejko & Speicher 1991	q-Gaussian $q \in [-1,1]$  	$X_1, X_2, \ldots$ "iid" $\mathbb{E}(X_i) = 0, \mathbb{E}(X^2) = 1$ $X_i X_j = s(j,i) X_j X_i$ $s(j,i) \in \{-1,1\}$ $\frac{X_1 + \ldots + X_N}{\sqrt{N}} \Longrightarrow a + a^*$ Speicher 1992  $q^{\#\text{crossings}}$

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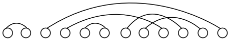
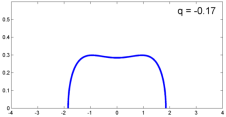
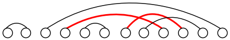
$a_i a_j^* - a_j^* a_i = \delta_{i,j}$ Fock 1932, Segal 1956	$\mathcal{N}(0,1)$	Classical CLT 
$a_i a_j^* - q a_j^* a_i = \delta_{i,j}$ Physics '50s/'70s Maths '90s Bożejko & Speicher 1991	q-Gaussian $q \in [-1,1]$ 	$X_1, X_2, \ldots$ "iid" $\mathbb{E}(X_i) = 0, \mathbb{E}(X^2) = 1$ $X_i X_j = s(j,i) X_j X_i$ $s(j,i) \in \{-1,1\}$ $\frac{X_1+\ldots+X_N}{\sqrt{N}} \implies a + a^*$ Speicher 1992  $q^{\#\text{crossings}}$

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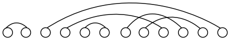
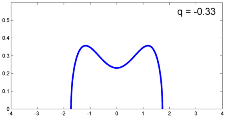
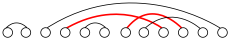
$a_i a_j^* - a_j^* a_i = \delta_{i,j}$ Fock 1932, Segal 1956	$\mathcal{N}(0,1)$	Classical CLT 
$a_i a_j^* - q a_j^* a_i = \delta_{i,j}$ Physics '50s/'70s Maths '90s Bożejko & Speicher 1991	q-Gaussian $q \in [-1,1]$ 	$X_1, X_2, \ldots$ "iid" $\mathbb{E}(X_i) = 0, \mathbb{E}(X^2) = 1$ $X_i X_j = s(j,i) X_j X_i$ $s(j,i) \in \{-1,1\}$ $\frac{X_1 + \ldots + X_N}{\sqrt{N}} \Longrightarrow a + a^*$ Speicher 1992  $q^{\#\text{crossings}}$

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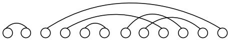
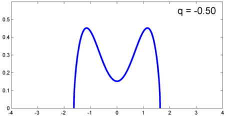
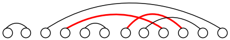
$a_i a_j^* - a_j^* a_i = \delta_{i,j}$ Fock 1932, Segal 1956	$\mathcal{N}(0,1)$	Classical CLT 
$a_i a_j^* - q a_j^* a_i = \delta_{i,j}$ Physics '50s/'70s Maths '90s Bożejko & Speicher 1991	q-Gaussian $q \in [-1,1]$  	$X_1, X_2, \ldots$ "iid" $\mathbb{E}(X_i) = 0, \mathbb{E}(X^2) = 1$ $X_i X_j = s(j,i) X_j X_i$ $s(j,i) \in \{-1,1\}$ $\frac{X_1 + \ldots + X_N}{\sqrt{N}} \implies a + a^*$ Speicher 1992  $q^{\#\text{crossings}}$

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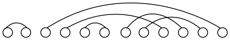
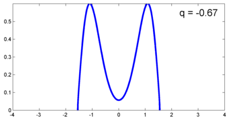
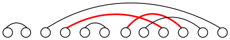
$a_i a_j^* - a_j^* a_i = \delta_{i,j}$ Fock 1932, Segal 1956	$\mathcal{N}(0,1)$	Classical CLT 
$a_i a_j^* - q a_j^* a_i = \delta_{i,j}$ Physics '50s/'70s Maths '90s Bożejko & Speicher 1991	q-Gaussian $q \in [-1,1]$  	$X_1, X_2, \ldots$ "iid" $\mathbb{E}(X_i) = 0, \mathbb{E}(X^2) = 1$ $X_i X_j = s(j,i) X_j X_i$ $s(j,i) \in \{-1,1\}$ $\frac{X_1 + \ldots + X_N}{\sqrt{N}} \Longrightarrow a + a^*$ Speicher 1992  $q^{\#\text{crossings}}$

