Minimal dimer models & maximal Riemann surfaces

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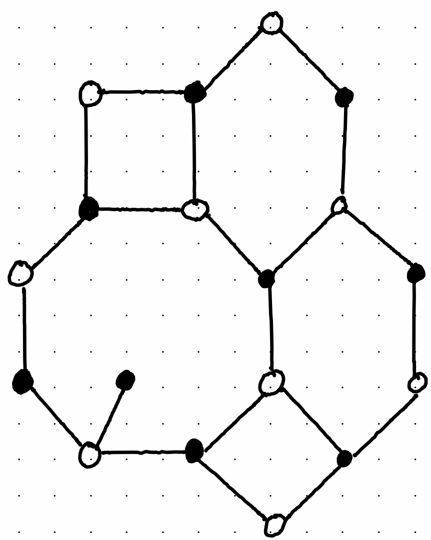
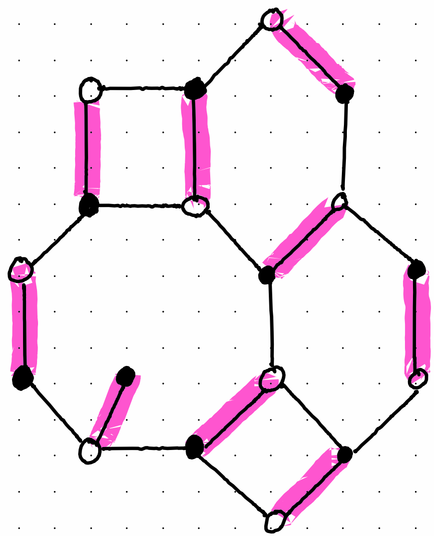
Dimers on planar bipartite graphs



$G = (V, E)$ finite graph

$V = \underset{\bullet}{B} \sqcup \underset{\circ}{W}$

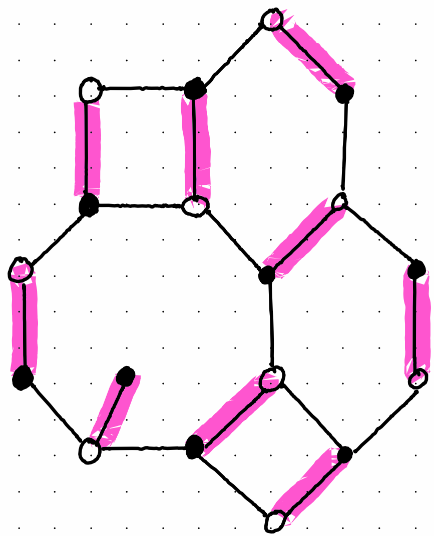
Dimers on planar bipartite graphs



- $G = (V, E)$ finite graph
- $V = \underset{\bullet}{B} \sqcup \underset{\circ}{W}$
- dimer configuration = perfect matching $\mathcal{C}$
- positive edge weights $(\nu_e)_{e \in E}$
- partition function

$$Z = \sum_{\mathcal{C} \text{ dimer config.}} \left(\prod_{e \in \mathcal{C}} \nu_e \right)$$

Dimers on planar bipartite graphs



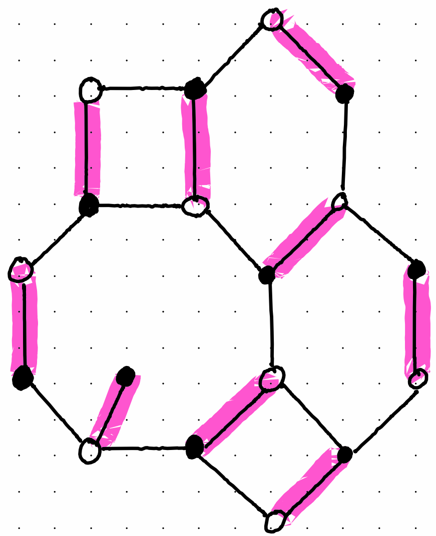
- $$Z = \sum_{\mathcal{C} \text{ dimer config.}} \left(\prod_{e \in \mathcal{C}} \nu_e \right)$$

- <u>Theorem</u> (Kasteleyn, Temperley–Fisher)

