Non-compact spin chains and integrable particle systems

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Based on collaborations with C. Giardinà (Unimore) and J. Kurchan (Ens Paris)
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Integrable simple exclusion models
Most famous stochastic particle processes are: **ASEP** and **SSEP**

- Integrable
- Nearest-neighbor hopping model
- One particle per site (exclusion)
- Closed or open boundary conditions

Hopping rates: $r$ and $l$ and $\alpha$, $\beta$, $\gamma$ and $\delta$

**SSEP:** $r = l = 1$

Some great reviews: [Derrida], [Schütz], [Blythe,Evans], [Crampé,Ragoucy,Vanicat], ...
Exclusion process is generated by Markov matrix

\[ M = B_1 + \sum_{i=1}^{N-1} \omega_{i,i+1} + B_N \]

**Bulk:**

\[ \omega = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & -l & r & 0 & 0 \\
0 & l & -r & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix} \]

**Boundary:**

\[ B_1 = \begin{pmatrix}
-\alpha & \gamma \\
\alpha & -\gamma
\end{pmatrix}, \quad B_N = \begin{pmatrix}
-\delta & \beta \\
\delta & -\beta
\end{pmatrix} \]

**Stochastic process:**

Sum over rows vanishes

Off-diagonal entries have opposite sign of diagonal entries
Stochastic process can be mapped to integrable spin chain

- ASEP ↔ XXZ spin chain
- SSEP ↔ XXX spin chain

Hamiltonian is related to Markov generator

\[ M = SHS^{-1} \]

Particle process can be studied using integrability tools: Coordinate Bethe ansatz, algebraic Bethe ansatz, ...

Multi-species generalisations from higher rank spin chains
ASEP/SSEP produces traffic jams!

Lot of effort to avoid traffic jams...
Multi-particle generalisations

Put particles on top of each other

Naïve observation:

- Higher spin integrable model: Hamiltonian not stochastic
- Higher spin stochastic model: Hamiltonian not integrable

→ Non-compact integrable spin chains [RF, Giardinà, Kurchan ’19]
SSEP within the quantum inverse scattering method
Quantum inverse scattering method  

**Starting point:** Yang-Baxter equation

\[ R_{12}(z_1 - z_2)R_{13}(z_1 - z_3)R_{23}(z_2 - z_3) = R_{23}(z_2 - z_3)R_{13}(z_1 - z_3)R_{12}(z_1 - z_2) \]

- Fundamental relation underlying integrable systems
- Each R-matrix \( R_{ij} \) acts on the tensor product of three spaces \( V_1 \otimes V_2 \otimes V_3 \) with

\[ R_{12}(z) = R(z) \otimes I, \ldots \]

- Fundamental R-matrix for SSEP / XXX Heisenberg spin chain

\[ R(z) = z + P \quad \text{with} \quad P = \sum_{a,b=1}^{2} e_{ab} \otimes e_{ba} \]

where \( (e_{ab})_{cd} = \delta_{ac}\delta_{bd} \), \( z \in \mathbb{C} \) and \( P \) acts as a permutation
Graphical notation

- **R-matrix:**

\[ R_{ij}(z_i - z_j) = \]

- **Multiplication of R-matrices:**

\[ R_{12}(z_1 - z_2)R_{13}(z_1 - z_3) = \]

- **Yang-Baxter equation:**

\[ \]
Spin chain monodromy

\[ \mathcal{M}(z) = R_{a,1}(z)R_{a,2}(z)\cdots R_{a,N}(z) = \begin{array}{cccc}
1 & 2 & \cdots & N \\
a & \cdots & & \\
\end{array} \]

- Multiplication of $2 \times 2$ matrices in auxiliary space and tensor product in quantum space
- Satisfies RTT-relation

\[ R(z_1 - z_2)(\mathcal{M}(z_1) \otimes \mathbb{I})(\mathbb{I} \otimes \mathcal{M}(z_2)) = (\mathbb{I} \otimes \mathcal{M}(z_2)) (\mathcal{M}(z_1) \otimes \mathbb{I}) R(z_1 - z_2) \]