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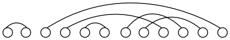
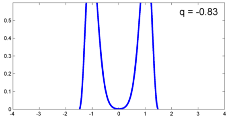
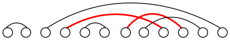
$a_i a_j^* - a_j^* a_i = \delta_{i,j}$ Fock 1932, Segal 1956	$\mathcal{N}(0,1)$	Classical CLT 
$a_i a_j^* - q a_j^* a_i = \delta_{i,j}$ Physics '50s/'70s Maths '90s Bożejko & Speicher 1991	q-Gaussian $q \in [-1,1]$  	$X_1, X_2, \dots$ "iid" $\mathbb{E}(X_i) = 0, \mathbb{E}(X^2) = 1$ $X_i X_j = s(j,i) X_j X_i$ $s(j,i) \in \{-1,1\}$ $\frac{X_1 + \ldots + X_N}{\sqrt{N}} \implies a + a^*$ Speicher 1992  $q^{\#\text{crossings}}$

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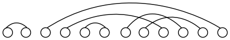
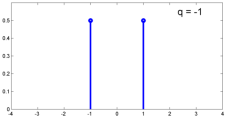
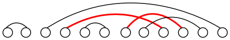
$a_i a_j^* - a_j^* a_i = \delta_{i,j}$ Fock 1932, Segal 1956	$\mathcal{N}(0,1)$	Classical CLT 
$a_i a_j^* - q a_j^* a_i = \delta_{i,j}$ Physics '50s/'70s Maths '90s Bożejko & Speicher 1991	q-Gaussian $q \in [-1,1]$  	$X_1, X_2, \ldots$ "iid" $\mathbb{E}(X_i) = 0, \mathbb{E}(X^2) = 1$ $X_i X_j = s(j,i) X_j X_i$ $s(j,i) \in \{-1,1\}$ $\frac{X_1 + \ldots + X_N}{\sqrt{N}} \implies a + a^*$ Speicher 1992  $q^{\#\text{crossings}}$

$$a_i a_j^* - q a_j^* a_i = t^N \delta_{i,j}$$

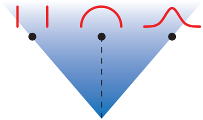
Physics '90s

Bożejko & Yoshida '06

B. 2012 JFA

'(q,t)-Gaussian'

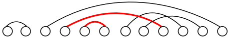
$$|q| \le t$$



$$X_i^{\epsilon} X_j^{\epsilon'} = \mu_{\epsilon',\epsilon}(j,i) X_j^{\epsilon'} X_i^{\epsilon}$$

$$\mu_{\epsilon',\epsilon}(j,i) \in \mathbb{R}$$

B. 2014 AIHP



$$q^{\#\text{crossings}}\ t^{\#\text{nestings}}$$

$$a_i a_j^* - q a_j^* a_i$$
$$= \delta_{i,j} + \alpha \langle e_i, \Pi_0 e_j \rangle q^{2N}$$

Bożejko, Ejsmont,

& Hasebe 2015

'type B' Gaussian

$$q, \alpha \in (-1,1)$$

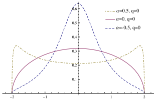


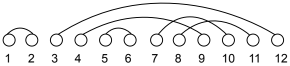
Fig. 1. $q = 0$.

$$X_i X_j = s(j,i) X_j X_i$$
$$Y_i Y_j = s(j,i) Y_j Y_i$$
$$X_i Y_j = r(j,i) Y_j X_i$$
$$s(j,i), r(j,i) \in \{-1, 1\}$$

B. - Ejsmont 2019 JMAA

Moments

$$\int_{-\infty}^{\infty} x^{2n} \frac{exp(-x^2/2)}{\sqrt{2\pi}}\, dx \quad = \quad \#\text{ perfect matchings on } [2n]$$



$$\int_{-2}^{2} x^{2n} \frac{\sqrt{4-x^2}}{2\pi}\, dx \quad = \quad \#\text{ non-crossing perfect matchings on } [2n]$$

$$\sum_{k\geq 0} k^n \frac{e^{-1}}{k!} \quad = \quad \#\text{ set partitions on } [n]$$

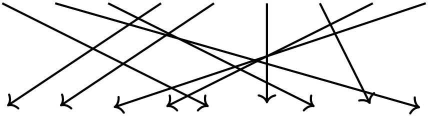
$$\int_{0}^{\infty} x^n exp(-x)\, dx \quad = \quad \#\text{ permutations on } [n]$$

Looking for:

- Unifying threads between known combinatorial constructions with ties to probability/analysis.
- Basic building blocks of more complicated combinatorial statistics.
- A combinatorial perspective on positivity.

