Limiting current distribution for a two-species particle model from first principles

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KPZ growth

- Stochastic growth normal to the surface
- Kardar-Parisi-Zhang (KPZ) 1986
- Basic object: (random) height function $h(x, t)$

KPZ equation (nonlinear stochastic PDE):

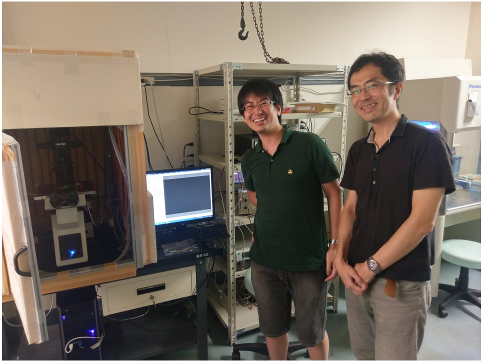
KPZ equation

$$\partial_t h(t,x) = \frac{1}{2}\partial_x^2 h(t,x) + \frac{1}{2}\left(\partial_x h(t,x)\right)^2 + \xi(t,x)$$

Claim:
Diffusion + non-linearity + space-time Gaussian white noise
correctly describes 1+1D surface growth

Realisation in liquid crystal growth: Takeuchi lab

1+1D Growth



Takeuchi and Sasamoto, Tokyo 2017

KPZ growth

Theorem (Non-Gaussian fluctuations)

$$h \sim \nu t + c t^{1/3}, \quad t \to \infty$$

Transformation to Stochastic Heat Equation (SHE):

$$h(t, x) := \log z(t, x).$$

SHE equation

$$\partial_t z(t, x) = \frac{1}{2} \partial_x^2 z(t, x) + \xi(t, x) z(t, x)$$

Fluctuations

The Laplace transform formula for $z(t,x)$ can be written as a Fredholm determinant

Theorem (Laplace transform of SHE)

$$\mathbb{E}[e^{-\zeta z(t,0)}] = \det(I - K_\zeta)_{L^2(\mathbb{R}_+)}$$

$$K_\zeta(\eta,\eta') = \int_{\mathbb{R}} f_\zeta(\xi,t)\mathrm{Ai}(\xi+\eta)\mathrm{Ai}(\xi+\eta')d\xi.$$

Theorem (Fluctuations of SHE)

$$\lim_{t\to\infty} P\left(\frac{h(t,x)-t}{t^{1/3}} < s\right) = F_{\mathrm{GUE}}(s).$$

$F_{\mathrm{GUE}}(s)$ is the Tracy-Widom distribution of the Gaussian Unitary Ensemble of random matrix theory.

ASEP

The asymmetric exclusion process on $\mathbb{Z}$:

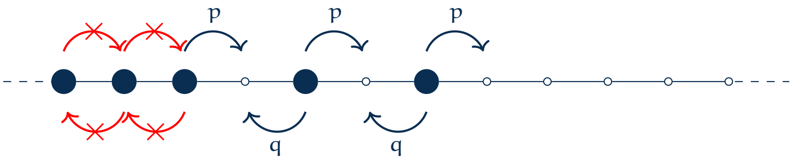


Figure: Configuration of particles and hopping rates in the ASEP on $\mathbb{Z}$

Markov chain:

$$\frac{d}{dt}\mathbb{P}(\nu;t) = \sum_{\lambda\neq\nu} W(\lambda \to \nu)\mathbb{P}(\lambda;t) - \sum_{\lambda\neq\nu} W(\nu \to \lambda)\mathbb{P}(\nu;t),$$

Initial condition:

$$\mathbb{P}(\mu \to \nu;0) = \prod_{i=1}^{n} \delta_{\nu_i,\mu_i}.$$

ASEP transition probability

One particle (Bethe ansatz) eigenfunction:

$$\varphi_z(\nu, t) = \exp\left(-t\frac{z(p-q)^2}{p(1+z)(1+z/\tau)}\right)\left(\frac{1+z}{1+z/\tau}\right)^{\nu-1}, \qquad \tau = \frac{p}{q}$$

Many particles (Tracy-Widom):

$$\mathbb{P}(\mu \to \nu; t) = \frac{1}{(2\pi i)^n}\oint_{-\tau} dz_1 \cdots \oint_{-\tau} dz_n \prod_{i=1}^{n} \frac{p-q}{(1+z_i/\tau)^2} \times \sum_{\pi \in S_n} \prod_{\pi_i < \pi_j} \frac{\tau z_i - z_j}{z_i - \tau z_j} \prod_{i=1}^{n} \varphi_{z_i}(\nu_{\pi_i} - \mu_i, t)$$

ASEP expectation values

$N_x(t)$: the number of particles to have crossed a given site x after time t.