K : twisted bipartite adjacency matrix
- rows / columns indexed by ○ / ●
- $|K_{w,b}| = \nu_e$ for $e = (w, b)$
- alternating product around faces have fixed signs

Then : $$Z = \det K$$

Dimers on planar bipartite graphs



Boltzmann measure :

- if $Z \neq 0$:

$$\mathbb{P}_\nu(\mathcal{C}) = \frac{1}{Z} \prod_{e \in \mathcal{C}} \nu_e$$

- if $e_1 = (w_1, b_1), \dots, e_k = (w_k, b_k)$ distinct

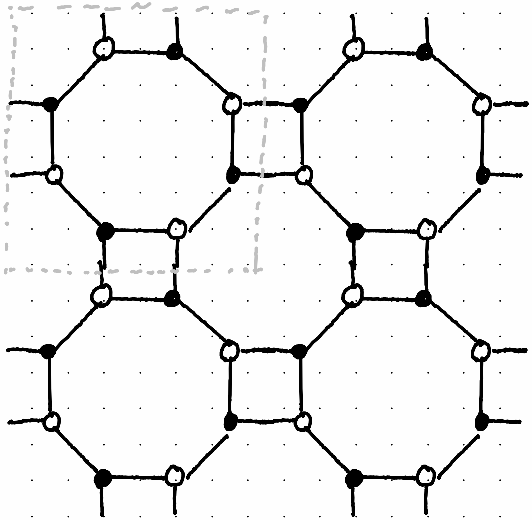
$$\mathbb{P}_\nu(e_1, \dots, e_k \text{ dimers}) = \left(\prod_{j=1}^{k} K_{w_j, b_j} \right) \det_{i,j} K^{-1}_{b_i, w_j}$$

- determinantal point process on edges E

What about infinite graphs?

Dimers on $\mathbb{Z}^2$-periodic planar bipartite graphs

Kenyon - Okounkov - Sheffield

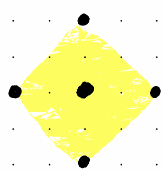


G, ν, K periodic

- $K(z,w)$: Fourier transform $(z,w) \in (\mathbb{C}^*)^2$
- characteristic polynomial
  $P(z,w) = \det K(z,w)$
- spectral curve $C = \{(z,w) \mid P(z,w) = 0\}$
  (Harnack curve)

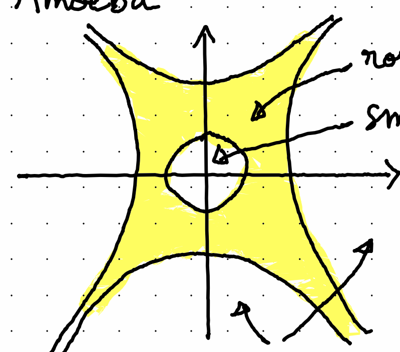
Newton polygon

$N(P)$



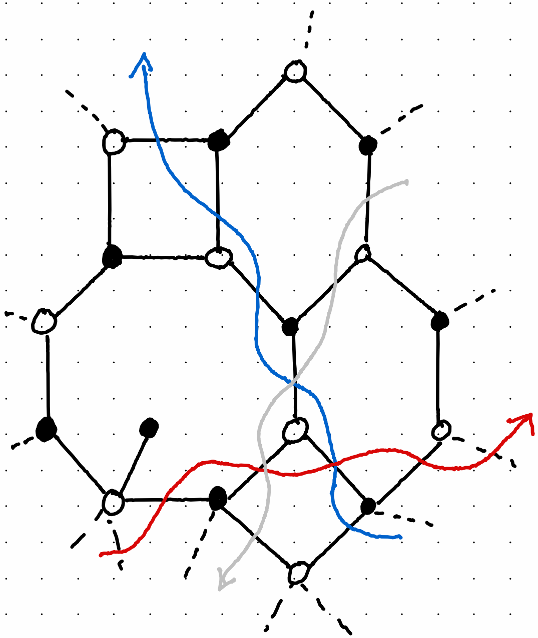
$\left\{ \begin{array}{l} \text{slopes for} \\ \text{height function} \end{array} \right\}$

