Non-compact spin chains and integrable particle systems

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Based on collaborations with C. Giardinà (Unimore) and J. Kurchan (Ens Paris)

Overview

Content

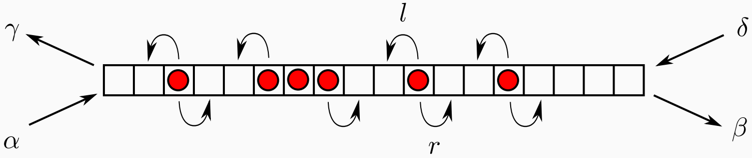
1. Review: Integrable simple exclusion models (ASEP/SSEP)
2. Quantum inverse scattering method
3. Non-compact spin chains as stochastic particle process
    - Non-compact XXX chain
    - Non-compact XXZ chain
    - Non-compact XXX chain with open boundaries
4. Construction of steady state
5. Outlook

Integrable simple exclusion models

Simple Exclusion Process

Most famous stochastic particle processes are: ASEP and SSEP

- Integrable
- Nearest-neighbor hopping model
- One particle per site (exclusion)
- Closed or open boundary conditions



Hopping rates: r and l and α, β, γ and δ

SSEP: $r = l = 1$

Some great reviews: [Derrida], [Schütz], [Blythe,Evans], [Crampé,Ragoucy,Vanicat], ...

Markov matrix of ASEP/SSEP

Exclusion process is generated by Markov matrix

$$M = \mathcal{B}_1 + \sum_{i=1}^{N-1} \omega_{i,i+1} + \mathcal{B}_N$$

Bulk:

$$\omega = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -l & r & 0 \\ 0 & l & -r & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Boundary:

$$\mathcal{B}_1 = \begin{pmatrix} -\alpha & \gamma \\ \alpha & -\gamma \end{pmatrix}, \qquad \mathcal{B}_N = \begin{pmatrix} -\delta & \beta \\ \delta & -\beta \end{pmatrix}$$

Stochastic process:

Sum over rows vanishes

Off-diagonal entires have opposite sign of diagonal entries

Relation to integrable spin chains

Stochastic process can be mapped to integrable spin chain

- ASEP $\leftrightarrow$ XXZ spin chain
- SSEP $\leftrightarrow$ XXX spin chain

Hamiltonian is related to Markov generator

$$M = SHS^{-1}$$

Particle process can be studied using integrability tools:
Coordinate Bethe ansatz, algebraic Bethe ansatz, ...

Multi-species generalisations from higher rank spin chains

Traffic jam

ASEP/SSEP produces traffic jams!

Lot of effort to avoid traffic jams...

Multi-particle generalisations

Put particles on top of each other

Naïve observation:

- Higher spin integrable model: Hamiltonian not stochastic
- Higher spin stochastic model: Hamiltonian not integrable

→ Non-compact integrable spin chains [RF, Giardinà, Kurchan '19]

SSEP within the quantum inverse scattering method

Quantum inverse scattering method Faddeev et al.

Starting point: Yang-Baxter equation

$$R_{12}(z_1 - z_2)R_{13}(z_1 - z_3)R_{23}(z_2 - z_3) = R_{23}(z_2 - z_3)R_{13}(z_1 - z_3)R_{12}(z_1 - z_2)$$

- Fundamental relation underlying integrable systems
- Each R-matrix R_{ij} acts on the tensor product of three spaces $V_1 \otimes V_2 \otimes V_3$ with

$$R_{12}(z) = R(z) \otimes \mathrm{I}, \ldots$$

- Fundamental R-matrix for SSEP / XXX Heisenberg spin chain

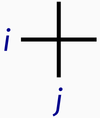
$$R(z) = z + P \quad \text{with} \quad P = \sum_{a,b=1}^{2} e_{ab} \otimes e_{ba}$$

where $(e_{ab})_{cd} = \delta_{ac}\delta_{bd}$, $z \in \mathbb{C}$ and P acts as a permutation

Graphical notation

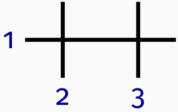
- R-matrix:

$$R_{ij}(z_i - z_j) =$$

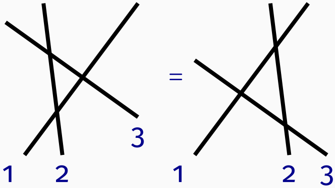


- Multiplication of R-matrices:

$$R_{12}(z_1 - z_2)R_{13}(z_1 - z_3) =$$



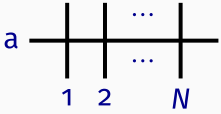
- Yang-Baxter equation:



Spin chain monodromy

Spin chain monodromy

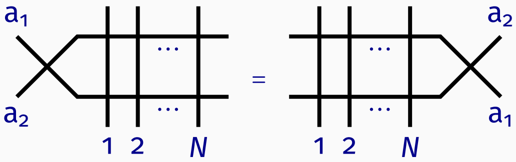
$$\mathcal{M}(z) = R_{\mathrm{a},1}(z)R_{\mathrm{a},2}(z)\cdots R_{\mathrm{a},N}(z) =$$



- Multiplication of 2×2 matrices in auxiliary space and tensor product in quantum space
- Satisfies RTT-relation

$$R(z_1 - z_2)(\mathcal{M}(z_1) \otimes \mathbb{I})(\mathbb{I} \otimes \mathcal{M}(z_2)) = (\mathbb{I} \otimes \mathcal{M}(z_2))(\mathcal{M}(z_1) \otimes \mathbb{I})R(z_1 - z_2)$$

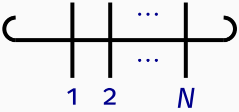
- Pictorially



Transfer matrix

Transfer matrix

$$T(z) = \mathrm{tr}_a \mathcal{M}(z) =$$



Markov generator / Hamiltonian

$$M^{cl.}_{SSEP} = \frac{\partial}{\partial z} \log T(z)|_{z=0} + const$$

Commuting family of operators (common eigenstates)

$$[T(z), T(z')] = 0, \qquad [T(z), M^{cl.}_{SSEP}] = 0$$

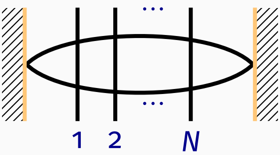
How to describe process with reservoir?

Open spin chains [Sklyanin]

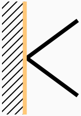
Transfer Matrix

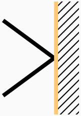
$$T(z) = tr\,\mathcal{K}_a(z)\,\underbrace{R_{a,1}(z)R_{a,2}(z)\cdots R_{a,N}(z)}_{\mathcal{M}(z)}\,\hat{\mathcal{K}}_a(z)\,\underbrace{R_{a,N}(z)\cdots R_{a,2}(z)R_{a,1}(z)}_{\hat{\mathcal{M}}(z)}$$

Graphically

$T(z) =$ 

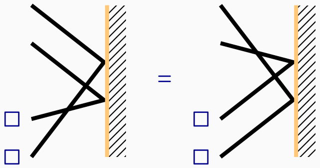
K-matrices

$\mathcal{K}(z) =$ 

$\hat{\mathcal{K}}(z) =$ 

QISM for boundary models

Boundary Yang-Baxter equation



$$R_{12}(z_1 - z_2)\hat{\mathcal{K}}_1(z_1)R_{12}(z_1 + z_2)\hat{\mathcal{K}}_2(z_2) = \hat{\mathcal{K}}_2(z_2)R_{12}(z_1 + z_2)\hat{\mathcal{K}}_1(z_1)R_{12}(z_1 - z_2)$$

And analogously for other boundary involving $\mathcal{K}(z)$

- Most general K-matrices

$$\mathcal{K}(z) = \begin{pmatrix} p_1 + p_2(z+1) & p_3(z+1) \\ p_4(z+1) & p_1 - p_2(z+1) \end{pmatrix}, \qquad \hat{\mathcal{K}}(z) = \begin{pmatrix} q_1 + q_2 z & z q_3 \\ z q_4 & q_1 - q_2 z \end{pmatrix}$$

Relation to SSEP

Adjust boundary parameters

$$q_1 = 1\,, \quad q_2 = \beta - \delta\,, \quad q_3 = 2\beta\,, \quad q_4 = 2\delta$$
$$p_1 = 1\,, \quad p_2 = \gamma - \alpha\,, \quad p_3 = 2\gamma\,, \quad p_4 = 2\alpha$$

Markov generator / Hamiltonian

$$M_{SSEP} = \frac{\partial}{\partial z} \log T(z)|_{z=0} + const.$$

Commuting transfer matrices

$$[T(z), T(z')] = 0, \qquad [T(z), M_{SSEP}] = 0$$

Expansion of $T(z)$ generates commuting charges

$$[M_{SSEP}, Q_k] = 0$$

Will become handy later...