- Pictorially
Transfer matrix

\[ T(z) = \text{tr}_a M(z) = \ldots \]

1 \hspace{1cm} 2 \hspace{1cm} \ldots \hspace{1cm} N

Markov generator / Hamiltonian

\[ M_{SSEP}^{\text{cl.}} = \frac{\partial}{\partial z} \log T(z)|_{z=0} + \text{const} \]

Commuting family of operators (common eigenstates)

\[ [T(z), T(z')] = 0, \hspace{1cm} [T(z), M_{SSEP}^{\text{cl.}}] = 0 \]

How to describe process with reservoir?
Open spin chains [Sklyanin]

Transfer Matrix

\[
T(z) = tr \, \mathcal{K}_a(z) \, R_{a,1}(z) R_{a,2}(z) \cdots R_{a,N}(z) \hat{\mathcal{K}}_a(z) \, R_{a,N}(z) \cdots R_{a,2}(z) R_{a,1}(z)
\]

Graphically

\[
T(z) = \begin{array}{c}
1 \quad 2 \quad \cdots \quad N \\
\end{array}
\]

K-matrices

\[
\mathcal{K}(z) = \begin{array}{c}
\langle \quad \rangle \\
\end{array} \quad \hat{\mathcal{K}}(z) = \begin{array}{c}
\rangle \quad \langle \\
\end{array}
\]
Boundary Yang-Baxter equation

\[ R_{12}(z_1 - z_2) \hat{K}_1(z_1) R_{12}(z_1 + z_2) \hat{K}_2(z_2) = \hat{K}_2(z_2) R_{12}(z_1 + z_2) \hat{K}_1(z_1) R_{12}(z_1 - z_2) \]

And analogously for other boundary involving \( \mathcal{K}(z) \)

- Most general K-matrices

\[
\mathcal{K}(z) = \begin{pmatrix}
p_1 + p_2(z + 1) & p_3(z + 1) \\
p_4(z + 1) & p_1 - p_2(z + 1)
\end{pmatrix}, \quad \hat{\mathcal{K}}(z) = \begin{pmatrix}
q_1 + q_2z & zq_3 \\
zq_4 & q_1 - q_2z
\end{pmatrix}
\]
Relation to SSEP

Adjust boundary parameters

\[ q_1 = 1, \quad q_2 = \beta - \delta, \quad q_3 = 2\beta, \quad q_4 = 2\delta \]
\[ p_1 = 1, \quad p_2 = \gamma - \alpha, \quad p_3 = 2\gamma, \quad p_4 = 2\alpha \]

Markov generator / Hamiltonian

\[ M_{SSEP} = \frac{\partial}{\partial z} \log T(z)|_{z=0} + \text{const.} \]

Commuting transfer matrices

\[ [T(z), T(z')] = 0, \quad [T(z), M_{SSEP}] = 0 \]

Expansion of \( T(z) \) generates commuting charges

\[ [M_{SSEP}, Q_k] = 0 \]

Will become handy later...
Non-compact integrable spin chains
Quantum space of non-compact chains with hws

\[ V = |m_1\rangle \otimes |m_2\rangle \otimes \ldots \otimes |m_N\rangle, \quad m_i = 0, 1, 2, \ldots \]

For spin \( s \) generators of \( \mathfrak{sl}(2) \) act locally as

\[ S_+|m\rangle = (m + 2s)|m + 1\rangle, \quad S_-|m\rangle = m|m - 1\rangle \quad S_0|m\rangle = (m + s)|m\rangle \]

Nearest-neighbor Hamiltonian density [Faddeev et al.]

\[ \mathcal{H} = 2 \left( \psi(S) - \psi(2s) \right) \]

where \( \psi(x) \) is Digamma function and \( S \) is related to the two-site Casimir operator via \( C_{[2]} = S(S - 1) \)

- First studied in high energy QCD [Lipatov; Faddeev, Korchemsky]
- Important subsector of the \( \mathcal{N} = 4 \) SYM spin chain! (\( s = \frac{1}{2} \))
- Integrable models [Derkachov]
The operator $S$

Consider tensor product decomposition

$$D_S \otimes D_S = \bigoplus_{j=0}^{\infty} D_{2s+j}$$

Operator $S$ acts diagonally on the irreps on the rhs

$$S|D_{2s+j}\rangle = (2s + j)|D_{2s+j}\rangle$$

Eigenvalues of Hamiltonian density are harmonic numbers $h_s$

$$\mathcal{H}|D_{2s+j}\rangle = 2 \sum_{k=1}^{j} \frac{1}{2s + k - 1}|D_{2s+j}\rangle$$

- Can’t tell if process is stochastic from eigenvalues
- A priory not known how $\mathcal{H}$ acts on the lhs...