A combinatorial perspective on positivity (B. & Steingrímsson '21)

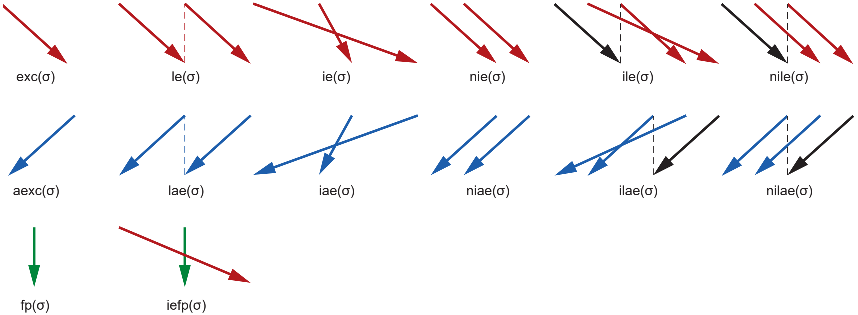
Represent $\sigma \in S_n$ in 2-line notation: i [top] $\mapsto \sigma(i)$ [bottom]



$$\sigma = 597126843$$

Definition (B. & Steingrímsson '21)

For $\sigma \in S_n$,



Definition For parameters $a, b, c, d, f, g, h, \ell, p, r, s, t, u, w \in \mathbb{R}$, let

$$\mathcal{C}(z) = \mathcal{C}_{a,b,c,d,f,g,h,\ell,p,r,s,t,u,w}(z) := \cfrac{1}{1 - \alpha_0 z - \cfrac{\beta_1 z^2}{1 - \alpha_1 z - \cfrac{\beta_2 z^2}{\ddots}}}$$

with

$$\alpha_n = u\,w^n + s\,[n]_{a,b} + t\,[n]_{f,g}, \qquad \beta_n = p\,r\,[n]_{c,d}\,[n]_{h,\ell}.$$

where $[n]_{x,y} = \sum_{k=0}^{n-1} x^k y^{n-1-k}$. (For $x \neq y$, $[n]_{x,y} = \frac{x^n - y^n}{x-y}$.)

$$
\begin{aligned}
\mathcal{C}(z) = \mathcal{C}_{a,b,c,d,f,g,h,\ell,p,r,s,t,u,w}(z) : &= \cfrac{1}{1 - \alpha_0 z - \cfrac{\beta_1 z^2}{1 - \alpha_1 z - \cfrac{\beta_2 z^2}{\ddots}}} \\
&= m_0 + m_1 z + m_2 z^2 + \dots
\end{aligned}
$$

with

$$
\alpha_n = u\, w^n + s\,[n]_{a,b} + t\,[n]_{f,g}, \qquad \beta_n = p\, r\,[n]_{c,d}\,[n]_{h,\ell} > 0
$$

$$\mathcal{C}(z) = \mathcal{C}_{a,b,c,d,f,g,h,\ell,p,r,s,t,u,w}(z) := \cfrac{1}{1 - \alpha_0 z - \cfrac{\beta_1 z^2}{1 - \alpha_1 z - \cfrac{\beta_2 z^2}{\ddots}}} = m_0 + m_1 z + m_2 z^2 + \ldots$$

with

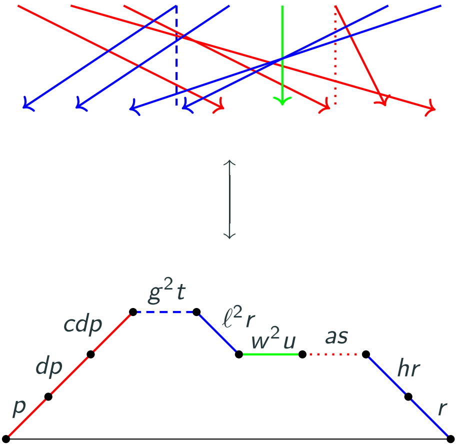
$$\alpha_n = u\, w^n + s\,[n]_{a,b} + t\,[n]_{f,g}, \qquad \beta_n = p\, r\,[n]_{c,d}\,[n]_{h,\ell} > 0$$

