Two-dimensional massive integrable models on a torus

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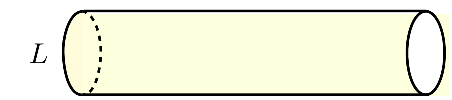
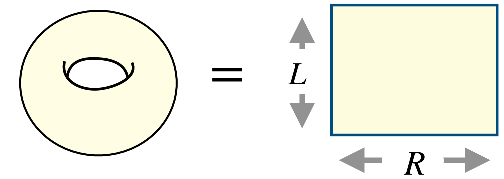
- Infinite-volume thermodynamics of a massive QFT can be expressed in terms of its S-matrix only
[R.Dashen,S.-k.Ma,andH.J.Bernstein (1969), for 2D: Al. Zamolodchikov, 90s



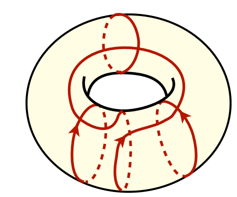
- Is that so for the **finite-volume** thermodynamics?



Yes, in principle. QSC ⇒ spectrum of excited states ⇒ torus partition function

- In this talk I will define the following claim:

The torus partition function is a grand canonical ensemble of loops with scattering factors associated with the crossings.



- The loop-gas representation of the partition function will be used to set up an effective field theory.

    1. Decouple the two-body interaction of the loops by a Hubbard-Stratonovich transformation

    2. Perform the path integral over the loops

    3. The result is an effective field theory for the HS fields defined in the complex rapidity plane. The limit $L \to \infty$ or $R \to \infty$ is a mean-field type limit with the mean field determined by the TBA equation

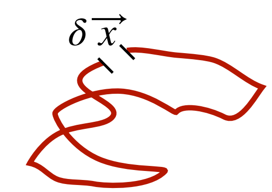
Path integral for a loop immersed in the torus $\mathbb{T} = \mathbb{R}^2/\boldsymbol{\Omega}$

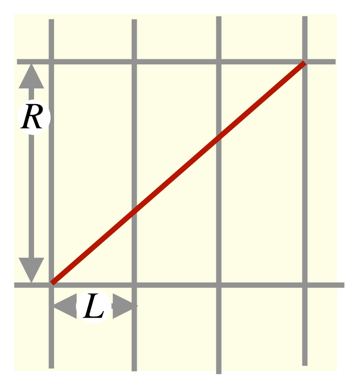
$\boldsymbol{\Omega} = L\mathbb{Z} \times R\mathbb{Z}$ - period lattice

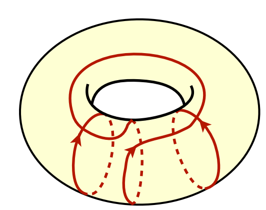
The path configuration space of loops splits into topological sectors labeled by winding numbers $w, w' \in \mathbb{Z}$:

$$\mathscr{F}(L,R) = \sum_{w,w' \in \mathbb{Z}} [\mathscr{F}(L,R)]_{w,w'}$$

1. Compute the path integral $\mathscr{F}(\overrightarrow{\delta x})$ for a loop with inserted discontinuity $\overrightarrow{\delta x}$
2. Evaluate the sum of $\mathscr{F}(\overrightarrow{\delta x})$ with $\overrightarrow{\delta x} \in \boldsymbol{\Omega}$







$w = 3,\ \bar{w} = 1$

1. $$\mathscr{F}(\overrightarrow{\delta x}) = -\tfrac{1}{2}\mathrm{Tr}[\log(-\nabla^2 + m^2)\, e^{\nabla \cdot \overrightarrow{\delta x}}] = -\tfrac{1}{2}\, RL \int \frac{d^2k}{(2\pi)^2} e^{i \vec{k} \cdot \overrightarrow{\delta x}} \log\left(\vec{k}^2 + m^2\right)$$

2. $$[\mathscr{F}(L,R)]_{w,\tilde{w}} = \mathscr{F}(\overrightarrow{\delta x}) \quad \text{with} \quad \delta x_1 = w' R, \;\; \delta x_2 = w L$$

Path integral wave functions of on-shell particles in physical and in mirror kinematics

$$\mathcal{F}(\Delta \vec{x}) = -\tfrac{1}{2}\, RL \int \frac{d^2k}{(2\pi)^2} \log\left(k_1^2 + k_2^2 + m^2\right)\, e^{ik_1\delta x_1 + ik_2\delta x_2}$$

$$= \tfrac{1}{2}\frac{L}{|\,\delta x_2\,|}\int_{\mathbb{R}} \frac{Rdk_1}{2\pi} e^{ik_1\delta x_1 - \sqrt{k_1^2+m^2}|\delta x_2|} \qquad (\delta x_2 \neq 0)$$

wave function of on-shell particle in the **direct** channel analytically continued to imaginary time $t = -\, i\delta x_2$

$$= \tfrac{1}{2}\frac{R}{|\,\delta x_1\,|}\int_{\mathbb{R}} \frac{Ldk_2}{2\pi} e^{ik_2\delta x_2 - \sqrt{k_2^2+m^2}|\delta x_1|} \qquad (\delta x_1 \neq 0)$$

wave function of on-shell particle in the **cross** channel analytically continued to imaginary time $t = -\, i\delta x_1$

