Minimal dimer models
& maximal Riemann surfaces

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Dimers on planar bipartite graphs

\[ G = (V, E) \text{ finite graph} \]

\[ V = B \cup W \]
Dimers on planar bipartite graphs

- $G = (V, E)$ finite graph
- $V = B \cup W$
- dimer configuration = perfect matching $\mathcal{C}$
- positive edge weights $(\nu_e)_{e \in E}$
- partition function

$$Z = \sum_{\mathcal{C} \text{ dimer config.}} \left( \prod_{e \in \mathcal{C}} \nu_e \right)$$
Dimers on planar bipartite graphs

- \[ Z = \sum_{\text{dimer config.}} \prod_{e \in e} \nu_e \]

- Theorem (Kasteleyn, Temperley-Fisher)
  - Twisted bipartite adjacency matrix
  - Rows/columns indexed by 0/1
  - \[ |K_{w,b}| = \nu_e \text{ for } e = (w,b) \]
  - Alternating product around faces have fixed signs

Then: \[ Z = \det K \]
**Dimers on planar bipartite graphs**

**Boltzmann measure:**

- if $Z \neq 0$:
  
  $$P_v(E) = \frac{1}{Z} \prod_{e \in E} \varphi_e$$

- if $e_1 = (w_1, b_1), \ldots, e_k = (w_k, b_k)$ distinct
  
  $$P_v(e_1, \ldots, e_k \text{ dimers}) = \left( \prod_{j=1}^{k} \varphi_{w_j, b_j} \right) \det K_{b_i, w_j}^{-1}$$

- determinantal point process on edges $E$

What about infinite graphs?
Dimers on $\mathbb{Z}^2$-periodic planar bipartite graphs

Kenyon - Okounkov - Sheffield

$G, D, K$ periodic

- $K(z, w)$: Fourier transform $(z, w) e^{iC\cdot C}$
- characteristic polynomial
  \[ P(z, w) = \det K(z, w) \]
- spectral curve
  \[ C = \{(z, w) \mid P(z, w) = 0\} \] (Harnack curve)

Newton polygon $N(P)$

Amoeba

\{ slopes for height function \}
Planar graphs and train-tracks (zig-zags)

G: infinite with bounded faces
Planar graphs and train-tracks (zig-zags)

$G$: infinite with bounded faces

- Minimal graphs (Thurston)
  - forbidden:
    - [Diagram of forbidden patterns]

- Isoradial graphs (Mercat, Kenyon)
  - forbidden:
    - [Diagram of forbidden patterns]

well defined partial cyclic order
for non parallel triplets of train-tracks.
Critical isoradial dimer models (Kenyon 2002)
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Explicit inverse:

$$K_{b,w}^{-1} = \frac{1}{4i\pi^2} \int \prod (\lambda - e^{i\theta})^{\pm 1} \log \lambda \, d\lambda$$

- Locality: depends only on the geometry of a path from $b$ to $w$

$$K_{w,b} = e^{i\beta} e^{i\alpha}$$

satisfies Kasteleyn condition.

- Can be used to define probabilities (de Tilière)
Critical isoradial dimer models

- Why does it work?
  - more general weights?
  - integrability?

- isoradial \& periodic
  1. inverse vs. 2-param. family?
Critical isoradial dimer models

- Why does it work?
  - more general weight?
  - integrability?
- isoradial / periodic
  - inverse vs. 2-param. family?

Spectral curve of genus 0

Partial answers

- trigonometric weights:
  - critical Laplacian (Kenyon)
  - critical Ising (B-de Tilière)
- elliptic weights:
  - massive Laplacian
  - non-critical Ising (B-de Tilière-Raschel)
  - elliptic dimers on minimal graphs, (B-Cimasoni-de Tilière)

Now: genus > 1, minimal graphs.
Our setting

- work with all minimal graphs simultaneously
- replace \( \hat{C} \) by higher genus Riemann surface \( \Sigma \)
  - “maximal curve”
  - plays the role of spectral curve, given a priori
- extra data (real point on \( \text{Jac}(\Sigma) \))

\( \widetilde{\omega} \) - Kasteleyn operators (Fock)

- \( \epsilon \)-parameter families of inverses
- probabilistic quantities read on \( \Sigma \)
Maximal curve $\Sigma$