Theorem (B.-Steingrímsson, 2021) As formal power series,

$$\mathcal{C}(z) = \sum_{n \geq 0} \sum_{\sigma \in \mathcal{S}_n} a^{\mathrm{ile}(\sigma)} b^{\mathrm{nile}(\sigma)} c^{\mathrm{ie}(\sigma) - \mathrm{ile}(\sigma)} d^{\mathrm{nie}(\sigma) - \mathrm{nile}(\sigma)} f^{\mathrm{ilae}(\sigma)} g^{\mathrm{nilae}(\sigma)} h^{\mathrm{iae}(\sigma) - \mathrm{ilae}(\sigma)} \times \ell^{\mathrm{niae}(\sigma) - \mathrm{nilae}(\sigma)} p^{\mathrm{exc}(\sigma) - \mathrm{le}(\sigma)} r^{\mathrm{aexc}(\sigma) - \mathrm{lae}(\sigma)} s^{\mathrm{le}(\sigma)} t^{\mathrm{lae}(\sigma)} u^{\mathrm{fp}(\sigma)} w^{\mathrm{iefp}(\sigma)} z^n.$$

Proof.

Exhibit a bijection:



😈 $\in$ bijection

Related to/extends:

Françon-Viennot 1979
Foata-Zeilberger 1990
Biane 1993
de Médicis-Viennot 1994
Simion-Stanton 1994
Clarke-Steingrímsson-Zeng 1996
Randrianarivony 1998
Elizalde 2018

Contemporaneous: Sokal & Zeng 2020 (arXiv:2003.08192, 122 p.).

Heilbronn Institute for Mathematical Research

Lancaster, 6-10 June 2022

Quick example

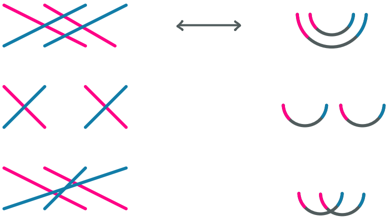
$c = s = t = u = 0$, $a = b = d = f = g = \ell = w = p = r = 1$.

Free parameter: h.

No: fixed points, linked excedances, linked antiexcedances, inversions among excedances.

Yes: $h^{\#\text{inversions among anti-excedances}}$

$n = 4$:



Another quick example

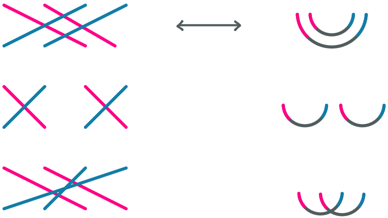
$c = s = t = u = 0$, $a = b = d = f = g = w = p = r = 1$.

Free parameters: h, ℓ.

No: fixed points, linked excedances, linked antiexcedances, inversions among excedances.

Yes: $h^{\#\text{inversions among anti-excedances}}\ \ell^{\#\text{restricted non-inversions among anti-excedances}}$

$n = 4$:



More examples

Set partitions by # blocks (Stirling, second kind)	$\longleftrightarrow$	Poisson
Non-crossing set partitions by # blocks (Narayana)		Free Poisson (Marchenko-Pastur)
Eulerian polynomials $\sum_\sigma q^{\mathrm{des}(\sigma)}$		Geometric
Derangements		e.g. Martin & Kearney '15
Alternating permutations		Sokal '18
Little Schroeder numbers		Młotowski & Penson '13

Positivity

Fix $a, b, c, d, f, g, h, \ell, p, r, s, t, u, w \in \mathbb{R}$ and let

$$m_n = \sum_{\sigma \in \mathcal{S}_n} a^{\mathrm{ile}(\sigma)} b^{\mathrm{nile}(\sigma)} c^{\mathrm{ie}(\sigma)-\mathrm{ile}(\sigma)} d^{\mathrm{nie}(\sigma)-\mathrm{nile}(\sigma)} f^{\mathrm{ilae}(\sigma)} g^{\mathrm{nilae}(\sigma)} h^{\mathrm{iae}(\sigma)-\mathrm{ilae}(\sigma)}$$

$$\times\, \ell^{\mathrm{niae}(\sigma)-\mathrm{nilae}(\sigma)} p^{\mathrm{exc}(\sigma)-\mathrm{le}(\sigma)} r^{\mathrm{aexc}(\sigma)-\mathrm{lae}(\sigma)} s^{\mathrm{le}(\sigma)} t^{\mathrm{lae}(\sigma)} u^{\mathrm{fp}(\sigma)} w^{\mathrm{iefp}(\sigma)}.$$