The two integrals are related by a **mirror transformation** = double Wick rotation exchanging the space and the time direction:

$$E \to -\, ip$$

$$p \to iE$$

In the parametrization with the rapidity

$$p(\theta) = m\sinh(\theta)$$

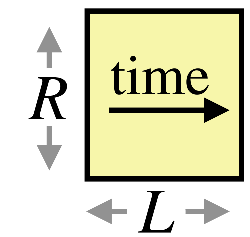
$$E(\theta) = \sqrt{p^2 + m^2} = m\cosh(\theta)$$

$$\theta \;\to\; i\pi/2 - \theta$$

Two possible descriptions of the winding loops:

- Description in physical kinematics (for loops winding at least once around the L-cycle):

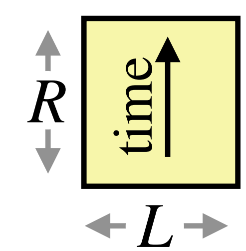
$$\mathscr{F}_{w,w'} = \frac{R}{2|w|}\int_{\mathbb{R}} \frac{dp(\theta)}{2\pi}\, e^{-|w|LE(\theta)+w'Rp(\theta)} \qquad (w \neq 0)$$



on-mass-shell winding particle in **physical** kinematics

- Description in mirror kinematics (for loops winding at least once around the R-cycle):

$$\tilde{\mathscr{F}}_{w',w} = \frac{L}{2|w'|}\int_{\mathbb{R}} \frac{dp(\theta)}{2\pi}\, e^{-|w'|R\,E(\theta)+iwL\,p(\theta)} \qquad (w' \neq 0)$$



… in the **mirror** kinematics

- $\mathscr{F}_{w,w'} = \tilde{\mathscr{F}}_{w',w} \qquad (w, w' \neq 0)$

Different choices for the kinematics lead to different but equivalent expressions for the free energy

Let us check how the loop gas description works for **free theory** (free massive boson)

$$\mathscr{Z} = \exp[\mathscr{F}] \qquad \mathscr{F} = \sum_{w,w'\in\mathbb{Z}} [\mathscr{F}]_{w,w'} = \mathscr{F}_{0,0} + \sum_{w\neq 0} \mathscr{F}_{w,0} + \sum_{w'\neq 0, w\in\mathbb{Z}} \tilde{\mathscr{F}}_{w',w}$$

one possible choice

divergent infinite-volume energy density, to be neglected

$$\mathrm{L}_k[x] \equiv \sum_{n=1}^{\infty} n^{k-2} e^{-nx} = (-1)^{k-1} \mathrm{Li}_{2-k}\left(\sigma e^{-x}\right) \qquad \mathrm{L}_1[x] = -\log\left(1 - e^{-x}\right) \qquad \mathrm{L}_2[x] = \frac{1}{e^x - 1}$$

$$\mathscr{F}^{(R,L)} = \int_{\mathbb{R}} \frac{R dp(\theta)}{2\pi} \; \mathrm{L}_1\left[LE(\theta)\right] - \oint_{\mathscr{C}_{\mathbb{R}}} \frac{L dp(\theta)}{2\pi} \mathrm{L}_1[RE(\theta)] \; \mathrm{L}_2[iLp(\theta)]$$

$\mathscr{C}_{\mathbb{R}}$ = contour enclosing the real axis $\mathbb{R}$

Take the contour integral by residues:

$$\mathscr{F}(L,R) = \frac{\pi}{6}\frac{R}{L} c_0(mL) - \sum_{n\in\mathbb{Z}} \log(1 - e^{-RE_n})$$

the **effective central charge**

$$c_0(mR) \equiv -\frac{6R}{\pi} \int_{\mathbb{R}} \frac{dp}{2\pi} \log\left(1 - e^{-R\sqrt{p^2+m^2}}\right)$$

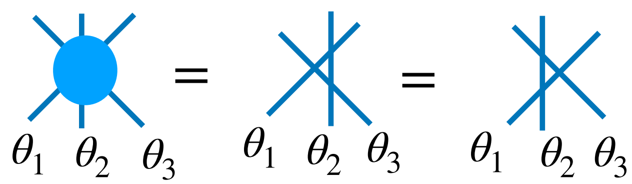
the **excited states**

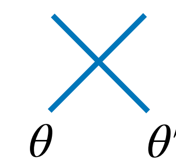
$$p_n = \frac{2\pi n}{R} \qquad E_n = \sqrt{p_n^2 + m^2}$$