Convenient observable (ASEP self-dual): $Q_x(t) = \tau^{N_x(t)}$ with $\tau = \frac{p}{q}$ and

$$\widetilde{Q}_x(t) = \frac{Q_x(t) - Q_{x-1}(t)}{\tau - 1} = \tau^{N_{x-1}(t)} \mathbf{1}_{x \in \nu_t}$$

Theorem (Borodin-Corwin-Sasamoto (step initial condition),...)

$$\mathbb{E}[\widetilde{Q}_{x_1}(t) \cdots \widetilde{Q}_{x_k}(t)] = \oint dz_1 \cdots \oint dz_n \prod_{1 \leqslant i < j \leqslant k} \frac{z_i - z_j}{z_i - \tau z_j} \prod_{\alpha=1}^{k} \varphi_{z_i}(x_i, t) \frac{1}{z_i + \tau}$$

Fluctuations of particle flow across the origin follow KPZ statistics given by the Tracy-Widom distribution:

Theorem (Fluctuations of ASEP)

$$\lim_{t \to \infty} P\left(\frac{N_0(t) - \nu t}{t^{1/3}} > -s \right) = F_{\mathsf{GUE}}(s).$$

Summary

Ingredients:

- Yang-Baxter integrable stochastic lattice model
- Observable expressed in terms of k-fold integral (Bethe Ansatz)
- Asympotics for large $k \rightarrow$ Fredholm determinant
- Saddle point analysis

New results:

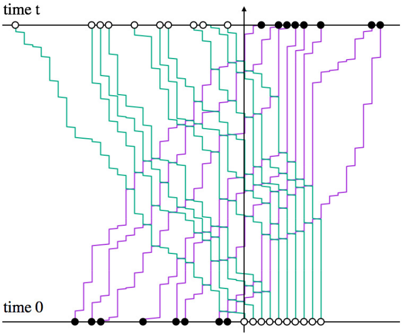
- Rank two model (two species of particles)
- Dynamic poles in integral
- Combination of Gaussian and GUE modes

AHR model

Introduced by Arndt-Heinzl-Rittenberg, the transition rates are

$$\begin{aligned} p: &\quad (+,0) \to (0,+) \\ q: &\quad (0,-) \to (-,0) \\ 1: &\quad (+,-) \to (-,+) \end{aligned}$$

Throughout we will take $p + q = 1$ (factorised steady state).



Nonlinear Fluctuating Hydrodynamics

Continuity equation

$$\frac{\partial \mathbf{u}(x,t)}{\partial t} + \frac{\partial \mathbf{j}(\mathbf{u}(x,t))}{\partial x} = 0,$$

where $\mathbf{u}(x,t) = (\rho_+, \rho_-)$ and $\mathbf{j}(\mathbf{u}) = (j_+, j_-)$ given by non-linear flows

$$\begin{aligned} j_+(\mathbf{u}) &= \rho_+(1-\rho_+-\rho_-) + 2\rho_+\rho_-, \\ j_-(\mathbf{u}) &= -(1-\rho_+-\rho_-)\rho_- - 2\rho_+\rho_- . \end{aligned}$$

Adding diffusion and noise, heuristic non-linear fluctuating hydrodynamics (NLFHD) leads to

$$P_{\text{crossing}}(t) \sim F_{\text{GUE}}(s_+) F_{\text{Gauss}}(s_-),$$

where $s_\pm$ are eigenmodes.

Aim of this work is to rigorously prove this.

Transition probability

Definition

The transition probability satisfies the master equation

$$\frac{d}{dt} G(t) = p \sum_i G(\vec{x}_i^-, t) + q \sum_i G(\vec{y}_i^+, t) - (np + mq) G(t) \qquad t > 0,$$

Scattering conditions

$$\textit{Exclusion:} \quad G(x, x; t) = G(x, x+1; t),$$

$$\textit{Exclusion:} \quad G(y, y; t) = G(y, y-1; t),$$

$$\textit{Scattering:} \quad G(x = y; y+1; t) = q G(x = y+1; y+1; t) + p G(x = y; y; t).$$

and initial condition

$$G(\vec{x}; \vec{y}; 0) = \prod_{i=1}^{n} \delta_{x_i, x_i^{(0)}} \prod_{j=1}^{m} \delta_{y_j, y_j^{(0)}}$$

- Explicit form can be determined by nested Bethe ansatz due to Yang-Baxter integrability.