Is (m_n) a moment sequence of some positive Borel measure on $\mathbb{R}$?

$$\mathcal{C}(z) = \cfrac{1}{1 - \alpha_0 z - \cfrac{\beta_1 z^2}{1 - \alpha_1 z - \cfrac{\beta_2 z^2}{\ddots}}} = m_0 + m_1 z + m_2 z^2 + \dots$$

with

$$\alpha_n = u\, w^n + s\,[n]_{a,b} + t\,[n]_{f,g}, \qquad \beta_n = p\, r\,[n]_{c,d}\,[n]_{h,\ell}$$

Answer:

If $\beta_1, \ldots, \beta_{k-1} > 0$ and $\beta_n = 0$ for all $n \geq k$, measure supported on k elements.

If $\beta_n > 0$ for all n, measure exists (need not be unique).

Answer:

If $\beta_1, \ldots, \beta_{k-1} > 0$ and $\beta_n = 0$ for all $n \geq k$, measure supported on k elements.

If $\beta_n > 0$ for all n, measure exists (need not be unique).

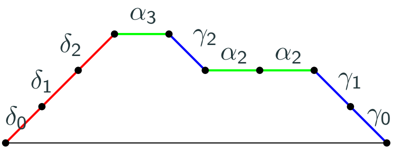
Perspective 1: Hamburger moment problem

$$H_n = \det \begin{bmatrix} m_0 & m_1 & \ldots & m_n \\ m_1 & m_2 & \ldots & m_{n+1} \\ \vdots & & & \vdots \\ m_n & m_{n+1} & \ldots & m_{2n} \end{bmatrix} = (\beta_1)^n (\beta_2)^{n-1} \ldots (\beta_{n-1})^2 \beta_n > 0$$

E.g. Partitions of $[n]$: $e_n = \sum_\pi x^{\#\text{blocks}(\pi)}$. Then, $H_n = x^{\binom{n+1}{2}} \prod_{k=1}^n k!$

E.g. Derangements of $[n]$: $d_n = \#\{\, \sigma \in S_n \mid \text{fp}(\sigma) = 0\}$. $H_n = \prod_{k=1}^n (k!)^2$

Perspective 2: tridiagonal operators



Summing over all weighted Motzkin paths p of length n:

$$\sum_p wt(p) = \langle A^n e_0, e_0 \rangle$$

where

$$A = \begin{bmatrix} \alpha_0 & \delta_1 & 0 & 0 & \dots \\ \gamma_1 & \alpha_1 & \delta_2 & 0 & \dots \\ 0 & \gamma_2 & \alpha_2 & \delta_3 & \dots \\ \vdots & & & \ddots & \end{bmatrix}$$

When $\delta_n \gamma_n \geq 0$ for all n, can symmetrize (redistribute weights):

$$\sum_p wt(p) = \langle A^n e_0, e_0 \rangle = \langle \tilde{A}^n e_0, e_0 \rangle$$

$$\tilde{A} = \begin{bmatrix} \alpha_0 & \sqrt{\delta_1\gamma_1} & 0 & 0 & \dots \\ \sqrt{\delta_1\gamma_1} & \alpha_1 & \sqrt{\delta_2\gamma_2} & 0 & \dots \\ 0 & \sqrt{\delta_2\gamma_2} & \alpha_2 & \sqrt{\delta_3\gamma_3} & \dots \\ \vdots & & & \ddots & \end{bmatrix}$$

Bounded case: self-adjoint extension and the spectral theorem.
In the unbounded case, may not have uniqueness.

$$\tilde{A} = \begin{bmatrix} \alpha_0 & \sqrt{\delta_1\gamma_1} & 0 & 0 & \dots \\ \sqrt{\delta_1\gamma_1} & \alpha_1 & \sqrt{\delta_2\gamma_2} & 0 & \dots \\ 0 & \sqrt{\delta_2\gamma_2} & \alpha_2 & \sqrt{\delta_3\gamma_3} & \dots \\ \vdots & & & \ddots & \end{bmatrix}$$