$\leadsto$ Clebsch Gordan decomposition
Harmonic action as stochastic process

Nearest neighbor hopping model for $s = \frac{1}{2}$

[Beisert; Braun, Derkachov, Manashov; Lipatov; Faddeev, Korchemsky]

$$\mathcal{H}|m\rangle \otimes |m'\rangle = (h(m) + h(m')) |m\rangle \otimes |m'\rangle - \sum_{k=1}^{m} \frac{1}{k} |m - k\rangle \otimes |m' + k\rangle$$

$$- \sum_{k=1}^{m'} \frac{1}{k} |m + k\rangle \otimes |m' - k\rangle$$

with the harmonic numbers $h(m) = \sum_{k=1}^{m} \frac{1}{k}$.

Hamiltonian density $\mathcal{H}$ is generator of Markov process!

[Giardinà, Kurchan, RF '19]

E.g. $m + m' = 2$:

$$\mathcal{H}_2 = \begin{pmatrix}
\frac{3}{2} & -1 & -\frac{1}{2} \\
-1 & 2 & -1 \\
-\frac{1}{2} & -1 & \frac{3}{2}
\end{pmatrix}$$
Harmonic action as stochastic process

Hamiltonian defined on $N$ sites as

$$H = \sum_{i=1}^{N-1} \mathcal{H}_{i,i+1}$$

Symmetric stochastic process without exclusion!

$\rightarrow$ $k$ particles jump with the rate $\varphi(k) = \frac{1}{k}$
Harmonic action as stochastic process

Hopping rates generalise to arbitrary spin $s > 0$ [Martins, Melo ’09]

$$\varphi_s(m, k) = \frac{1}{k} \frac{\Gamma(m + 1)\Gamma(m - k + 2s)}{\Gamma(m - k + 1)\Gamma(m + 2s)}$$

Again we find a symmetric particle process!

$\leftrightarrow$ Rates depend on number of particles at departing site

Up to now only reinterpreting results of others...

Add a particle current (non-equilibrium models):

- q-analog/XXZ-analog $\rightarrow$ asymmetric (drift) process
- Rational case with boundary reservoirs
Non-compact XXZ spin chain as stochastic particle process
Non-compact $\mathcal{U}_q(sl_2)$ invariant XXZ chain

Commutation relations $\mathcal{U}_q(sl_2)$

$$\left[S_+, S_- \right] = -\left[2S_0 \right], \quad \left[S_0, S_\pm \right] = \pm S_\pm$$

with q-number $[x] = \frac{q^x - q^{-x}}{q - q^{-1}}$.

Generators of $\mathcal{U}_q(sl_2)$ act locally as

$$S_+ |m\rangle = [m + 2s] |m + 1\rangle, \quad S_- |m\rangle = [m] |m - 1\rangle, \quad S_0 |m\rangle = (m + s) |m\rangle$$

Hamiltonian density of XXZ chain with $|q| < 1$ [Bytsko]

$$\mathcal{H} = \frac{\psi_q(S) - \psi_q(2s)}{-q^{4s} \log(q)}$$

with q-Digamma function $\psi_q$ and $S$ is related to the co-product of the Casimir operator via $\Delta(C) = [S][S - 1]$. 
Some definitions and special functions...