Transition probability

Initial conditions: assume $x_j^{(0)} < y_k^{(0)}$, i.e. at $t = 0$ all $+$ particles are to the left of all $-$ particles.

Final condition: $x_j > y_k$, i.e. at time t all $+$ particles have passed all $-$ particles

Then

$$G(\mathbf{x}, \mathbf{y}, t) = \oint \prod_{j=1}^{n} \mathrm{d}\, z_j \prod_{k=1}^{m} \mathrm{d}\, w_k \, e^{\Lambda t} \prod_{k=1}^{m} \prod_{j=1}^{n} \frac{1}{q z_j + p w_k}$$
$$\times \det \left(\left(\frac{z_j - 1}{z_i - 1} \right)^{j-1} z_i^{x_j} \right) z_j^{-x_j^{(0)} - 1}$$
$$\times \det \left(\left(\frac{w_k - 1}{w_\ell - 1} \right)^{m-k} w_\ell^{-y_k} \right) w_k^{y_k^{(0)} - 1},$$

with all contours around the origin, and with eigenvalue

$$\Lambda = p \sum_{j=1}^{n} (z_j^{-1} - 1) + q \sum_{k=1}^{m} (w_k^{-1} - 1).$$

- It is possible to give an explicit expression for any intial and final condition.

Current distribution: step initial condition

Given the following step initial condition

$$\cdots 0000 \overbrace{+ + \cdots + +}^{n\text{times}} \overbrace{- - \cdots - -}^{m\text{times}} 0000 \cdots ,$$

Then

$$\mathbb{P}(x_1(t) \geqslant s) = \sum_{x_1=s}^{\infty} \cdots \sum_{x_n=x_{n-1}+1}^{\infty} \sum_{y_1=-\infty}^{-n} \cdots \sum_{y_n=y_{n-1}+1}^{-1} G(\{x_j\};\{y_k\};t),$$

Proposition

$$\mathbb{P}(x_1(t) \geqslant 0) =$$

$$\oint \prod_{j=1}^{n} dz_j \prod_{k=1}^{m} dw_k \, e^{\Lambda t} \frac{\prod\limits_{1\leqslant i<j\leqslant n} (z_i - z_j) \prod\limits_{1\leqslant k<l\leqslant m} (w_l - w_k) \prod\limits_{j=1}^{n} z_j^{n-j+s} \prod\limits_{k=1}^{m} w_k^{k-1}}{\prod\limits_{j=1}^{n} (z_j - 1)^{n+1-j} \prod\limits_{k=1}^{m} (w_k - 1)^k \prod\limits_{j=1}^{n} \prod\limits_{k=1}^{m} \left(qz_j + pw_k\right)},$$

with all contours around the origin.

Current distribution

- $e^{\Lambda t}$ produces an essential singularity at origin: $\Lambda = p\sum_{j=1}^{n}(z_j^{-1}-1) + q\sum_{k=1}^{m}(w_k^{-1}-1)$.
- Deform w-contours to lie around poles other than the origin
- Only (simple) poles at $w=1$ give nonzero contribution

After evaluating the residues in w, we get

Proposition

$$\mathbb{P}(x_1(t) \geqslant 0) = \oint \prod_{j=1}^{n} \frac{d z_j}{2\pi i}\, e^{\tilde{\Lambda} t} \frac{\prod_{1\leqslant i<j\leqslant n}(z_i - z_j) \prod_{j=1}^{n} z_j^{n-j}}{\prod_{j=1}^{n}(z_j-1)^{n+1-j} \prod_{j=1}^{n}\left(q z_j + p\right)^{m}}$$

TASEP limit

Corollary

When $n = m$ *and* $p = q = 1/2$ *we retrieve the same distribution as for the single species TASEP under step initial condition, i.e.*

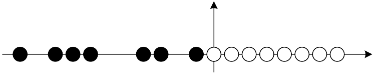
$$P_{n,n,1}(t) = \frac{1}{n!} \oint \prod_{j=1}^{n} dx_j \, e^{\mathcal{E} t} \frac{\prod_{1 \leqslant i < j \leqslant n} (x_i - x_j)^2}{\prod_{j=1}^{n} (x_j - 1)^n},$$

where the contours are still around the origin and $\mathcal{E} = \sum_{j=1}^{n} (x_j^{-1} - 1)$.

This is made explicit by symmetrising and the simple change of variable $z_j = x_j/(2 - x_j)$.

From known analysis (Tracy-Widom) this probability converges to the GUE distribution as $n, t \to \infty$.

