The rough-smooth boundary in dimer models

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Galileo Galilei Institute, May 26, 2022
Two-periodic Aztec diamond
Two-periodic Aztec diamond
Aztec diamond

A *domino tiling* of an Aztec diamond shape corresponds to a *dimer configuration* on the Aztec graph.
Probability measure

Let $\nu(e) > 0$ be the weight of the edge $e$ in the graph $G$. The probability of a certain dimer cover $C$, i.e. each vertex is covered exactly once, is

$$\frac{1}{Z} \prod_{e \in C} \nu(e).$$

$Z$ is the partition function.
The **two-periodic weighting** of the Aztec diamond is defined in the following way. For a two-colouring of the faces, the edge weights around a particular coloured face alternate between \( a \) and \( b \), we have **a-edges** and **b-edges**. E.g. for a size 4 Aztec diamond...
Aztec diamond height function

To each tiling of an Aztec diamond we can associate a **height function**. The heights sit on the faces of the Aztec graph. The height differences between two faces are given by

- $+3$ if we cross a dimer with a white vertex to the right
- $-1$ if we do not cross a dimer and have a white vertex to the right
Two-periodic Aztec diamond height function
Two-periodic Aztec diamond height function

Picture by B. Young
Variational principle for the limit height shape

The limiting height function solves

\[ \inf_{h} \int_{\Omega} \sigma(\nabla h) \, dx, \]

where \( \sigma \) is the surface tension. (Cohn-Kenyon-Propp).

Properties investigated by Kenyon and Okounkov.

Recent breakthrough work by Astala-Duse-Prause-Zhong, *Dimer models and Conformal structures*. Investigate possible geometries and prove regularity results. **Pokrovsky-Talapov law**: height function \( \sim d^{3/2} \) at typical boundary point of the rough region.
Phases

The curve in the picture is a degree 8 curve with two real components. We get three regions which are called **frozen, rough and smooth**.
Kenyon, Okounkov and Sheffield have characterized the different limiting translation invariant Gibbs measures that are possible for bipartite dimer models on the plane.

There are three classes of Gibbs measures, frozen, rough and smooth.

Correlations between dominos decay polynomially with distance in the rough region, and exponentially in the smooth region.
Frozen-Rough boundary
Frozen-Rough boundary
Airy kernel point process

Figure: The Airy line ensemble. The top path is the Airy process.
Airy kernel point process

The extended Airy point process is a determinantal point process on parallel lines \( \{\tau_q\} \times \mathbb{R}, 1 \leq q \leq L_1 \) in \( \mathbb{R}^2 \). We can think of it as a random measure \( \mu_{Ai} \) defined via a Laplace transform. Let \( A_p, 1 \leq p \leq L_2 \), be disjoint intervals in \( \mathbb{R} \), \( w_{p,q} \in \mathbb{C} \),

\[
\mathbb{E} \left[ \exp \left( \sum_{p=1}^{L_2} \sum_{q=1}^{L_1} w_{p,q} \mu_{Ai} (\{\tau_q\} \times A_p) \right) \right] \\
= \det (I + (e^\Psi - 1) K_{\text{extAi}})_{L^2(\{\tau_1,\ldots,\tau_q\} \times \mathbb{R})},
\]

where

\[
\psi(x) = \sum_{q=1}^{L_1} \sum_{p=1}^{L_2} w_{p,q} \mathbb{1}_{\{\tau_q\} \times A_p} (x).
\]
Airy kernel point process

\[
\mathbb{E} \left[ \exp \left( \sum_{p=1}^{L_2} \sum_{q=1}^{L_1} w_{p,q} \mu_{Ai} \left( \{ \tau_q \} \times A_p \right) \right) \right]
= \det \left( I + (e^\Psi - 1) K_{\text{extAi}} \right)_{L^2(\{\tau_1, \ldots, \tau_q\} \times \mathbb{R})},
\]

where

\[
\psi(x) = \sum_{q=1}^{L_1} \sum_{p=1}^{L_2} w_{p,q} I_{\{ \tau_q \}} \times A_p (x).
\]