Amoeba



Planar graphs and train-tracks (zig-zags)

G : infinite with bounded faces



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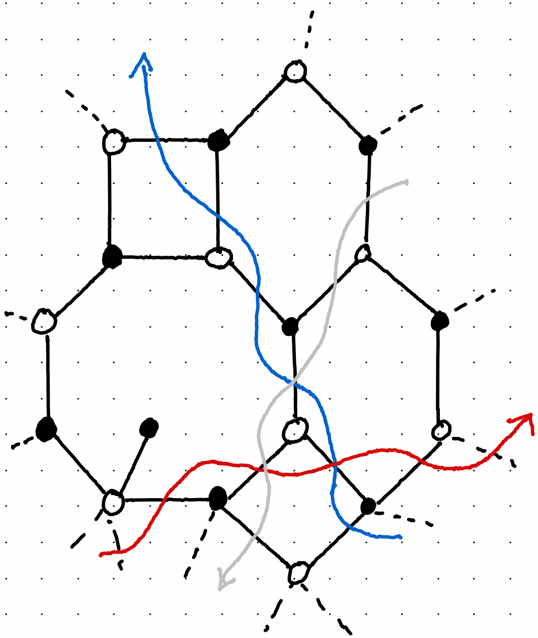
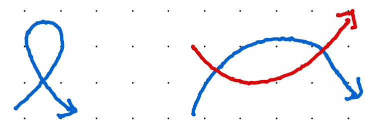
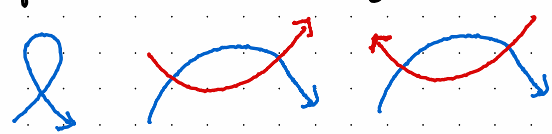


G : infinite with bounded faces

- minimal graphs (Thurston)

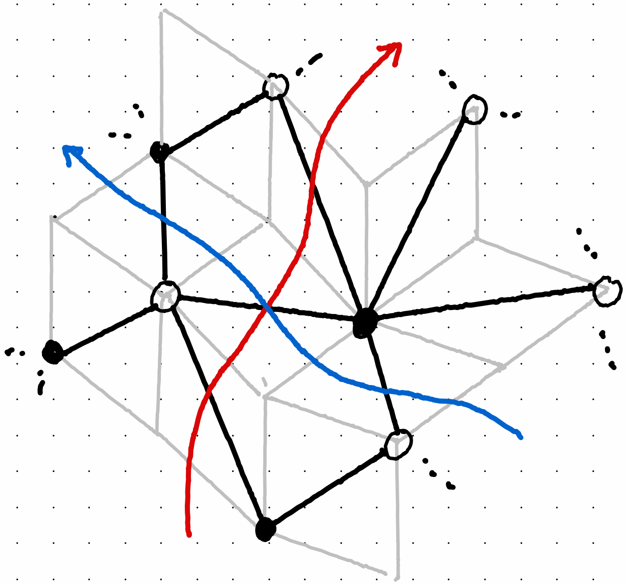
  forbidden : 

- isoradial graphs (Mercat, Kenyon)

  forbidden : 

well defined partial cyclic order
for non parallel triplets of train-tracks.

Critical isoradial dimer models (Kenyon 2002)



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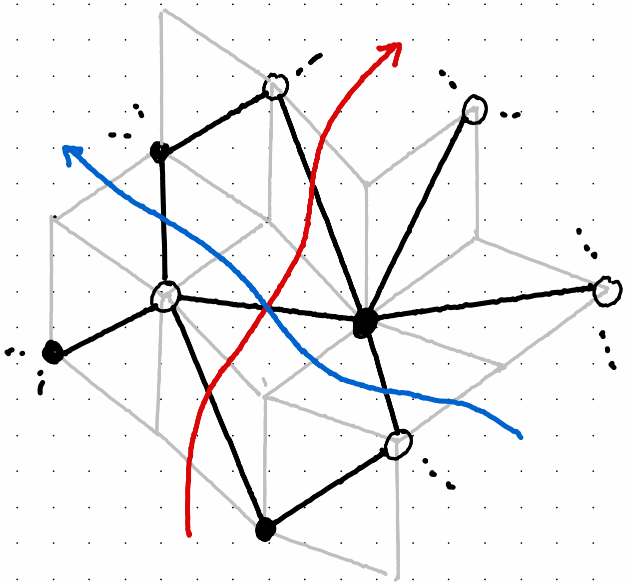
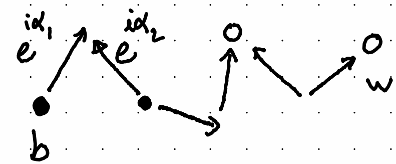


explicit inverse:

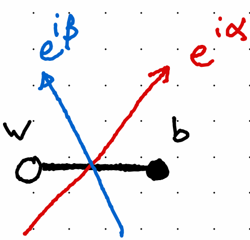
$$K^{-1}_{b,w} = \frac{1}{4 i \pi^2} \oint \prod (\lambda - e^{i\alpha_j})^{\pm 1} \log \lambda \, d\lambda$$



- locality: depends only on the geometry of a path from b to w

- can be used to define probabilities (de Tilière)

$$K_{w,b} = e^{i\beta} - e^{i\alpha}$$



satisfies Kasteleyn condition.

Critical isoradial dimer models

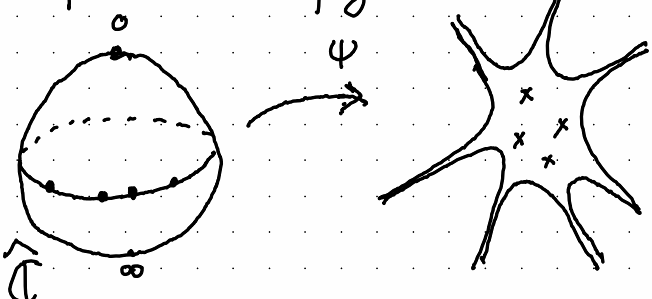
- Why does it work ?

  - more general weights ?
  - integrability ?

- isoradial $\cap$ periodic

  1 inverse vs. 2-param. family ?

spectral curve of genus 0



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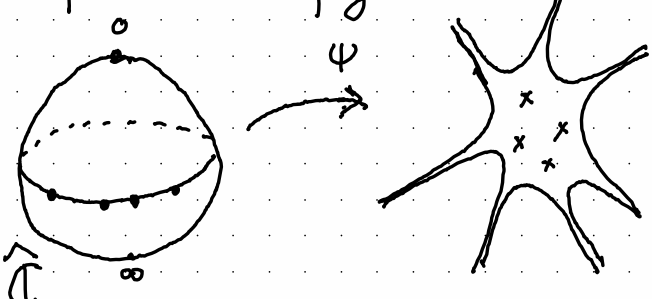

partial answers

- trigonometric weights:
  - critical Laplacian (Kenyon)
  - critical Ising (B-de Tilière)
- elliptic weights:
  - massive Laplacian
  - non-critical Ising (B-de Tilière-Raschel)
  - elliptic dimers on minimal graphs (B-Cimasoni-de Tilière)

now: genus ≥ 1, minimal graphs.

Our setting

- work with all minimal graphs simultaneously
- replace $\hat{\mathbb{C}}$ by higher genus Riemann surface Σ:
    - "maximal curve"
    - plays the rôle of spectral curve, given a priori
- extra data (real point on $\mathrm{Jac}(\Sigma)$)

$\rightsquigarrow$ - Kasteleyn operators (Fock)

- ℓ-parameter families of inverses
- probabilistic quantities read on Σ

Maximal curve Σ

- Abstract compact Riemann surface of genus $g \geq 1$
- Antiholomorphic involution σ
- "Real locus" = fixed points of σ
  $g+1$ topological circles $A_0, \dots, A_g$

Maximal curve Σ

- Abstract compact Riemann surface of genus $g \geqslant 1$
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  $g+1$ topological circles $A_0, \dots, A_g$