- Abstract compact Riemann surface of genus $g \geq 1$
- Anti-holomorphic involution $\sigma$
- "Real locus" = fixed points of $\sigma$
  $g+1$ topological circles $A_0, \ldots, A_g$
Maximal curve $\Sigma$

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  - $g+1$ topological circles $A_0, \ldots, A_g$

- Complete $A_1, A_2, \ldots A_g, B_1, \ldots B_g$ : symplectic basis for homology

- Adapted basis of holomorphic 1-forms $\omega_1, \ldots, \omega_g$
  - $\int_{A_i} \omega_j = \delta_{ij}$

- Riemann matrix $\Omega = (\Omega_{ij}) = \left( \int_{B_i} \omega_j \right)$
  - Pure imaginary, $\text{Im } \Omega$ symmetric, positive definite
• construction of $\text{Jac}(\Sigma)$

from formal linear combination of points of $\Sigma$

$$\sum_i u_i - \nu_i \sim \left( \sum_i \omega_i^u, \ldots, \sum_i \omega_i^\nu \right) \in \mathbb{C}^g \bigg/ \mathbb{Z}^g + \Omega \mathbb{Z}^g = \text{Jac}(\Sigma)$$

• Riemann Theta function

$$\Theta(z) = \sum_{n \in \mathbb{Z}^g} \exp \left( i \pi \left( n \cdot \Omega n + z \cdot n \right) \right) \quad z \in \mathbb{C}^g$$

quasi-periodic function on $\text{Jac}(\Sigma)$

• Prime form on $\Sigma$

$$E(u,v) : \text{basic bloc to construct meromorphic functions}$$

with prescribed zero/poles on $\Sigma$

$$E(u,v) = 0 \iff u = v \quad \text{(higher genus analogue of } u - v \text{)}$$
Parameters on minimal graphs

- partial cyclic order on train-tracks

\[ \{\alpha\}: \{\text{train-tracks}\} \rightarrow A_0 \]

\[ T \rightarrow \alpha_T \]

preserving the order
Parameters on minimal graphs

- Partial cyclic order on train-tracks

\{ \alpha \} : \{ train-tracks \} \rightarrow A_0

\[ T \rightarrow \alpha_T \]

preserving the order

- Discrete Abel map
  - Defined on faces/white/black
  - Linear comb. of points of \( A_0 \)

\[ \eta(b) = \eta(f) + \beta \]
\[ \eta(w) = \eta(f) - \alpha \]
\[ \eta(f) = \eta(b) - \alpha = \eta(w) + \beta \]

\eta : discrete antiderivative of \( \{ \alpha \} \)
Fock's Kasteleyn operator

- \( \Sigma \) maximal curve
- \( t \) real point of \( \text{Jac}(\Sigma) \) (think \( \in \mathbb{R}^g \))
- \( G \) minimal graph, \( \{a\}, \eta \)

\[
K_{w,b} = \frac{E(\alpha, \beta)}{\Theta(t+\eta(f))\Theta(t+\eta(f'))}
\]
Fock's Kasteleyn operator

- $\Sigma$ maximal curve
- $t$ real point of $\text{Jac}(\Sigma)$ (think $t \in \mathbb{R}^g$)
- $G$ minimal graph, $\{a\}, \eta$

\[
K_{w, b} = \frac{E(\alpha, \beta)}{\Theta(t + \eta(f)) \Theta(t + \eta(f'))}
\]

Lemma: Under the geometric hypotheses above

$K$ satisfies the Kasteleyn condition

(sign of alternating products around faces)
Special functions in $\text{Ker } K$

$$g_b(u) = \frac{\theta(-t+(u-\eta(b)))}{E(\beta,u)} = g_b^{-1}(u)$$

$$g_{f,b}(u) = \frac{\theta(t+u+\eta(b))}{E(\alpha,u)} = g_{f,b}^{-1}(u)$$

$$b = x_0, x_1, \ldots, x_n = w$$

Path alternating between vertices of $G/G^*$

$$g_b,w(u) = \prod_{j=1}^{n} g_{x_{j-1},x_j}(u)$$
Special functions in $\ker K$

\[ g_{b,f}(u) = \frac{\theta(-t+(u-\eta(b)))}{E(\beta,u)} = g^{-1}_{f,b}(u) \]

\[ g_{f,w}(u) = \frac{\theta(t+u+\eta(w))}{E(\alpha,u)} = g^{-1}_{w,f}(u) \]