Non-compact integrable spin chains

Non-compact spin chains

Quantum space of non-compact chains with hws

$$V = |m_1\rangle \otimes |m_2\rangle \otimes \ldots \otimes |m_N\rangle\,, \qquad m_i = 0, 1, 2, \ldots$$

For spin s generators of $\mathfrak{sl}(2)$ act locally as

$$S_+|m\rangle = (m+2s)|m+1\rangle\,, \quad S_-|m\rangle = m|m-1\rangle \quad S_0|m\rangle = (m+s)|m\rangle$$

Nearest-neighbor Hamiltonian density [Faddeev et al.]

$$\mathcal{H} = 2\left(\psi(\mathbb{S}) - \psi(2s)\right)$$

where $\psi(x)$ is Digamma function and $\mathbb{S}$ is related to the two-site Casimir operator via $C_{[2]} = \mathbb{S}(\mathbb{S}-1)$

- First studied in high energy QCD [Lipatov;Faddeev,Korchemsky]
- Important subsector of the $\mathcal{N}=4$ SYM spin chain! ($s=\frac{1}{2}$)
- Integrable models [Derkachov]

The operator $\mathbb{S}$

Consider tensor product decomposition

$$D_s \otimes D_s = \bigoplus_{j=0}^{\infty} D_{2s+j}$$

Operator $\mathbb{S}$ acts diagonally on the irreps on the rhs

$$\mathbb{S}|D_{2s+j}\rangle = (2s+j)|D_{2s+j}\rangle$$

Eigenvalues of Hamiltonian density are harmonic numbers h_s

$$\mathcal{H}|D_{2s+j}\rangle = 2\sum_{k=1}^{j} \frac{1}{2s+k-1}|D_{2s+j}\rangle$$

- Can't tell if process is stochastic from eigenvalues
- A priory not known how $\mathcal{H}$ acts on the lhs...
  $\rightsquigarrow$ Clebsch Gordan decomposition

Harmonic action as stochastic process

Nearest neighbor hopping model for $s = \frac{1}{2}$

[Beisert; Braun,Derkachov,Manashov; Lipatov; Faddeev,Korchemsky]

$$\mathcal{H}|m\rangle \otimes |m'\rangle = (h(m) + h(m'))\,|m\rangle \otimes |m'\rangle - \sum_{k=1}^{m} \frac{1}{k}|m-k\rangle \otimes |m'+k\rangle - \sum_{k=1}^{m'} \frac{1}{k}|m+k\rangle \otimes |m'-k\rangle$$

with the harmonic numbers $h(m) = \sum_{k=1}^{m} \frac{1}{k}$.

Hamiltonian density $\mathcal{H}$ is generator of Markov process!

[Giardinà, Kurchan, RF '19]

E.g. $m + m' = 2$:

$$\mathcal{H}_2 = \begin{pmatrix} \frac{3}{2} & -1 & -\frac{1}{2} \\ -1 & 2 & -1 \\ -\frac{1}{2} & -1 & \frac{3}{2} \end{pmatrix}$$

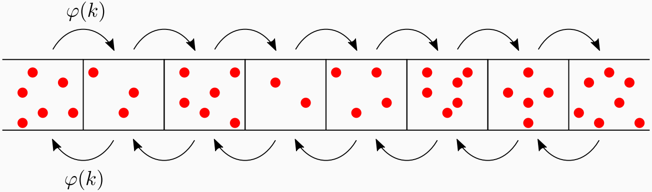
Harmonic action as stochastic process

Hamiltonian defined on N sites as

$$H = \sum_{i=1}^{N-1} \mathcal{H}_{i,i+1}$$

Symmetric stochastic process without exclusion!

$\rightarrow$ k particles jump with the rate $\varphi(k) = \frac{1}{k}$



Harmonic action as stochastic process

Hopping rates generalise to arbitrary spin $s > 0$ [Martins,Melo '09]

$$\varphi_s(m,k) = \frac{1}{k}\frac{\Gamma(m+1)\Gamma(m-k+2s)}{\Gamma(m-k+1)\Gamma(m+2s)}$$

Again we find a symmetric particle process!
$\hookrightarrow$ Rates depend on number of particles at departing site

Up to now only reinterpreting results of others...

Add a particle current (non-equilibrium models):

- q-analog/XXZ-analog $\longrightarrow$ asymmetric (drift) process
- Rational case with boundary reservoirs

Non-compact XXZ spin chain as stochastic particle process

Non-compact $\mathcal{U}_q(\mathfrak{sl}_2)$ invariant XXZ chain

Commutation relations $\mathcal{U}_q(\mathfrak{sl}_2)$

$$[S_+, S_-] = -[2S_0]\,, \qquad [S_0, S_\pm] = \pm S_\pm$$

with q-number $[x] = \frac{q^x - q^{-x}}{q - q^{-1}}$.