- complete $A_1, A_2, \dots A_g, B_1, \dots B_g$: symplectic basis for homology
- adapted basis of holomorphic 1-forms $\omega_1, \dots, \omega_g$ $\qquad \int_{A_i} \omega_j = \delta_{ij}$
- Riemann matrix $\Omega = (\Omega_{ij}) = \left(\int_{B_i} \omega_j \right)$

  pure imaginary, $\mathrm{Im}\, \Omega$ symmetric, positive definite

- construction of $\mathrm{Jac}(\Sigma)$

from formal linear combination of points of Σ

$$\sum_i u_i - v_i \quad \rightsquigarrow \quad \left(\sum_i \int_{u_i}^{v_i} \omega_1, \dots, \sum_i \int_{u_i}^{v_i} \omega_g\right) \in \mathbb{C}^g \Big/ \mathbb{Z}^g + \Omega\,\mathbb{Z}^g = \mathrm{Jac}(\Sigma)$$

- Riemann Theta function

$$\Theta(z) = \sum_{n \in \mathbb{Z}^g} \exp\Big(i\pi\big(n\cdot\Omega n + 2z\cdot n\big)\Big) \qquad z \in \mathbb{C}^g$$

quasi-periodic function on $\mathrm{Jac}(\Sigma)$

- Prime form on Σ

$E(u,v)$: basic bloc to construct meromorphic functions with prescribed zeros/poles on Σ

$E(u,v) = 0 \Leftrightarrow u = v$ (higher genus analogue of $u-v$)

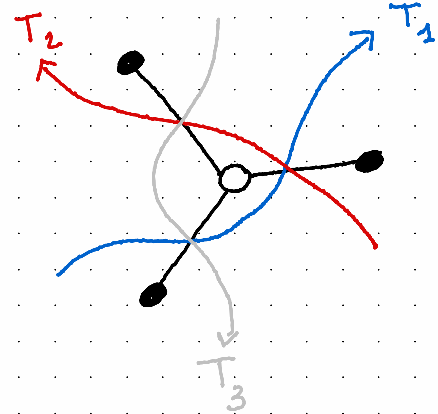
Parameters on minimal graphs

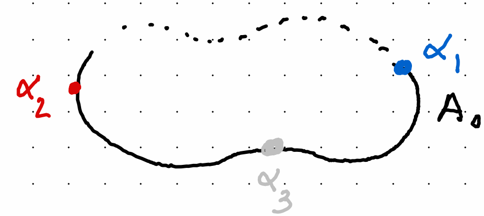
- partial cyclic order on train-tracks

- $\{\alpha\} : \{\text{train-tracks}\} \longrightarrow A_0$

  $\qquad\qquad T \longmapsto \alpha_T$

  preserving the order





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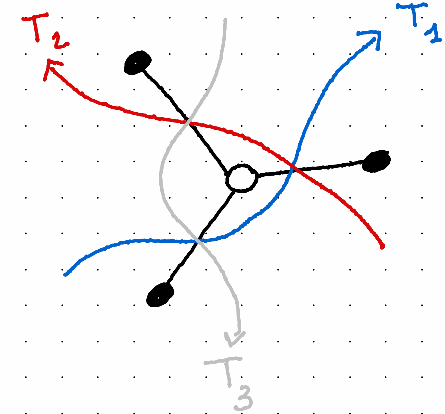

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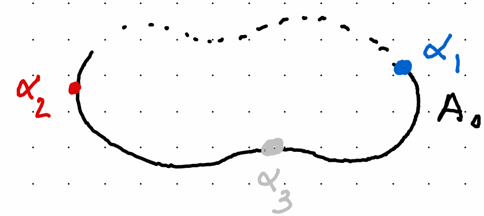
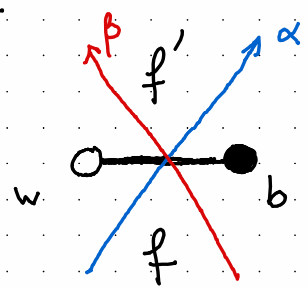


- discrete Abel map
  - defined on faces/white/black
  - linear comb. of points of A_0
  -