$b = x_0, x_1, \ldots, x_n = w$

path alternating between vertices of $G/G^*$

\[ g_{b,w}(u) = \prod_{j=1}^{n} g_{x_{j-1},x_j}(u) \]

Proposition:

$\mu \mapsto g_{b,w}(\mu)$ meromorphic 1-form on $\Sigma$

- $\forall b, b', \forall \mu \in \Sigma$
  \[ \sum_{w} g_{bw}(u) K_{wb'} = 0 \]

- $\forall w, w'$
  \[ \sum_{b} K_{wb} g_{bw}(u) = 0 \]

Proof: Fay's identity
Divisor of a white vertex

**Definition:**

For \( w \) a white vertex, \( \text{div}(w) \) is the divisor of \( u \rightarrow \Theta(t + u + \eta(w)) \) (2.666).

- By properties of theta functions: \( g \) points on \( \Sigma \).
- \( \Sigma \) maximal and \( t \) real: one point on each \( A_1, \ldots, A_g \).
- \( \forall j \in \{1, \ldots, g\} \): \( \forall b \) black vertex

\[ g_{bw}(u_j) = 0 \]
**Theorem**

1. Let $u_0 \in \Sigma^+$
2. $A_{b,w}^{u_0} = \frac{1}{2 \pi i} \int_{C_{b,w}} g_{b,w}(u) \mathrm{d}u$ is an inverse of $K$
3. Locality property (modulo $\eta$)
**Theorem**

- Let \( \mu_0 \in \Sigma^+ \)

\[
A_{b,w}^{\mu_0} = \frac{1}{\det \Pi} \int g_b(w) \text{ is an inverse of } K
\]

- Locality property (modulo \( \eta \))

- Under assumption \((\mathcal{A})\), the formula

\[
\left( \prod_{j=1}^{k} K_{w_j b_j} \right) \det \left( A_{b,w}^{\mu_0} \right)
\]

defines a prob. measure on dimers

- if \( \mu_0 \in A_0 \): solid
- if \( \mu_0 \in A_j \): smooth
- otherwise: rough \( \rightarrow \Sigma^+ \): phase diagram
The periodic case

- \( G \) minimal \( \mathbb{Z}^2 \)-periodic graph
- train-track \( T \) of \( G_1 \) loop on the torus with hom. class \( (h_T, v_T) \in \mathbb{Z}^2 \)
- \( \mathcal{N}(G) \) convex polygon with boundary vectors \( \{(h_T, v_T)\} \)
- \( \{\alpha\} \) periodic
  \( \Rightarrow \) not enough for \( \eta, \kappa \) to be periodic
- Forcing periodicity + non-degeneracy:
  \( \Leftrightarrow \) picking \( g \) distinct points in the interior of \( \mathcal{N}(G) \)
\[ \psi: \mu \in \Sigma \mapsto \left( \frac{g_{b'b}(u)}{z(u)}, \frac{g_{bb'}(u)}{w(u)} \right) \]

explicit parametrization of spectral curve

\[ P(z,w) = 0 \]
\[ \psi : \mu \in \Sigma \mapsto \left( \frac{g_{b,b'}(w)}{z(w)}, \frac{g_{b,b'}(w)}{w(w)} \right) \]

Explicit parametrization of spectral curve

\[ P(z,w) = 0 \]

More: (spectral theorem, Kenyon-Okounkov)

weights on $G$ \[ \sim \]
gauge transform \[ \sim \]
Harnack curves with $N(P) = N(G)$ + standard divisor

**Theorem:** Fix a Harnack curve $C$ with standard divisor $D$

There exist $\Sigma$, $G$, \( \{ \alpha \} \), $t$ such that

- $C$ is the spectral curve
- $D$ divisor of a fixed white vertex

Moreover, $t \mapsto D$ is a bijection
Conclusion and perspectives

- Many other probabilistic quantities have an expression on $\Sigma$
  - average slope of the height function
  - free energy
  - surface tension

- Better understanding of Kenyon's critical case (geometric degenerates)

- Integrability: Fay's identity / spider move

- Special cases:
  - $G$ double of isoradial graph $G$: relation $K \leftrightarrow \Delta G$
  (in relation with spectral theorem for periodic networks (Georgi))