Step-Bernoulli condition



Let the distance among positive particles be independently distributed with parameter ρ',

Proposition

The total exchange probability $P_{n,m,\rho}(t)$ *with Bernoulli initial data is given by*

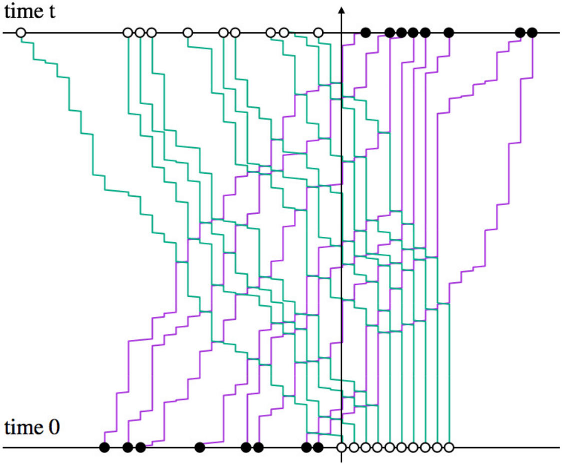
$$P_{n,m,\rho}(t) = \oint \prod_{j=1}^{n} \mathrm{d} z_j \prod_{k=1}^{m} \mathrm{d} w_k \, e^{\Lambda_{n,m} t} \times$$

$$\frac{\rho^n \prod_{1 \leqslant i < j \leqslant n} (z_i - z_j) \prod_{1 \leqslant k < l \leqslant m} (w_l - w_k) \prod_{j=1}^{n} z_j^{n-j} \prod_{k=1}^{m} w_k^{k-1}}{\prod_{j=1}^{n} (z_j - 1)^{n+1-j} (1 - \rho' z_j) \prod_{k=1}^{m} (w_k - 1)^k \prod_{j=1}^{n} \prod_{k=1}^{m} \left(q z_j + p w_k \right)},$$

with all contours around the origin.

The w-contours can be readily evaluated if $n > m$ but not when $n < m$

Exchange



Asymptotics

Non-linear fluctuating hydrodynamics (KPZ formalism) suggests a scaling limit of the form

$$n = j_1 t + \alpha t^{1/3} + \beta t^{1/2},$$
$$m = j_2 t + \gamma t^{1/3} + \delta t^{1/2},$$

where $j_{1,2}$, α, β, γ, δ are known functions of ρ', and $n < m$.

Need to analyse

$$P_{n,m,\rho}(t) = \underbrace{\oint \dots \oint}_{n \times m} \text{factorised integrand}$$

where n, m, t are large.

Trick: Convert to Fredholm determinant:

$$P_{n,m,\rho}(t) = \det(\mathbb{I} - AB)_{m\times m} = \det(\mathbb{I} - BA)_{L^2(\mathbb{R})},$$

where n, m, t all occur as parameters in BA.

Asymptotics

Need to calculate integrals like

$$\mathcal{J}_2 = \oint_1 \mathrm{d}^{n-1} z\, L(\vec{z}) \det(\mathbb{I} - K(\vec{z}))_{\ell^2(\mathbb{N})}$$

with

$$K(x, y; \vec{z}) = \oint_1 \frac{\mathrm{d}\zeta}{2\pi \mathrm{i}} F(\zeta, x) \prod_{j=1}^{n-1} \frac{1 + z_j \zeta}{1 + \zeta} \oint_C \frac{\mathrm{d}w}{2\pi \mathrm{i}} G(w, y) \prod_{j=1}^{n-1} \frac{1 + w}{1 + z_j w} \frac{1}{w - \zeta},$$

Proposition

For any $(x_1, x_2, \ldots, x_k) \in \mathbb{N}^k$, $\rho \in (0, 1)$, $t > 0$ *and* $n, m \in \mathbb{N}$, *the following equality holds:*

$$\oint_1 \mathrm{d}^{n-1} z\, L(\vec{z}) \det\left[K(x_i, x_j, \vec{z})\right]_{1 \leqslant i,j \leqslant k}$$

$$= \oint_1 \mathrm{d}^{n-1} z\, L(\vec{z}) \det\left\{K_W(x_i, x_j) - \left[\sum_{l=1}^{n-1} \prod_{k=1}^{l} (z_k - 1) A_l(x_i)\right] B(x_j)\right\}_{1 \leqslant i,j \leqslant k}.$$

Asymptotics

$$\mathcal{J}_2 = \mathcal{J}_z \det\left(\mathbb{I} - \mathrm{K}(\vec{z} = \vec{1})\right)_{\ell^2(\mathbb{N})} + \text{ lower order}$$