**Co-product**

\[ \Delta(S_0) = S_0 \otimes 1 + 1 \otimes S_0, \quad \Delta(S_{\pm}) = S_{\pm} \otimes q^{-S_0} + q^{S_0} \otimes S_{\pm} \]

**q-Gamma function**

\[ \Gamma_q(x) = q^{\frac{1}{2}x(1-x)} (q^{-1} - q)^{1-x} \frac{(q^2; q^2)_\infty}{(q^{2x}; q^2)_\infty} \]

with \((a; q)_n = \prod_{k=0}^{n-1}(1 - aq^k)\)

**q-Digamma function**

\[ \psi_q(x) = \partial_x \log \Gamma_q(x) \]
Harmonic action for XXZ chain

Use Clebsch-Gordan decomposition to obtain nearest neighbor hopping action on two sites [RF '19]

\[
\mathcal{H}|m\rangle \otimes |m'\rangle = (\alpha_+(m) + \alpha_-(m')) |m\rangle \otimes |m'\rangle - \sum_{k=1}^{m} \rho(m, k) |m - k\rangle \otimes |m' + k\rangle \\
- \sum_{k=1}^{m'} \rho(m', k) |m + k\rangle \otimes |m' - k\rangle
\]

with diagonal entries

\[
\alpha_{\pm}(m) = \frac{\psi_q(m + 2s) - \psi_q(2s) \pm m \log(q)}{-2q^{4s} \log(q)}
\]

and off-diagonal entries

\[
\rho(m, k) = \frac{q^{2ks}}{q^{4s}(1 - q^{2k})} \frac{(q^2; q^2)_m (q^{4s}; q^2)_{m-k}}{(q^2; q^2)_{m-k} (q^{4s}; q^2)_m}
\]
Relation to stochastic q-Hahn process

As in ASEP, Hamiltonian density $\mathcal{H}$ is not a Markov matrix!

Similarity transformation yields Markov matrix

$$
\mathcal{M} = \begin{pmatrix}
\alpha_+(n) + \alpha_-(0) & -\beta_-(1,1) & -\beta_-(2,2) & \cdots & -\beta_-(n,n) \\
-\beta_+(n,1) & \alpha_+(n-1) + \alpha_-(1) & -\beta_-(2,1) & \cdots & -\beta_-(n,n-1) \\
-\beta_+(n,2) & -\beta_+(n-1,1) & \alpha_+(n-2) + \alpha_-(2) & \cdots & -\beta_-(n,n-2) \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
-\beta_+(n,n) & -\beta_+(n-1,n-1) & -\beta_+(n-2,n-2) & \cdots & \alpha_+(0) + \alpha_-(n)
\end{pmatrix}.
$$

with

$$
\beta_{\pm}(m, k) = \frac{\mu^{1/2}k(1\pm 1)}{\mu(1-\gamma^k)} \frac{(\gamma; \gamma)_m(\mu; \gamma)_{m-k}}{(\gamma; \gamma)_{m-k}(\mu; \gamma)_m},
$$

where $\gamma = q^2$ and $\mu = q^{4s}$

Coincides with rates of q-Hahn process introduced by

[Povolotsky; Barraquand-Corwin; Sasamoto-Wadati] without reference to XXZ chain!
Non-compact spin chains and stochastic particle processes

- q-Hahn process
  - $\mu = q^{4s}$
  - $\gamma = q^2$
  - $\mu \rightarrow \gamma$

- MADM
  - $\gamma = q^2$

- non-compact XXZ spin $s$
  - $s = \frac{1}{2}$
  - $q \rightarrow 1$

- non-compact XXX spin $s$
  - $s = \frac{1}{2}$
  - $q \rightarrow 1$

Harmonic processes
Non-compact XXX chain with boundaries
Stochastic process with boundary reservoirs

Add stochastic boundary conditions to rational process

\[ H = B_1 + \sum_{i=1}^{N-1} \mathcal{H}_{i,i+1} + B_N. \]

Guess boundary terms for \( 0 < \beta_i < 1 \) and \( s = \frac{1}{2} \) [RF, Giardinà, Kurchan '19]

\[ B_i|m_i\rangle = \left( h(m_i) + \sum_{k=1}^{\infty} \frac{\beta_i^k}{k} \right) |m_i\rangle - \sum_{k=1}^{m_i} \frac{1}{k} |m_i - k\rangle - \sum_{k=1}^{\infty} \frac{\beta_i^k}{k} |m_i + k\rangle \]