Now consider a theory of non-trivial factorised scattering interaction by scattering:
(for simplicity one neutral particle, no bound states)

Relativistic QFT’s with factorized scattering matrix

Factorized scattering:





$S(\theta-\theta')$ - two-particle scattering matrix

$S(\theta)S(-\theta)=1$	unitarity
$S(\theta)^* = S(-\theta^*)$	real analyticity
$S(\theta) = S(i\pi-\theta)$	crossing
$\sigma \equiv S(0) = \pm 1$	“TBA statistics”

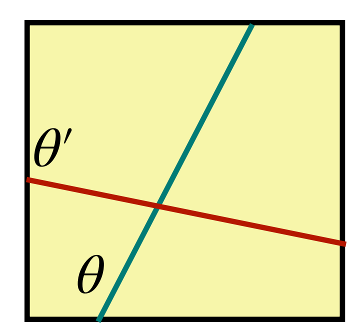
Mirror transformation= double Wick rotation exchanging the space and the time direction:

$$\theta \;\to\; i\pi/2-\theta$$

$$E(\theta) \to E(i\pi/2-\theta) = -ip(\theta)$$

$$p(\theta) \to p(i\pi/2-\theta) = iE(\theta)$$

We will strongly use the analyticity and will consider scattering processes for complex rapidities. E.g. scattering matrix between a particle with rapidity θ in the direct channel and a particle with rapidity θ' in the cross channel is $W(\theta+\theta') = S(\theta+\theta'-i\pi/2)$



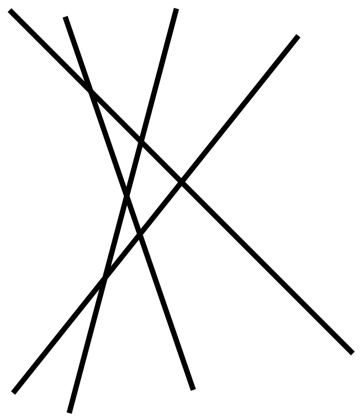
A recipe to introduce the interaction by scattering in the path integral for N loops:

"From loops to scattering particles and then back to loops by analytical continuation"

- $\overrightarrow{\delta x_j} = \overrightarrow{x_j} - \overrightarrow{x'_j}, \;\; j = 1,\ldots,N$

- Analytical continue to Minkowski space, with $\overrightarrow{x}_1, \ldots, \overrightarrow{x}_N$ in the far past and $\overrightarrow{x}'_1, \ldots, \overrightarrow{x}'_N$ in the far future, well separated in space

- Introduce the interaction by factorised scattering $\displaystyle\int_{\mathbb{R}} \frac{dp_1}{2\pi} \cdots \frac{dp_N}{2\pi} \, \mu(p_1, \ldots, p_N) \prod_{i<j} S(p_i - p_j)$

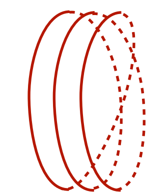


- Analytically continue back to the Euclidean lattice Ω

Boltzmann weights and measure for the partition function of loops:

- weight= $\displaystyle\prod_{1}^{N} \frac{e^{-|w_j|LE(\theta_j)+iw_j'Rp(\theta_j)}}{2|w_j|} \prod_{k=1}^{\tilde N} \frac{e^{-R|\tilde w_k|E(\tilde\theta_k)+i\tilde w_k' Lp(\tilde\theta_k)}}{2|\tilde w_k|}$

  $\displaystyle\times \prod_{j} \sigma^{w_j+w_j'-1} \prod_{k} \sigma^{\tilde w_k+\tilde w_k'-1}$ ← factors $\sigma = S(0)$ from self-intersections



  $\times\, W(\theta_j+\tilde\theta_k)^{-|w_j||\tilde w_k|+w_j'\tilde w_k'}\; S(\theta_j-\tilde\theta_k)^{w_j'|\tilde w_k|-|w_j|\tilde w_k'}$ ← Two-body interaction $W(\theta) \equiv S(\theta - i\pi/2)$

The integration measure is assumed to be the flat measure for the phase shifts:

$$\mu(p_1,\ldots,p_N,\tilde p_1,\ldots,\tilde p_{\tilde N}) = d\phi_1 \wedge d\phi_2 \ldots \wedge d\phi_N \wedge d\tilde\phi_1 \wedge d\tilde\phi_2 \ldots \wedge d\tilde\phi_{\tilde N}$$

$$\phi_j = Rp(\theta_j) - i\sum_{j'=1}^{N} |w_{j'}| \log S(\theta_j-\theta_{j'}) - i\sum_{k=1}^{\tilde N} \tilde w_k' \log W(\theta_j+\tilde\theta_k), \quad (j=1,\ldots,N)$$

$$\tilde\phi_k = Lp(\tilde\theta_k) - i\sum_{j=1}^{N} w_j' \log W(\tilde\theta_k+\theta_j) - i\sum_{k'=1}^{\tilde N} |\tilde w_{k'}| \log S(\tilde\theta_k-\tilde\theta_{k'}) \quad (k=1,\ldots,\tilde N)$$