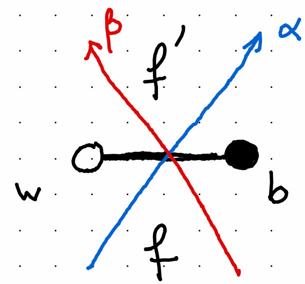
$$\eta(b) = \eta(f) + \beta$$
$$\eta(w) = \eta(f) - \alpha$$
$$\eta(f') = \eta(b) - \alpha = \eta(w) + \beta$$

η : discrete antiderivative of $\{\alpha\}$

Fock's Kasteleyn operator

- Σ maximal curve
- t real point of $\mathrm{Jac}(\Sigma)$ (think $\in \mathbb{R}^g$)
- G minimal graph, $\{\alpha\}$, η

$$K_{w,b} = \frac{E(\alpha,\beta)}{\theta(t+\eta(f))\,\theta(t+\eta(f'))}$$



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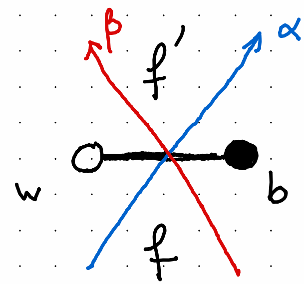
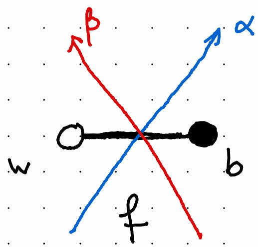


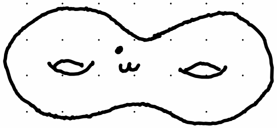
Lemma: Under the geometric hypotheses above

K satisfies the Kasteleyn condition

(sign of alternating products around faces)

Special functions in Ker K



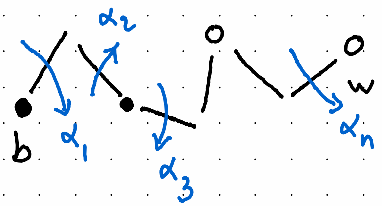


$$g_{b,f}(u) = \frac{\theta(-t+(u-\eta(b)))}{E(\beta,u)} = g^{-1}_{f,b}(u)$$

$$g_{f,w}(u) = \frac{\theta(t+u+\eta(w))}{E(\alpha,u)} = g^{-1}_{w,f}(u)$$

$b = x_0, x_1, \ldots\ldots x_n = w$

path alternating between vertices of G/G^*



$$g_{b,w}(u) = \prod_{j=1}^{n} g_{x_{j-1},x_j}(u)$$

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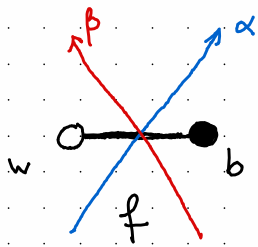

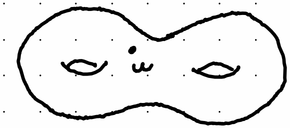

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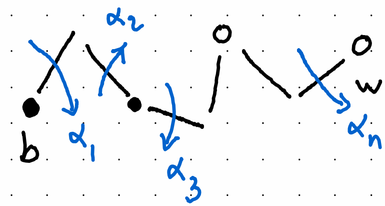


$$g_{b,w}(u) = \prod_{j=1}^{n} g_{x_{j-1},x_j}(u)$$

<u>Proposition</u>

- $u \mapsto g_{b,w}(u)$ meromorphic 1-form on Σ
- $\forall b,b'$ $\forall u \in \Sigma$ $\displaystyle\sum_w g_{bw}(u) K_{wb'} = 0$
- $\forall w,w'$ $\displaystyle\sum_b K_{wb}\, g_{bw}(u) = 0$

Proof: Fay's identity

Divisor of a white vertex

<u>definition</u>:

> For w a white vertex, $\mathrm{div}(w)$ is the divisor (zeros) of $u \mapsto \theta(t+u+\eta(w))$