Generators of $\mathcal{U}_q(\mathfrak{sl}_2)$ act locally as

$$S_+|m\rangle = [m+2s]|m+1\rangle\,, \quad S_-|m\rangle = [m]|m-1\rangle \quad S_0|m\rangle = (m+s)|m\rangle$$

Hamiltonian density of XXZ chain with $|q| < 1$ [Bytsko]

$$\mathcal{H} = \frac{\psi_q(\mathbb{S}) - \psi_q(2s)}{-q^{4s}\log(q)}$$

with q-Digamma function ψ_q and $\mathbb{S}$ is related to the co-product of the Casimir operator via $\Delta(C) = [\mathbb{S}][\mathbb{S}-1]$.

Co-product and special functions

Some definitions and special functions...

Co-product

$$\Delta(S_0) = S_0 \otimes 1 + 1 \otimes S_0 \,, \qquad \Delta(S_\pm) = S_\pm \otimes q^{-S_0} + q^{S_0} \otimes S_\pm$$

q-Gamma function

$$\Gamma_q(x) = q^{\frac{1}{2}x(1-x)}(q^{-1} - q)^{1-x}\frac{(q^2;q^2)_\infty}{(q^{2x};q^2)_\infty}$$

with $(a;q)_n = \prod_{k=0}^{n-1}(1 - aq^k)$

q-Digamma function

$\psi_q(x) = \partial_x \log \Gamma_q(x)$

Harmonic action for XXZ chain

Use Clebsch-Gordan decomposition to obtain nearest neighbor hopping action on two sites [RF '19]

$$\mathcal{H}|m\rangle \otimes |m'\rangle = (\alpha_+(m) + \alpha_-(m'))\,|m\rangle \otimes |m'\rangle - \sum_{k=1}^{m} \rho(m,k)|m-k\rangle \otimes |m'+k\rangle$$
$$- \sum_{k=1}^{m'} \rho(m',k)|m+k\rangle \otimes |m'-k\rangle$$

with diagonal entries

$$\alpha_\pm(m) = \frac{\psi_q\,(m+2s) - \psi_q\,(2s) \pm m\log(q)}{-2q^{4s}\log(q)}$$

and off-diagonal entries

$$\rho(m,k) = \frac{q^{2ks}}{q^{4s}\left(1-q^{2k}\right)} \frac{(q^2;q^2)_m (q^{4s};q^2)_{m-k}}{(q^2;q^2)_{m-k}(q^{4s};q^2)_m}$$

Relation to stochastic q-Hahn process

As in ASEP, Hamiltonian density $\mathcal{H}$ is not a Markov matrix!

Similarity transformation yields Markov matrix

$$\mathcal{M} = \begin{pmatrix} \alpha_+(n)+\alpha_-(0) & -\beta_-(1,1) & -\beta_-(2,2) & \cdots & -\beta_-(n,n) \\ -\beta_+(n,1) & \alpha_+(n-1)+\alpha_-(1) & -\beta_-(2,1) & \cdots & -\beta_-(n,n-1) \\ -\beta_+(n,2) & -\beta_+(n-1,1) & \alpha_+(n-2)+\alpha_-(2) & \cdots & -\beta_-(n,n-2) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -\beta_+(n,n) & -\beta_+(n-1,n-1) & -\beta_+(n-2,n-2) & \cdots & \alpha_+(0)+\alpha_-(n) \end{pmatrix}.$$

with

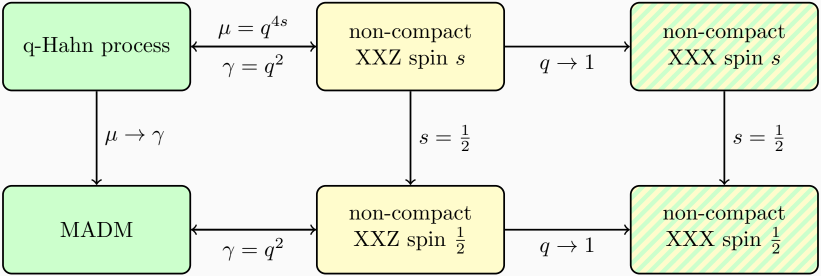
$$\beta_\pm(m,k) = \frac{\mu^{\frac{1}{2}k(1\pm 1)}}{\mu\left(1-\gamma^k\right)} \frac{(\gamma;\gamma)_m(\mu;\gamma)_{m-k}}{(\gamma;\gamma)_{m-k}(\mu;\gamma)_m},$$

where $\gamma = q^2$ and $\mu = q^{4s}$

Coincides with rates of q-Hahn process introduced by [Povolotsky;Barraquand-Corwin;Sasamoto-Wadati] without reference to XXZ chain!