In order to perform asymptotic analysis, we define the rescaled functions

$$\xi = x/\lambda_c t^{1/3}, \qquad \zeta = y/\lambda_c t^{1/3}$$

such that

$$\mathcal{K}(\xi, \zeta) = (w_c + c)^{\lambda_c t^{1/3}(\xi - \zeta)} \lambda_c t^{1/3} \mathrm{K}(\lambda_c t^{1/3} \xi, \lambda_c t^{1/3} \zeta),$$

The rescaled kernel is explicitly described as

$$\mathcal{K}(\xi, \zeta) = \lambda_c t^{1/3} \oint_1 \frac{dz}{2\pi i} e^{f(z,t,\xi) - f(w_c,t,\xi) + g(z)} \times \oint_{0,-\rho',-1} \frac{dw}{2\pi i} e^{-f(w,t,\zeta) + f(w_c,t,\zeta) + h(w)} \frac{1}{w - z},$$

Sadlle point analysis

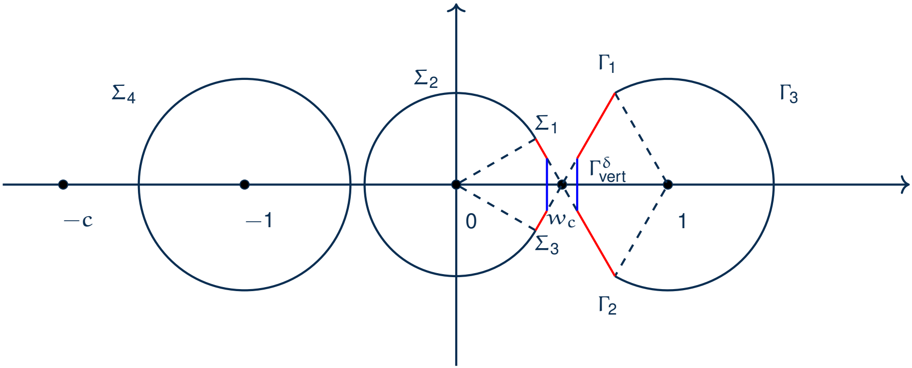


Figure: Saddle point contour ensuring uniform convergence

Theorem

$$\lim_{t\to\infty} \det(1-\mathcal{K})_{\ell^2(\mathbb{N}/(\lambda_c t^{1/3}))} = \lim_{t\to\infty} \det(1-\mathcal{K})_{L^2(0,\infty)} = \det(1-\mathcal{A})_{L^2(s,\infty)} = F_2(s)$$

$$\mathcal{A}(x,y) = \int_0^\infty \mathrm{Ai}(x+\lambda)\,\mathrm{Ai}(y+\lambda)\,\mathrm{d}\lambda$$

and

$$s = \frac{1}{c_2 t^{1/3}}\Big((1+\rho)n - (3-\rho)m + \tfrac{1}{2}(1-\rho)(1-(1-\rho)^2/4)t\Big)$$

Recall

$$\mathcal{J}_2 = \mathcal{I}_z \det\left(\mathbb{I} - K(\vec{z}=\vec{1})\right)_{\ell^2(\mathbb{N})} + \text{ lower order}$$

The integral $\mathcal{I}_z$ converges to a Gaussian

$$\mathcal{J}_2 \to \big(1-F_G(s')\big)F_2(s) \quad \text{as } t\to\infty$$

Final result

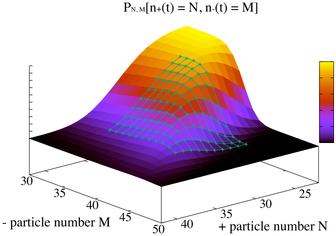
Theorem

In the appropriate scaling limit

$$\lim_{t\to\infty} \mathbb{P}_{n,m,\rho}(t) = F_{\mathrm{GUE}}(s) F_{\mathrm{Gauss}}(s'),$$

$$s(n,m;t) =: \frac{1}{c_2 t^{1/3}} \Big((1+\rho)n - (3-\rho)m + \tfrac{1}{2}(1-\rho)(1-(1-\rho)^2/4)t \Big),$$

$$s'(n,m;t) =: \frac{1}{c_g t^{1/2}} \Big(-2(2-\rho)n + 2\rho m + (2-\rho)(1-\rho)\rho t \Big),$$



Conclusion

- (First) proof of Nonlinear Fluctuating Hydrodynamics for a two-component mixture
- Using integrability
- Mix of Gaussian and KPZ modes
- Dynamic poles in integrand