Recall that the extended Airy kernel is given by

\[
K_{\text{extAi}}(\tau_1, \xi_1; \tau_2, \xi_2) = -\phi_{\tau_1, \tau_2}(\xi_1, \xi_2) + \tilde{K}_{\text{extAi}}(\tau_1, \xi_1; \tau_2, \xi_2),
\]

where

\[
\tilde{K}_{\text{extAi}}(\tau_1, \xi_1; \tau_2, \xi_2) = \int_0^\infty e^{-\lambda(\tau_1 - \tau_2)} \text{Ai} (\xi_1 + \lambda) \text{Ai} (\xi_2 + \lambda) \, d\lambda.
\]
The rough-smooth boundary
The rough-smooth boundary

To the left part of frozen-rough boundary, to the right part of rough-smooth boundary.
The rough-smooth boundary

Particles in the two-periodic Aztec diamond.
The rough-smooth boundary

What are the ”long paths” that we see in the picture? Can we define a boundary that converges to the Airy process?
The Kasteleyn method

For the Aztec diamond graph we define the **Kasteleyn matrix** by

\[
K(b, w) = \begin{cases} 
\nu(b, w) & \text{if } e = (b, w) \text{ is horizontal} \\
\imath \nu(b, w) & \text{if } e = (b, w) \text{ is vertical} \\
0 & \text{otherwise (i.e. no edge between } b \text{ and } w) 
\end{cases}
\]

**Theorem (Montroll-Potts-Ward, Kenyon)**

If \( e_i = (b_i, w_i) \), then the probability that \( e_1, \ldots, e_m \) belong to a dimer cover is

\[
P(e_1, \ldots, e_m) = \det \left( K(b_i, w_i)K^{-1}(w_i, b_j) \right)_{1 \leq i, j \leq m}
\]

This means that the dimers form a **determinantal point process** with correlation kernel \( K(e_i, e_j) = K(b_i, w_i)K^{-1}(w_i, b_j) \), \( e_i = (b_i, w_i) \).
Dimer-dimer correlations at the rough-smooth boundary

(Based on joint work with S. Mason, *Dimer-dimer correlations at the rough-smooth boundary*, arXiv:2110.14505; formula for the inverse Kasteleyn matrix from J., Chhita.)

Consider a size $n$ two-periodic Aztec diamond with $n$ very large. Formula for the inverse Kasteleyn matrix

$$K_{a,1}^{-1}(x, y) = \mathbb{K}_{1,1}^{-1}(x, y) - C_{\omega_c}(x, y) + R(x, y) + B^*(x, y).$$
Dimer-dimer correlations at the rough-smooth boundary

\[ B^* \text{ is exponentially small in } n. \text{ Write} \]

\[ x = n(1 + \xi)(1, 1) + (2a_1 - 1, 2a_2), \quad y = n(1 + \xi)(1, 1) + (2b_1, 2b_2 - 1). \]

\[ \xi = \xi_c = -\frac{1}{2} \sqrt{1 - 2c}, \quad c = \frac{a}{1+a^2}, \text{ gives asymptotic boundary.} \]
We are close to the boundary: $\xi_c - \xi \to 0$ as $n \to \infty$.
Let $G(w) = \frac{1}{\sqrt{2c}}(w - \sqrt{w^2 + 2c})$ and

$$g_\xi(w) = \log w - \xi \log G(w) + \xi \log G(w^{-1}).$$

Then $g'_\xi(w)$ has roots at $\pm \omega_c$, $\pm \bar{\omega}_c$, where $\omega_c$ is in the first quadrant. $\omega_c = i$ iff $\xi = \xi_c$. $R$ is an error term for our purposes.

$$|R(x, y)| \leq C |G(\omega_c)|^{b_1-b_2+a_2-a_1} \min\left(\frac{1}{n^{1/3}}, \frac{1}{\sqrt{n\sqrt{\xi_c - \xi}}}\right),$$

for $|a_i|, |b_i| \leq \max(n^{1/3}, \sqrt{n\sqrt{\xi_c - \xi}}).$
Dimer-dimer correlations at the rough-smooth boundary