- measure = $\displaystyle\prod_{j=1}^{N} d\phi_j \prod_{k=1}^{\tilde N} d\tilde\phi_k = \prod_{j=1}^{N} d\theta_j \prod_{k=1}^{\tilde N} d\tilde\theta_k \;\det \begin{bmatrix} \partial\phi_j/\partial\theta_{j'} & \partial\phi_j/\partial\tilde\theta_k \\ \partial\tilde\phi_k/\partial\theta_j & \partial\tilde\phi_k/\partial\tilde\theta_{k'} \end{bmatrix}$ ← Gaudin determinant

Decouple the two-body interaction of loops by a **Hubbard-Stratonovich transformation**

- HS auxiliary gaussian fields $\varphi(\theta), \bar{\varphi}(\theta)$ associated with the two cycles of $\mathbb{T}$:

— classical values: $\langle \varphi(\theta) \rangle = Lm \cosh\theta, \quad \langle \tilde{\varphi}(\theta) \rangle = Rm \cosh\theta$

— 2pt function: $\langle \varphi(\theta)\tilde{\varphi}(\theta') \rangle = -\log W(\theta - \theta')$ phys/mir

$\Rightarrow \langle \varphi(\theta)\tilde{\varphi}^{\pm}(\theta') \rangle = \mp \log S(\theta - \theta')$ phys/phys mir/mir

$\varphi^{\pm}(\theta) \equiv \varphi(\theta \pm i\pi/2)$

- A second 'Faddeev-Popov ghost' field $\psi(\theta), \bar{\psi}(\theta)$ is needed to generate the Jacobian for the measure:

$$"d\varphi(\theta)" = (\partial\varphi(\theta) - \tilde{\psi}(\theta)\partial\psi(\theta))d\theta \qquad \langle \psi(\theta)\tilde{\psi}(\theta') \rangle = -\log W(\theta - \theta')$$

- **Operator loop amplitudes:**

$$\mathbf{F}_{w,w'} = \frac{1}{2}\sigma^{w+w'-1}\int_{\mathbb{R}} \frac{d\theta}{2\pi} \exp\left(-|w|\varphi - w'\tilde{\varphi}^{[-]}\right)\left(\frac{\partial_\theta \tilde{\varphi}^-}{|w|} - \tilde{\psi}^- \partial_\theta \psi(\theta)\right) \qquad (w \neq 0)$$

$$\tilde{\mathbf{F}}_{w',w} = \frac{1}{2}\sigma^{w+w'-1}\int_{\mathbb{R}} \frac{d\theta}{2\pi} \exp\left(-|w'|\tilde{\varphi} - w\varphi^+\right)\left(\frac{\partial_\theta \varphi^+}{|w'|} - \psi^+ \partial_\theta \tilde{\psi}\right) \qquad (w' \neq 0)$$

$$\mathbf{F}_{w,w'} = \tilde{\mathbf{F}}_{w',w} \qquad (w, w' \neq 0)$$

Effective field theory for the partition function

- Boltzmann weights of the loop gas as expectation values of HS fields:

$$\text{weight} = \left\langle \prod_{j=1}^{N} \mathbf{F}_{w_j,w_j'} \prod_{j=1}^{\tilde N} \tilde{\mathbf{F}}_{\tilde w_j',\tilde w_j} \right\rangle \qquad \mathscr{Z}^{(L,R)}_{\text{tor}} = \sum_{N,\tilde N=0}^{\infty} \sum_{\{w_j\neq 0\}} \sum_{\{\tilde w_j\neq 0,\tilde w_j'\}} \int \frac{\text{weight}\times\text{measure}}{N!\ \tilde N!}$$

one possible choice

- The sum inside the expectation value exponentiates and the exponent is expressed in terms of the functions

$$\sum_{n=1}^{\infty} \sigma^{n-1} n^{k-2} e^{-nx} \equiv \mathrm{L}^{\sigma}_{k}[x] = (-1)^{k-1}\mathrm{Li}_{2-k}\left(\sigma e^{-x}\right)$$

$$\mathscr{Z}^{(L,R)}_{\text{tor}} = \langle \exp[\mathbf{F}_{\text{tor}}]\rangle$$