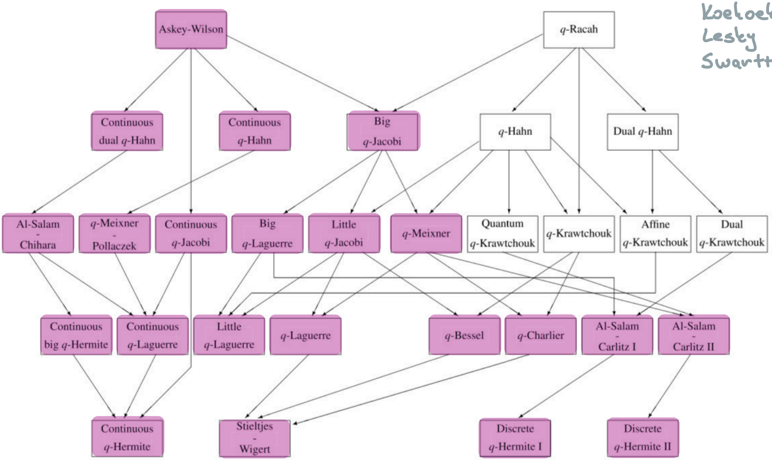
Bounded case: self-adjoint extension and the spectral theorem.
In the unbounded case, may not have uniqueness.

Perspective 3: Orthogonal polynomials

$$P_0(x) = 1, P_1(x) = x - \alpha_0, P_{n+1}(x) = (x - \alpha_n)P_n(x) - \beta_n P_{n-1}(x)$$

Stieltjes, Favard, Shohat, ...

Consequences: (1) New Interpretations



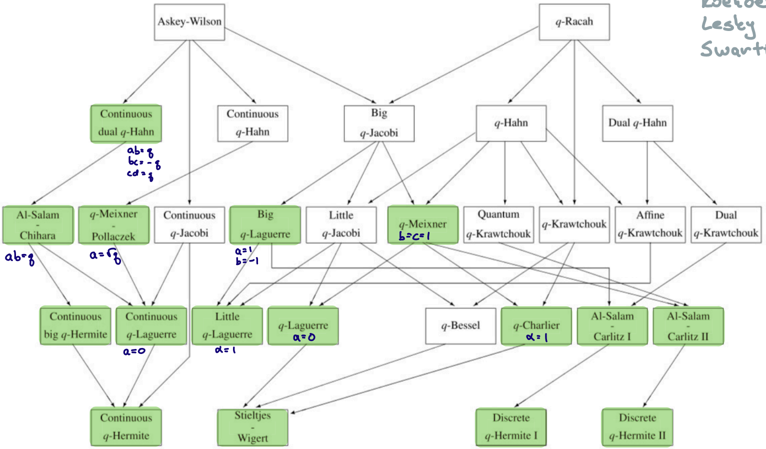
Koekoek
Lesky
Swarttouw

Corteel & Williams '11/'12:

$$m_n = \frac{(abcd;q)_\infty}{(q,ab,ac,ad,bc,bd,cd;q)_\infty} \sum_{\ell=0}^{n} (-1)^{n-\ell} \binom{n}{\ell} \left(\frac{1-q}{2}\right)^{\ell} \frac{Z_\ell}{\prod_{i=0}^{\ell-1}(\alpha\beta - \gamma\delta q^i)}.$$

Consequences: (1) New Interpretations

Koekoek
Lesky
Swarttouw

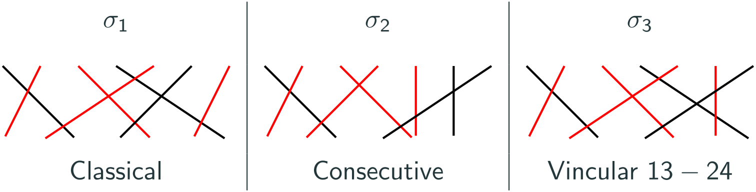


$$m_n = \sum_{\sigma \in \mathcal{S}_n} a^{\mathrm{ile}(\sigma)} b^{\mathrm{nile}(\sigma)} c^{\mathrm{ie}(\sigma)-\mathrm{ile}(\sigma)} d^{\mathrm{nie}(\sigma)-\mathrm{nile}(\sigma)} f^{\mathrm{ilae}(\sigma)} g^{\mathrm{nilae}(\sigma)} h^{\mathrm{iae}(\sigma)-\mathrm{ilae}(\sigma)}$$