- by properties of theta functions : g points on Σ
- Σ maximal and t real : one point on each $\underset{u_1}{\overset{}{A_1}}, \dots, \underset{u_g}{A_g}$ ($u_1 \in A_1, \dots, u_g \in A_g$)
- $\forall j \in \{1, \dots, g\}$ $\forall b$ black vertex

$$g_{bw}(u_j) = 0$$

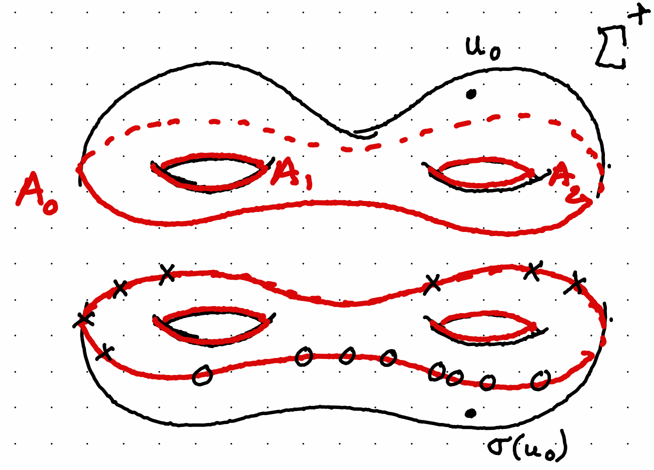
Inverses of K

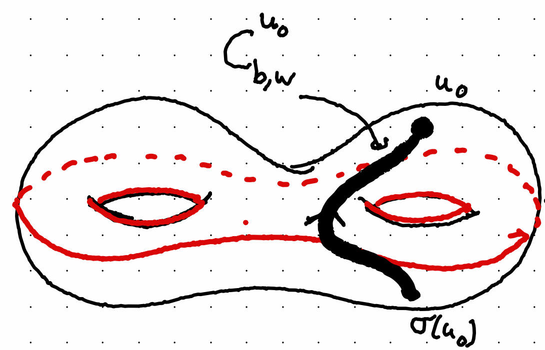
Theorem

- Let $u_0 \in \Sigma^+$

$$A^{u_0}_{b,w} = \frac{1}{2i\pi}\int_{C^{u_0}_{b,w}} g_{b,w}(u)$$ is an inverse of K

- locality property (modulo η)





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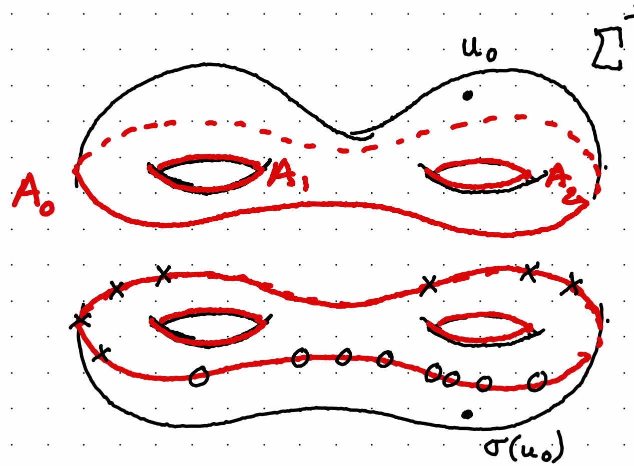

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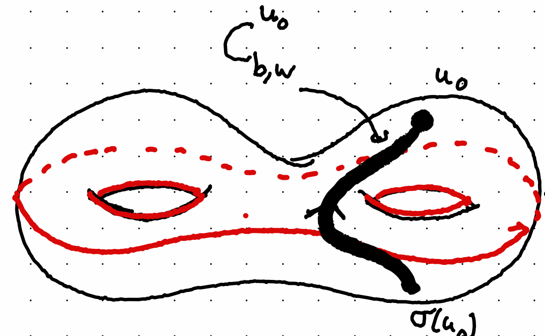
- locality property (modulo η)

- Under assumption ($\star$), the formula

$$\left(\prod_{j=1}^{k} K_{w_j, b_j} \right) \det_{1 \leq i,j \leq k} \left(A^{u_0}_{b_i, w_j} \right)$$

defines a prob. measure on dimers

- if $u_0 \in A_0$: solid
- if $u_0 \in A_j$: smooth
- otherwise : rough $\rightsquigarrow$ Σ^+ : phase diagram