$$\mathbf{F}_{\text{tor}} = \int_{\mathbb{R}} \frac{d\theta}{2\pi i}\left[\mathrm{L}^{\sigma}_{1}[\varphi]\,\partial_\theta\tilde\varphi^{-} + \mathrm{L}^{\sigma}_{2}[\varphi]\,\tilde\psi^{-}\partial_\theta\psi\right] + \oint_{\mathscr{C}_{\mathbb{R}}} \frac{d\theta}{2\pi i}\left(\mathrm{L}^{\sigma}_{1}[\tilde\varphi]\mathrm{L}^{\sigma}_{2}[\varphi^{+}]\ \partial_\theta\tilde\varphi^{+} + \mathrm{L}^{\sigma}_{2}[\tilde\varphi]\ \psi^{+}\partial_\theta\tilde\psi\right)$$

$$\mathrm{L}^{\sigma}_{1}[x] = -\,\sigma\log(1-\sigma e^{-x})$$

$$\mathrm{L}^{\sigma}_{2}[x] = \frac{1}{e^{x}-\sigma}$$

$$\mathrm{L}^{\sigma}_{3}[x] = \frac{e^{x}}{(e^{x}-\sigma)^2}$$

$$\langle\varphi(\theta)\rangle = Lm\cosh\theta,\quad \langle\tilde\varphi(\theta)\rangle = Rm\cosh\theta$$

$$\langle\varphi(\theta)\tilde\varphi(\theta')\rangle = \langle\psi(\theta)\tilde\psi(\theta')\rangle = -\log W(\theta-\theta')$$

Oscillator representation

$$\varphi(\theta) = \sum_{n \text{ odd}} \mathbf{a}_n \frac{e^{-n\theta}}{n}, \quad \tilde{\varphi}(\theta) = \sum_{n \text{ odd}} \tilde{\mathbf{a}}_n \frac{e^{-n\theta}}{n}$$

$$\psi(\theta) = \sum_{n \text{ odd}} \mathbf{b}_n \frac{e^{-n\theta}}{n}, \quad \tilde{\psi}(\theta) = \sum_{n \text{ odd}} \tilde{\mathbf{b}}_n \frac{e^{-n\theta}}{n}$$

$$\langle 0 | 0 \rangle = 1$$

$$\langle 0 | \mathbf{a}_{-n} = \langle 0 | \tilde{\mathbf{a}}_{-n} = 0 \qquad \mathbf{a}_n | 0 \rangle = \tilde{\mathbf{a}}_n | 0 \rangle = 0$$

$$\langle 0 | \mathbf{b}_{-n} = \langle 0 | \tilde{\mathbf{b}}_{-n} = 0 \qquad \mathbf{b}_n | 0 \rangle = \tilde{\mathbf{b}}_n | 0 \rangle = 0$$

$$(n > 0, \quad \text{odd})$$

$$\mathbf{a}_n \tilde{\mathbf{a}}_m - \tilde{\mathbf{a}}_m \mathbf{a}_n = -\, n W_n \delta_{m+n,0}$$

$$\mathbf{a}_m \mathbf{a}_n = \mathbf{a}_n \mathbf{a}_m, \quad \tilde{\mathbf{a}}_m \tilde{\mathbf{a}}_n = \tilde{\mathbf{a}}_n \tilde{\mathbf{a}}_m$$

$$\mathbf{b}_n, \tilde{\mathbf{b}}_m + \tilde{\mathbf{b}}_m \mathbf{b}_n = -\, n W_n \delta_{m+n,0}$$

$$\mathbf{b}_m \mathbf{b}_n = \mathbf{b}_n \mathbf{b}_m, \quad \tilde{\mathbf{b}}_m \tilde{\mathbf{b}}_n = \tilde{\mathbf{b}}_n \tilde{\mathbf{b}}_m \qquad (n, m = \text{odd})$$

$$\log W(\theta) = \sum_{k \geq 1, \text{odd}}^{\infty} \frac{W_n}{n} e^{-n\theta} \qquad (\Re\theta > 0)$$

$$= \sum_{k \geq 1, \text{odd}}^{\infty} \frac{W_n}{n} e^{n\theta} \qquad (\Re\theta < 0)$$

The scattering matrix is encoded in the canonical commutation relations

$$\mathcal{Z}_{\text{tor}}^{(L,R)} = \langle 0 | e^{\mathbf{H}_+} \, e^{\mathbf{F}_{\text{tor}}} \, e^{-\mathbf{H}_-} | 0 \rangle$$

$$\mathbf{H}_- = \frac{m}{2W_1} \left(L \tilde{\mathbf{a}}_{-1} + R \mathbf{a}_{-1} \right) \qquad \mathbf{H}_+ = \frac{m}{2W_1} \left(L \tilde{\mathbf{a}}_1 + R \mathbf{a}_1 \right)$$