Introduces reservoirs at left and right end of the chain:

Is this process integrable?
Construct the fundamental transfer matrix

\[ T(x) = \text{tr} K(x)M(x)\hat{K}(x)\hat{M}(x) \]

with the monodromies

\[ M(x) = R_1(x) \cdots R_N(x), \quad \hat{M}(x) = R_N(x) \cdots R_1(x) \]

where

\[ R(x) = (-1)^s \frac{\Gamma(2s - x)\Gamma(s + x)}{\Gamma(2s + x)\Gamma(s - x)} \]

Hamiltonian is logarithmic derivative of \( T(x) \) at permutation point

\[ H = \partial_x \log T(x)|_{x=0} \]

But: Closed expression of K-matrix unknown!
Quantum Inverse Scattering Method

Derive the universal K-matrix from BYBE [RF, Giardinà, Kurchan '19]

\[
\mathcal{L}(x - y) \hat{K}(x) \mathcal{L}(x + y) \hat{K}(y) = \hat{K}(y) \mathcal{L}(x + y) \hat{K}(x) \mathcal{L}(x - y)
\]

Lax matrix and K-matrix in fundamental representation

\[
\mathcal{L}(x) = \begin{pmatrix} x + \frac{1}{2} + S_o & -S_- \\ S_+ & x + \frac{1}{2} - S_o \end{pmatrix}, \quad \hat{K}(x) = \begin{pmatrix} q_1 + xq_2 & xq_3 \\ xq_4 & q_1 - xq_2 \end{pmatrix}
\]

Solve for \( \hat{K}(x) \)...
Universal solution to BYBE

1. Introduce useful parametrisation of boundary variables

\[ q_1 = \delta, \quad q_2 = \frac{1}{2} (1 + 2\alpha\beta)\gamma, \quad q_3 = -(1 + \alpha\beta)\beta\gamma, \quad q_4 = \alpha\gamma \]

2. Make the ansatz

\[ \hat{\mathcal{K}}(x) = e^{\beta S_+} e^{-\alpha S_-} \hat{\mathcal{K}}_0(S_0; x) e^{\alpha S_-} e^{-\beta S_+} \]

Yields difference equation for \( \hat{\mathcal{K}}_0(S_0; x) \) which can be solved

\[ \hat{\mathcal{K}}_0(S_0; x) = \frac{\Gamma\left(\frac{1}{2} + s + 2\frac{\delta}{\gamma} - x\right) \Gamma\left(\frac{1}{2} + S_0 + 2\frac{\delta}{\gamma} + x\right)}{\Gamma\left(\frac{1}{2} + s + 2\frac{\delta}{\gamma} + x\right) \Gamma\left(\frac{1}{2} + S_0 + 2\frac{\delta}{\gamma} - x\right)} \]

Other boundary obtained via

\[ \mathcal{K}(x) = \frac{1}{\hat{\mathcal{K}}(x + 1)} \]
To derive stochastic boundary conditions for Hamiltonian fix

\[ \frac{2}{\gamma} = s - \frac{1}{2}, \quad \alpha = \frac{1}{1 - \beta} \]

and compute the logarithmic derivative of the transfer matrix

\[ \frac{\partial}{\partial x} \ln T(x) \bigg|_{x=0} = \frac{\text{tr}_a \mathcal{K}_a(O)}{\text{tr}_a \mathcal{K}_a(O)} + 2 \frac{\text{tr}_a \mathcal{K}_a(O) \mathcal{H}_{a,1}}{\text{tr}_a \mathcal{K}_a(O)} + \frac{\hat{\mathcal{K}}'_N(O)}{\hat{\mathcal{K}}_N(O)} + 2 \sum_{k=1}^{N-1} \frac{\partial}{\partial x} \ln \mathcal{R}_{k,k+1}(x) \bigg|_{x=0}, \]