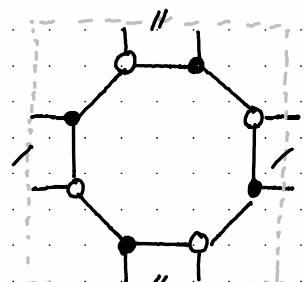
The periodic case

- G minimal $\mathbb{Z}^2$-periodic graph $\qquad G_1 = G/\mathbb{Z}^2$ on the torus
- train-track T of G_1: loop on the torus with hom. class $(h_T, v_T) \in \mathbb{Z}^2$
- $N(G)$: convex polygon with boundary vectors $\{(h_T, v_T)\}$
- $\{\alpha\}$ periodic

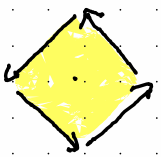
  $\rightsquigarrow$ not enough for η, K to be periodic
- Forcing periodicity + non degeneracy:

  $\Leftrightarrow$ picking g distinct points

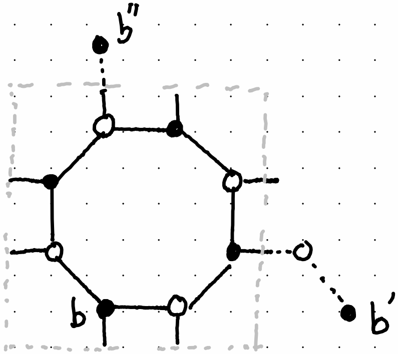
  in the interior of $N(G)$



G_1

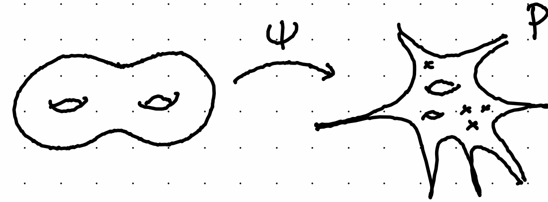


$N(G)$

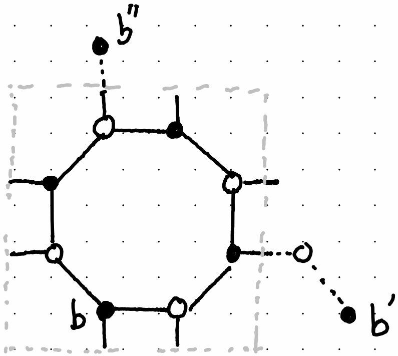


$$\Psi : u \in \Sigma \longmapsto \Big(\underbrace{g_{b,b'}(u)}_{z(u)} , \underbrace{g_{b,b''}(u)}_{w(u)} \Big)$$

explicit parametrization of spectral curve



$P(z,w) = 0$



$$\psi : u \in \Sigma \longmapsto \Big(\underbrace{g_{b,b'}(u)}_{z(u)} \, , \, \underbrace{g_{b,b''}(u)}_{w(u)} \Big)$$

explicit parametrization of spectral curve

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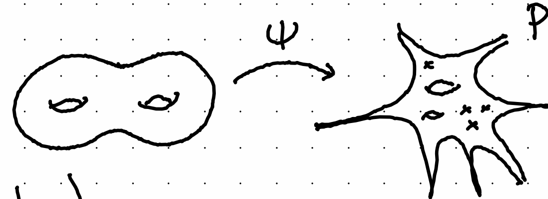

More: (spectral theorem, Kenyon-Okounkov)

$$\text{weights on } G \big/ \text{gauge transf.} \cong \text{Harnack curves with } N(P) = N(G) + \text{standard divisor}$$

<u>Theorem</u>: Fix a Harnack curve C with standard divisor D

There exists Σ, G, $\{\alpha\}$, t such that

- C is the spectral curve
- D divisor of a fixed white vertex

Moreover, $t \mapsto D$ is a bijection.

Conclusion and perspectives

- Many other probabilistic quantities have an expression on Σ
  - average slope of the height function
  - free energy
  - surface tension
- Better understanding of Kenyon's critical case (geometric degeneration)
- Integrability: Fay's identity / spider move
- special cases:
  - G double of isoradial graph $\mathcal{G}$: relation $K \leftrightarrow \Delta_{\mathcal{G}}$

    (in relation with spectral theorem for periodic networks (George))