The two periods are encoded in two “Hamiltonians” transforming the Fock vacua

Example: Sinh-GORDON model

$$\mathscr{A} = \int_{\mathbb{T}} d^2x \left[\frac{1}{4\pi} (\nabla\phi)^2 + 2\mu \cosh(2b\phi) \right]$$

$$S(\theta) = \frac{\sinh(\theta) - i\sin(\pi\alpha)}{\sinh(\theta) + i\sin(\pi\alpha)} \longleftarrow \alpha = \frac{b^2}{1+b^2}$$

$$\log W(\theta) = \sum_{n\geq 1,\mathrm{odd}} \frac{W_n}{n} e^{-n\theta}, \quad W_n = 4\cos\frac{n\pi a}{2} \longleftarrow a = 1 - 2\alpha = \frac{1-b^2}{1+b^2}$$

Remark 1: curiously the operator representation reproduces the infinite-volume energy density

$$\mathscr{Z}_{\mathrm{tor}}^{(L,R)} \underset{R,L\to\infty}{=} \langle 0 | e^{\mathbf{H}_+} e^{-\mathbf{H}_-} | 0 \rangle = \exp[LR\epsilon_0] \qquad \epsilon_0 = \frac{m^2}{2W_1} = \frac{m^2}{8\sin\pi\alpha}.$$

[Destri-De Vega, 1991]

Remark 2: With this specific S-matrix one can write the Ward identity for φ as a finite-difference equation

$$\Big\langle \varphi(\theta + i\pi/2) + \varphi(\theta - i\pi/2) \Big\rangle_{\mathrm{tor}} = \Big\langle \log(1 + e^{-\varphi(\theta + i\pi a/2)}) + \log(1 + e^{-\varphi(\theta - i\pi a/2)}) \Big\rangle_{\mathrm{tor}}$$

TBA limit $R \to \infty$ = mean field limit

$$\mathcal{Z}_{\rm cyl} = \langle 0 | e^{\mathbf{H}_+} \, e^{\mathbf{F}_{\rm cyl}} \, e^{-\mathbf{H}_-} | 0 \rangle \qquad \mathbf{F}_{\rm cyl} = \int_{\mathbb{R}} \frac{d\theta}{2\pi i} \left[\log\left(1 + e^{-\varphi}\right) \partial_\theta \tilde{\varphi}^- + \frac{\psi \partial_\theta \tilde{\psi}^-}{1 + e^{\varphi}} \right]$$

The Feynman diagram technique ends at one loop. Fermionic and bosonic loops cancel. Hence no gaussian fluctuations, pure mean field theory. The fields can be replaced by their expectation values $\epsilon(\theta) = \left\langle \varphi(\theta) \right\rangle_{\rm cyl}$ and $\phi(\theta) = -\, i \langle \varphi(\theta - i\pi/2) \rangle_{\rm cyl}$

$$\mathcal{Z}_{\rm cyl} = \exp[\mathcal{F}_{\rm cyl}]$$

$$\mathcal{F}_{\rm cyl} = \langle \mathbf{F}_{\rm cyl} \rangle = -\, \sigma R \int \frac{d\theta}{2\pi} \log\left(1 - \sigma e^{-\epsilon(\theta)}\right) \partial p(\theta)$$

Relation to the TBA approach:

ρ_p, ρ_h- particle and hole densities

$$\epsilon = \log \frac{\rho_h}{\rho_p}, \quad \partial_\theta \phi = R(\rho_p + \rho_h)$$

Ward identities:

$$\epsilon(\theta) = LE(\theta) - \int_{-\infty}^{\infty} \frac{d\theta'}{2\pi} K(\theta - \theta') \log(1 + e^{-\epsilon(\theta')})$$

→ TBA equation for the pseudoenergy ϵ

$$\partial \phi(\theta) = R \partial p(\theta) + \int_{-\infty}^{\infty} \frac{d\theta'}{2\pi} K(\theta - \theta') \, \frac{\partial \phi(\theta')}{e^{\epsilon(\theta')} - \sigma}$$

→

$$\rho_p(\theta) + \rho_h(\theta) = \partial \tilde{p}(\theta) + \int_{\mathbb{R}} \frac{d\theta'}{2\pi} K(\theta, \theta') \, \rho_p(\theta)$$

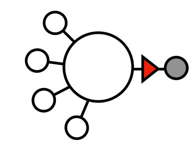
Bethe equation in terms of densities

$$K(\theta, \theta') = -\, \langle \partial \tilde{\varphi}^-(\theta) \varphi(\theta') \rangle_c = -\, i \partial_\theta \log S(\theta - \theta')$$

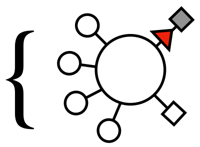
scattering kernel

Feynman rules ($\sigma = -1$):

$$\mathbf{F}_{\rm cyl} = \sum_{n=0}^{\infty} \int_{\mathbb{R}} \frac{d\theta}{2\pi i} \left[\mathrm{v}_{n+1} \frac{\varphi^n}{n!} \partial_\theta \tilde{\varphi}^- + \mathrm{v}_{n+2} \frac{\varphi^n}{n!} \tilde{\psi} \partial_\theta \psi \right] = \sum_{n=0}^{\infty} \int_{\mathbb{R}} \frac{d\theta}{2\pi} \Big[n \Big\{$$