Full Hamiltonian

\[ H = \mathcal{B}_1 + \sum_{i=1}^{N-1} \mathcal{H}_{i,i+1} + \mathcal{B}_N \]

with algebraic expression for boundaries

\[ \mathcal{B}_i = e^{-S[i]} e^{\rho_i S[i]} \left( \psi(S_0^{[i]} + s) - \psi(2s) \right) e^{-\rho_i S[i]} e^{S[i]} \quad \text{for } i \in \{1, N\}. \]

where \( \rho_i = \frac{\beta_i}{1 - \beta_i} \).
A longer computation shows that we obtain the spin $s$ version of desired boundary terms!

\[
\mathcal{B}_i|m_i\rangle = \left( h^{(s)}(m_i) + \sum_{k=1}^{\infty} \frac{\beta_i^k}{k} \right) |m_i\rangle - \sum_{k=1}^{m_i} \frac{1}{k} \frac{\Gamma(m_i + 1)\Gamma(m_i - k + 2s)}{\Gamma(m_i - k + 1)\Gamma(m_i + 2s)} |m_i - k\rangle
\]

\[
- \sum_{k=1}^{\infty} \frac{\beta_i^k}{k} |m_i + k\rangle,
\]

- Process is integrable!
- Derived stochastic boundaries for arbitrary spin $s$
Steady state of harmonic process with boundaries
Derrida solution

SSEP solved in 1993 using matrix product ansatz [Derrida et al.]

Representation of steady state $H|\mu\rangle = 0$

$$|\mu\rangle = \frac{1}{\langle W|(E + D)^N|V\rangle} \begin{pmatrix} \langle W|E\cdots EEE|V\rangle \\ \langle W|E\cdots EED|V\rangle \\ \langle W|E\cdots EDE|V\rangle \\ \vdots \\ \langle W|D\cdots DDD|V\rangle \end{pmatrix}$$

DEHP algebra

- Bulk relation: $DE - ED = D + E$
- Boundary relations:
  $$\langle W|(\alpha E - \gamma D) = \langle W|, \quad (\beta D - \delta E)|V\rangle = |V\rangle$$

MPA difficult as there are not only two operators $E$ and $D$
Steady state

Follow alternative route applied for SSEP in [RF ‘19; RF, Giardina, Kurchan ‘20], inspired by [Alcaraz, Droz, Henkel, Rittenberg], [Melo, Ribeiro, Martins], [Essler, de Gier], [Crampé, Ragoucy, Vanicat]

1. SSEP generator can be brought to a block triangular form

\[
H_\Delta = G^{-1}HG = \begin{pmatrix} -\alpha - \gamma & \Delta \\ 0 & 0 \end{pmatrix} + \sum_{i=1}^{N-1} \omega_{i,i+1} \begin{pmatrix} -\beta - \delta \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \end{pmatrix}
\]

with \( \Delta = \frac{(\alpha+\gamma)(\alpha\beta-\gamma\delta)}{\beta+\delta} \) and \( G \) only depends on \( S_{a}^{\text{tot}} \).

2. \( H_\Delta \) is isospectral to diagonal Hamiltonian \( H^\circ = H_{\Delta=0} \) with \( \Delta = 0 \)

3. Determine non-local transformation \( W_\Delta \) s.t.

\[
H^\circ = W_\Delta^{-1}H_\Delta W_\Delta
\]

4. Obtain closed-form of steady state from pseudovacuum

\[
|\psi\rangle = GW_\Delta |\Omega\rangle
\]

Same logic works for non-compact boundary model [Frassek, Giardina ‘21]
Transformations for the non-compact model

Local transformation that block triangularises $H$:

$$G = \prod_{i=1}^{N} e^{-S_{i}^{-}} e^{\rho N S_{i}^{+}}$$

Non-local transformation that block diagonalises $H_{\Delta}$:

$$W_{\Delta} = \sum_{k=0}^{\infty} \Delta^{k} \frac{Q_{+}^{k}}{k!} \frac{\Gamma(2(S_{0}^{\text{tot}} + s))}{\Gamma(k + 2(S_{0}^{\text{tot}} + s))}$$

with

$$Q_{+} = s S_{+}^{\text{tot}} + \sum_{i=1}^{N} S_{i}^{[i]} \left(S_{i}^{[i]} + 2 \sum_{j=i+1}^{N} S_{j}^{[j]}\right)$$