$$\times\, \ell^{\mathrm{niae}(\sigma)-\mathrm{nilae}(\sigma)} p^{\mathrm{exc}(\sigma)-\mathrm{le}(\sigma)} r^{\mathrm{aexc}(\sigma)-\mathrm{lae}(\sigma)} s^{\mathrm{le}(\sigma)} t^{\mathrm{lae}(\sigma)} u^{\mathrm{fp}(\sigma)} w^{\mathrm{iefp}(\sigma)} z^n.$$

Definition. Permutation pattern: $\pi \in S_n$ and a containment rule

Example: $\pi = 1324$ classical, consecutive, or vincular



Let $\mathrm{Av}_\pi(n) := \#$ permutations on $[n]$ avoiding π.

Theorem (Simion & Schmidt '85).
For any classical pattern π of length 3,

$$\mathrm{Av}_\pi(n) = \frac{1}{n+1}\binom{2n}{n}.$$

Conjecture (Stanley-Wilf) / Theorem (Marcus & Tardos '04)

$$(\mathrm{Av}_\pi(n))^{1/n} \to c_\pi > 0 \text{ as } n \to \infty.$$

Only 3 symmetry classes for patterns of length 4:

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1234 Gessel '90 (exact) |

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Only 3 symmetry classes for patterns of length 4:

1234 Gessel '90 (exact) | 1342 Bona '97 (exact) |

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$$(\mathrm{Av}_\pi(n))^{1/n} \to c_\pi > 0 \text{ as } n \to \infty.$$

Only 3 symmetry classes for patterns of length 4:

1234 Gessel '90 (exact) | 1342 Bona '97 (exact) | 1324 wide open, $c = ?$

Theorem (Rains '98). Let $\pi = 1\,2\,3 \ldots k$ classical ($k \in \mathbb{N}$). Then,

$$\mathrm{Av}_\pi(n) = \mathbb{E}_{U \in \mathbb{U}(k)}(|\,Tr(U)^2|^n).$$

Conjecture (B. & Steingrímsson, Elvey Price's PhD Thesis for 1324).
For any classical pattern π, $(\mathrm{Av}_\pi(n))_n$ is a moment sequence.

Numerical evidence for patterns of length 4: Bostan, Elvey Price, Guttmann, Maillard '20

Numerical evidence for patterns of length 5: Clisby, Conway, Guttmann, Inoue '21+

$$\mathcal{C}(z) = \mathcal{C}_{a,b,c,d,f,g,h,\ell,p,r,s,t,u,w}(z) := \cfrac{1}{1 - \alpha_0 z - \cfrac{\beta_1 z^2}{1 - \alpha_1 z - \cfrac{\beta_2 z^2}{\ddots}}}$$

with

$$\alpha_n = u\, w^n + s\, [n]_{a,b} + t\, [n]_{f,g}, \qquad \beta_n = p\, r\, [n]_{c,d}\, [n]_{h,\ell}.$$

Theorem (B. & Steingrímsson, '21) For any pattern of length 3 (classical, consecutive, vincular), $(\mathrm{Av}_\pi(n))_n$ is a moment sequence $\iff \sum_{n\geq 0} \mathrm{Av}_\pi(n) z^n$ is a special case of $\mathcal{C}(z)$.

Refines further:

Theorem Set $s = qx$, $p = x$, all other parameters $= 1$,

$$\mathcal{C}(z) = \sum_{n\geq 0}\sum_{\sigma\in S_n} x^{\mathrm{des}(\sigma)} q^{\mathrm{occ}_{321}(\sigma)} z^n.$$

(Previously in Elizalde '17.)

Theorem Set $b = d = g = l = q, s = qx, p = u = x$ all other parameters $= 1$,

$$\mathcal{C}(z) = \sum_{n\geq 0}\sum_{\sigma\in S_n} x^{\mathrm{des}(\sigma)+1} q^{\mathrm{occ}_{2-31}(\sigma)} z^n.$$

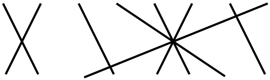
(Previously in Claesson & Mansour '02.)

Further work in progress with Slim Kammoun and Einar Steingrímsson.