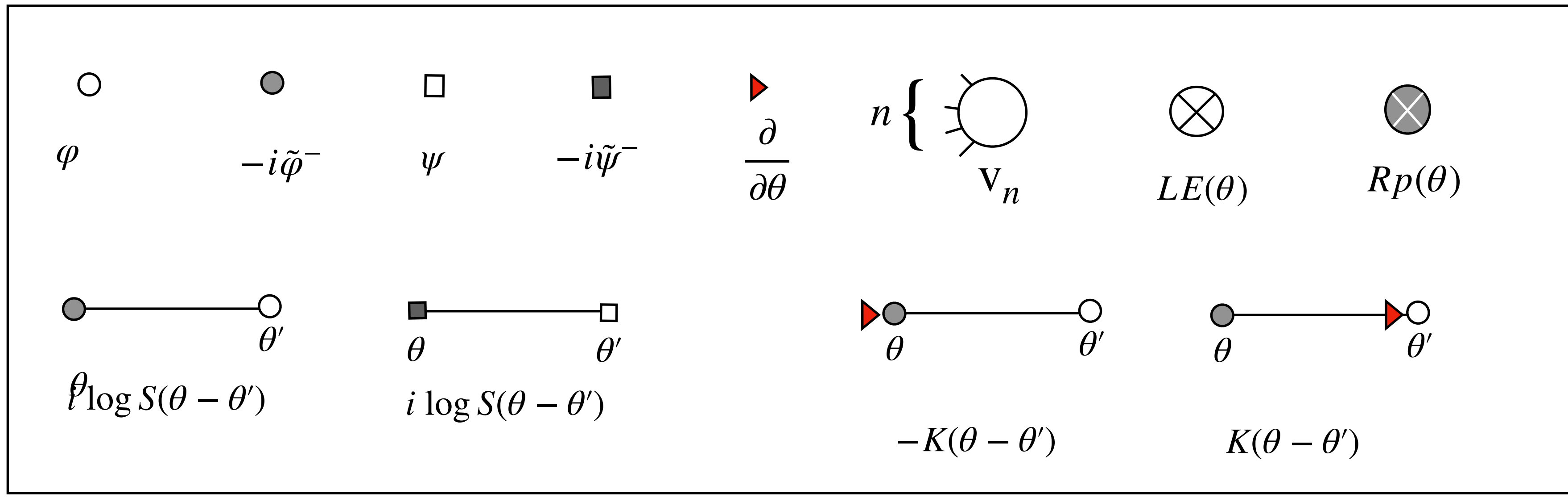


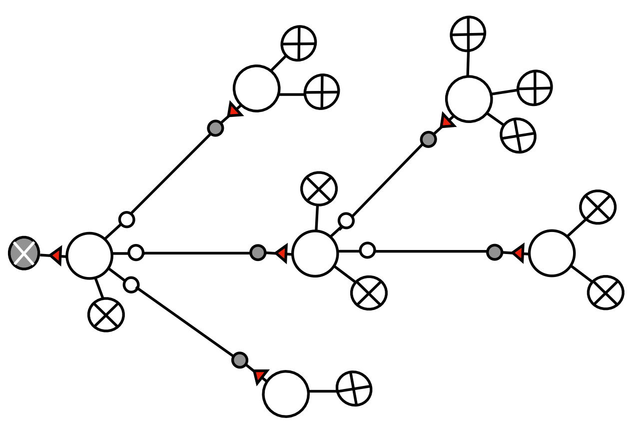
$+ \; n\Big\{$



$\Big]$

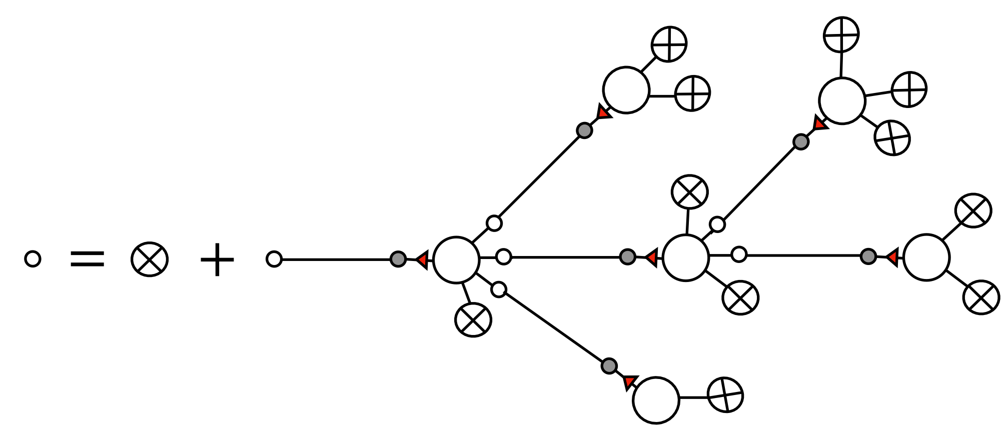
$\mathrm{v}_k \equiv (-1)^k \mathrm{Li}_{2-k}(-1)$