$Q_{+}$ is obtained from the transfer matrix at leading order in spectral parameter
Evaluation of the steady state
Factorial moments

Steady state

\[ \langle m|\mu \rangle = \langle m|GW_\Delta|\Omega \rangle = \sum_{n \geq m} F(n) \prod_{i=1}^{N} \left( \frac{(-1)^{n_i-m_i}}{n_i!} \binom{n_i}{m_i} \frac{\Gamma(2s+n_i)}{\Gamma(2s)} \right) \]

with factorial moments

\[ F(n) = \sum_{k=0}^{\lfloor |n| \rfloor} \rho_N^{|n|-k} (\rho_1 - \rho_N)^k f_n(k) \]

where

\[ f_n(k) = \sum_{|w|=k} \prod_{i=1}^{N} \binom{n_i}{w_i} \prod_{j=1}^{w_i} \frac{2s(N+1-i) - j + \sum_{k=i}^{N} w_k}{2s(N+1) - j + \sum_{k=i}^{N} w_k} . \]
Steady state for length $N=1$

$N = 1$ and $s = 1/2$

$$\langle m_1 | \mu \rangle = \frac{(\beta_L - 1)(\beta_R - 1)}{\beta_L - \beta_R}\left(\gamma_{\beta_L}(m_1 + 1) - \gamma_{\beta_R}(m_1 + 1)\right).$$

with $\beta_L = \beta_1$ and $\beta_R = \beta_N$ and

$$\gamma_{\beta}(n) = \sum_{k=n}^{\infty} \frac{\beta^k}{k}.$$
Steady state for length $N=2$

$N = 2$ and $s = 1/2$

$$\langle m_1, m_2 | \mu \rangle = 2 \frac{(\beta_L - 1)^2(\beta_R - 1)^2}{(\beta_L - \beta_R)^2} \left( \phi_{\beta_L}(m_1, m_2) - \kappa(m_1, m_2) + \phi_{\beta_R}(m_2, m_1) \right)$$

where

$$\phi_{\beta}(m_1, m_2) = \frac{1}{2} \gamma_{\beta}^2 (1 + m_1) - \sum_{k=m_1+1}^{m_2} \frac{1}{k} \gamma_{\beta}(m_1 + k + 1) + \sum_{k=m_2+1}^{m_1} \frac{1}{k} \gamma_{\beta}(m_1 + k + 1)$$

and

$$\kappa(m_1, m_2) = \gamma_{\beta_L}(1 + m_1) \gamma_{\beta_R}(1 + m_2).$$
Other eigenstates of $H$ can be obtained from standard Bethe ansatz for $H^\circ$:

$$|\Psi\rangle = GW_\Delta |\Psi^\circ\rangle$$

Process can be mapped to equilibrium $H^{eq}$ with $\rho = \rho_1 = \rho_N$ such that

$$H = G_{\rho_N} W_\Delta G_\rho^{-1} H^{eq} G_\rho W_\Delta^{-1} G_{\rho_N}^{-1}$$

Observed macroscopically in [Tailleur, Kurchan, Lecomte ’07]
Conclusion & Outlook
Conclusion & Outlook

Conclusion

• Interesting connections between high energy physics, quantum groups, statistical mechanics and probability theory
• QISM is powerful tool to study integrable stochastic processes

Work in progress

• Boundary K-matrices for non-compact XXZ
• $W_\Delta$ for ASEP? Interesting works by [Nichols, Rittenberg, de Gier]
• Role of Baxter Q-operator and relation to [Lazarescu, Pasquier]
• Generalisation to $\mathfrak{su}_q(n, 1)$ and relation to stochastic R-matrix [Kuniba, Mangazeev, Maruyama, Okado]

Implications for AdS/CFT? [Olivucci, Vieira ’21]
Thank you!

References

arXiv:1904.01048 “Non-compact quantum spin chains as integrable stochastic particle processes”
with C. Giardinà and J. Kurchan


with C. Giardinà and J. Kurchan

arXiv:2107.01720 “Exact solution of an integrable non-equilibrium particle systems“
with C. Giardinà

with C. Franceschini and C. Giardinà