(3) New definitions

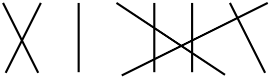
Definition (B. & Steingrímsson '21)
***k*-arrangement**: $\sigma \in S_n$ and a k-coloring of the fixed points in σ ($k \in \mathbb{N}_0$)

$k = 0$



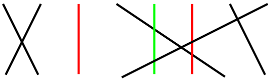
Derangements

$k = 1$



Permutations

$k = 2$



Arrangements (Comtet '74)
Decorated permutations (Postnikov '06)

Same form of CF, replace uw^n by $(u_1 + \ldots + u_k)w^n$.

$\rightsquigarrow$ Same 14 statistics + distinguishes the color of the fixed points.

$$A_k(n) := \begin{cases} 1, & n = 0 \\ \#k\text{-arrangements on } [n], & n \in \mathbb{N} \end{cases}.$$

Easy Proposition:

- $A_k(n) = nA_k(n-1) + (k-1)^n$.
- $A_k(n) = \operatorname{perm} \begin{bmatrix} k & 1 & 1 & \cdots & 1 \\ 1 & k & 1 & \cdots & 1 \\ 1 & 1 & k & \cdots & 1 \\ \vdots & \vdots & \vdots & & 1 \\ 1 & 1 & 1 & \ldots & k \end{bmatrix}.$
- $A_k(n) = \sum_{i \geq 0} \binom{n}{i} A_{k-1}(i)$.

Fix a classical permutation pattern π of length 3. Let:

$$\mathsf{Av}_\pi(n;k) := \#k\text{-arrangements on } [n] \text{ avoiding } \pi.$$

Proposition (Simion & Schmidt '85)

$$\mathsf{Av}_\pi(n;1) = C_n = \frac{1}{n+1}\binom{2n}{n}$$

Proposition (B. & Steingrímsson '21)

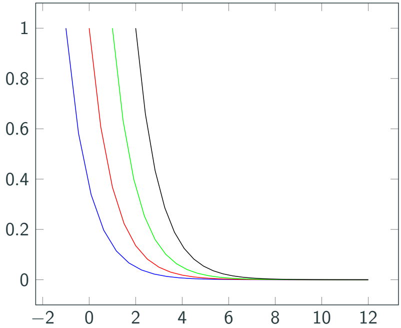
$$\mathsf{Av}_\pi(n;2) = C_{n+1}$$

~~**Conjecture** (B. & Steingrímsson '21)~~ **Theorem (Fu, Han, Lin '20)**

$$\mathsf{Av}_\pi(n;3) = C_{n+2} - 2^n$$

Proposition (B. & Steingrímsson, '21)

$$\#k\text{-arrangements on } [n] = \int_{k-1}^{\infty} x^n e^{-x+(k-1)} dx$$

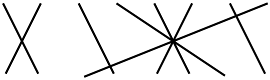


Positivity previously observed for:

- $k = 0$: Martin & Kearney '15
- $k = 2$: Ardila, Rincón, Williams '16 (# positroids)

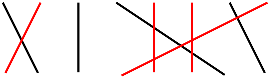
(4) Further generalizations

S_n



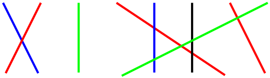
Permutations

B_n



Signed permutations

$\mathbb{Z}_k \wr S_n$



k-colored permutations

Steingrímsson '94 defines fixed points, excedances, anti-excedances, and inversions.

Corollary (B. & Steingrímsson '21)
For $s = p = kx$, $t = r = ky$, $u = (k-1)x + q$ *(all other parameters =1),*

$$\mathcal{C}(z) = \sum_{n \geq 0} \sum_{\sigma \in S_n^k} x^{exc(\sigma)} y^{aexc(\sigma)} q^{fxpt(\sigma)} z^n.$$

(Refines further.)

Corollary (B. & Steingrímsson '21)
For $a = c = h = r = q$, $b = f = d = \ell = t = q^2$, $g = w = 0$, $p = u = 1$, $s = 2q$,

$$\mathcal{C}(z) = \sum_{n \geq 0} \sum_{\sigma \in S_n^k} q^{inv(\sigma)} z^n.$$

Recovers Biane '93 for $k = 1$.

Conclusion (for now)

Multiparameter combinatorial frameworks can:

- unite
- distinguish
- explain
- suggest new definitions and
- new points of view on familiar things

THANK YOU

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