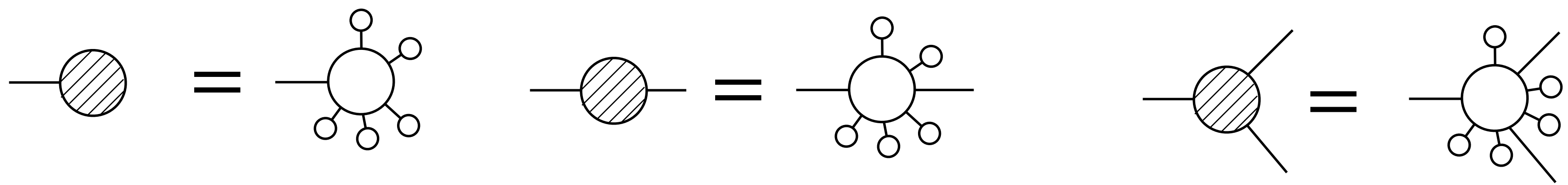
Vacuum Feynman graphs (boson and fermionic loops cancel, only trees survive)

[I.K., D. Serban, D.-L. Vu , 2017, 2018]



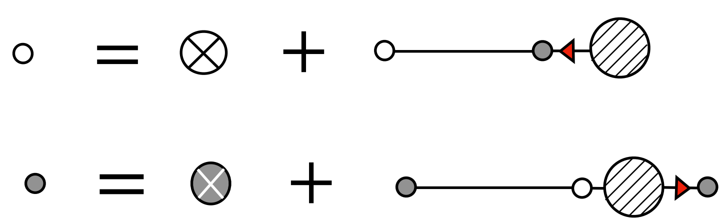
Ward identity for the expectation value $\epsilon = \langle \varphi \rangle_{\rm cyl}$ (TBA equation)

Dressed vertices:



$$L_2^-[\epsilon] = \frac{1}{e^\epsilon + 1} \qquad L_1^-[\epsilon] = \log(1 + e^{-\epsilon}) \qquad L_3^-[x] = \frac{e^\epsilon}{(e^\epsilon + 1)^2}$$

Ward identities:



Free energy:

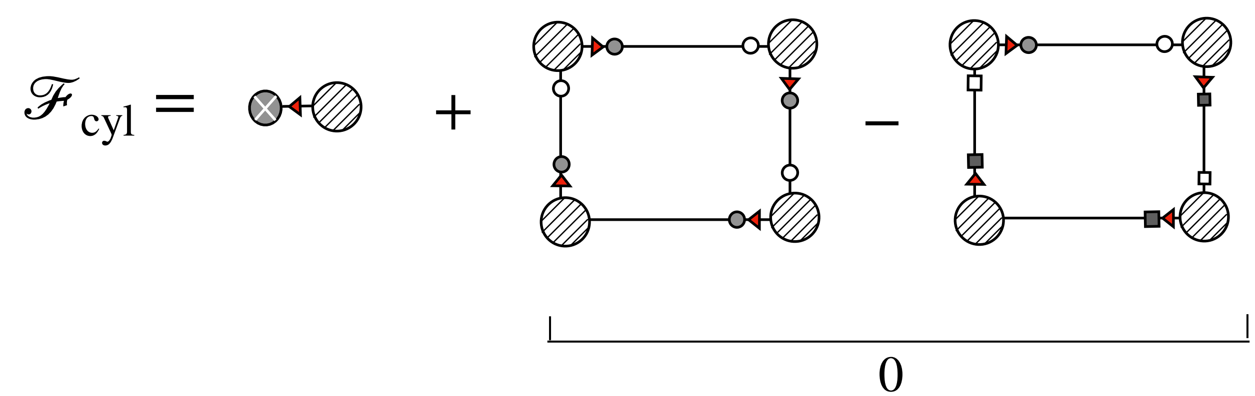


Diagram technique for the torus - complicated. Loops of all orders will contribute

What could be done next:

- Learn how to perform systematic perturbative expansion above the mean field (TBA) limit
- Generalisation to diagonal scattering matrices (type ADE) easy
- Generalisation to non-diagonal scattering and bound states - need new insight
- The EQFT for the torus can be formulated with little effort for the finite cylinder with integrable boundaries.
- Leclair-Mussardo formula for the torus