In the region we are investigating,

$$K^{-1}_{a,1}(x, y) = K^{-1}_{1,1}(x, y) - C_{\omega_c}(x, y) + \text{negligible}$$

$$= K^{-1}_{s_1,s_2}(x, y) + \text{negligible},$$

where $K^{-1}_{s_1,s_2}$ gives a rough Gibbs measure in the whole plane. $K^{-1}_{1,1}$ can be expressed in terms of the integrals

$$E_{k,\ell} = \frac{i^{-k-\ell}}{4(1 + a^2)\pi i} \int_{|w|=1} \frac{G(w)^\ell G(1/w)^k}{\sqrt{w^2 + 2c\sqrt{1/w^2 + 2c}}} \frac{1}{w} dw.$$

$C_{\omega_c}(x, y)$ can be expressed in terms of the integral the same integral but integrated over $\Gamma_{\omega_c}$ which consists of two short arcs on the unit circle around $i$ and $-i$ of length $c\sqrt{\xi_c - \xi}$. $k \approx (x_2 - y_2)/2$ and $\ell \approx (x_1 - y_1)/2.$
Consider two dimers along the main diagonal oriented orthogonally to the diagonal. Think of $n$ as very large but fixed and consider growing $r$.

Assume $n^{-2/3} \ll \xi_c - \xi$ (not right at the boundary) and $\xi_c - \xi < \delta_n \to 0$ (not fully in the rough region).

- $r_{\text{min}} < r < c_1 \log \frac{1}{\sqrt{\xi_c - \xi}}$: $\text{corr} \sim c e^{-r/\alpha}$ (exponential decay)
- $c_1 \log \frac{1}{\sqrt{\xi_c - \xi}} < r < c_2 \frac{1}{\sqrt{\xi_c - \xi}}$: $\text{corr} \sim c(\xi_c - \xi)$ (constant)
- $c_2 \frac{1}{\sqrt{\xi_c - \xi}} < r << \sqrt{n\sqrt{\xi_c - \xi}}$ and $r \sim \frac{d}{\sqrt{\xi_c - \xi}}$: $\text{corr} \sim (\xi_c - \xi) \frac{\sin^2 d}{d^2}$ (power law and oscillatory)
The rough-smooth boundary. Correlation asymptotics

**Two length scales**

1) the lattice spacing
2) the distance \( \frac{1}{\sqrt{\xi_c - \xi}} \)

The results can be thought of as the decay of correlations for infinite volume Gibbs measures in the rough phase close to the smooth phase.
What are the "long paths" that we see in the picture? Can we define a boundary that converges to the Airy process?
Squishing
(Based on joint work with Beffara and Chhita.)

An \textbf{a-dimer} is a dimer that covers an \textit{a}-edge. They are \textit{oriented} from white to black.

\textbf{Figure:} The red dimers are \textit{a}-dimers, and the black \textit{b}-dimers.
Squishing

We let the $b$-faces become smaller, go to zero in size.
Squishing

We get double edges, loops and paths.
Paths and Loops

To get a unique split between paths and loops and get well-defined loops we need a convention. We use **mirrors**.
Paths

The paths go between the boundaries.

Figure: After squishing.
Paths

The paths go between the boundaries.

Figure: After squishing, $n = 300, a = 0.5$. 
What we would like to prove

With high probability, if we go along the main diagonal there is a last path in the third quadrant close to the asymptotic rough-smooth boundary and this path converges to the Airy process.
What we can prove

Let $h(f)$ be the height at the face $f$ in the Aztec diamond. Then we can split it into two parts:

$$h(f) = h_\ell(f) + h_c(f)$$

where $h_\ell(f)$ is the loop height and $h_c(f)$ is the corridor height.
What we can prove

Assume that \( a < 1/3 \). Imbed the interval \( A \) as a discrete interval of length \( \sim m^{1/3} \) in the Aztec diamond at the rough-smooth boundary. Define the random signed measure

\[
\kappa_m(\{\beta\} \times A) = \frac{1}{4} (h_c(F_+) - h_c(F_-)),
\]

where \( F_+ \) and \( F_- \) are the end-faces of the discrete imbedded interval. Then \( \kappa_m(\{\beta\} \times A) \) converges in terms of Laplace transforms to \( \mu_{Ai}(\{\beta\} \times A) \) as \( m \to \infty \), where \( \mu_{Ai} \) is the Airy kernel point process.

We expect that with high probability \( \kappa_m \) is actually a positive measure. We should think of \( \kappa_m \) as counting the number of paths between the two faces.
Thank you for your attention!