Summary

Non-compact spin chains and stochastic particle processes



Harmonic processes

Non-compact XXX chain with boundaries

Stochastic process with boundary reservoirs

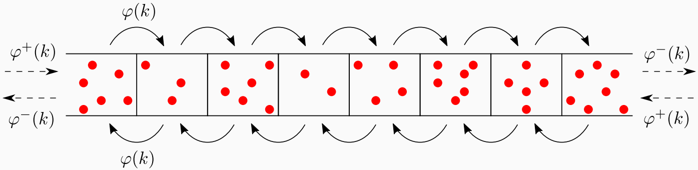
Add stochastic boundary conditions to rational process

$$H = \mathcal{B}_1 + \sum_{i=1}^{N-1} \mathcal{H}_{i,i+1} + \mathcal{B}_N \,.$$

Guess boundary terms for $0 < \beta_i < 1$ and $s = \frac{1}{2}$ [RF, Giardinà, Kurchan '19]

$$\mathcal{B}_i |m_i\rangle = \left(h(m_i) + \sum_{k=1}^{\infty} \frac{\beta_i^k}{k} \right) |m_i\rangle - \sum_{k=1}^{m_i} \frac{1}{k} |m_i - k\rangle - \sum_{k=1}^{\infty} \frac{\beta_i^k}{k} |m_i + k\rangle$$

Introduces reservoirs at left and right end of the chain:



Is this process integrable?

Quantum Inverse Scattering Method [Sklyanin]

Construct the fundamental transfer matrix

$$T(x) = \operatorname{tr} \mathcal{K}(x)\mathcal{M}(x)\hat{\mathcal{K}}(x)\hat{\mathcal{M}}(x)$$

with the monodromies

$$\mathcal{M}(x) = \mathcal{R}_1(x)\cdots\mathcal{R}_N(x)\,, \qquad \hat{\mathcal{M}}(x) = \mathcal{R}_N(x)\cdots\mathcal{R}_1(x)$$

where

$$\mathcal{R}(x) = (-1)^{\mathbb{S}}\frac{\Gamma(2s-x)\Gamma(\mathbb{S}+x)}{\Gamma(2s+x)\Gamma(\mathbb{S}-x)}$$

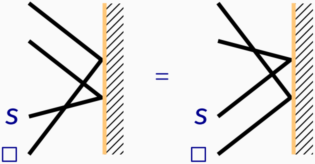
Hamiltonian is logarithmic derivative of $T(x)$ at permutation point

$$H = \partial_x \log T(x)|_{x=0}$$

But: Closed expression of K-matrix unknown!

Quantum Inverse Scattering Method

Derive the universal K-matrix from BYBE [RF, Giardinà, Kurchan '19]



$$\mathcal{L}(x-y)\hat{\mathcal{K}}(x)\mathcal{L}(x+y)\hat{K}(y) = \hat{K}(y)\mathcal{L}(x+y)\hat{\mathcal{K}}(x)\mathcal{L}(x-y)$$

Lax matrix and K-matrix in fundamental representation

$$\mathcal{L}(x) = \begin{pmatrix} x+\frac{1}{2}+S_0 & -S_- \\ S_+ & x+\frac{1}{2}-S_0 \end{pmatrix}, \quad \hat{K}(x) = \begin{pmatrix} q_1+xq_2 & xq_3 \\ xq_4 & q_1-xq_2 \end{pmatrix}$$

Solve for $\hat{\mathcal{K}}(x)$...

Universal solution to BYBE

1. Introduce useful parametrisation of boundary variables

$$q_1 = \delta\,, \quad q_2 = \frac{1}{2}(1 + 2\alpha\beta)\gamma\,, \quad q_3 = -(1 + \alpha\beta)\beta\gamma\,, \qquad q_4 = \alpha\gamma$$

2. Make the ansatz

$$\hat{\mathcal{K}}(x) = e^{\beta S_+}\, e^{-\alpha S_-}\, \hat{\mathcal{K}}_0(S_0; x)\, e^{\alpha S_-}\, e^{-\beta S_+}$$

Yields difference equation for $\hat{\mathcal{K}}_0(S_0; x)$ which can be solved

$$\hat{\mathcal{K}}_0(S_0; x) = \frac{\Gamma(\frac{1}{2} + s + 2\frac{\delta}{\gamma} - x)}{\Gamma(\frac{1}{2} + s + 2\frac{\delta}{\gamma} + x)} \frac{\Gamma\left(\frac{1}{2} + S_0 + 2\frac{\delta}{\gamma} + x\right)}{\Gamma\left(\frac{1}{2} + S_0 + 2\frac{\delta}{\gamma} - x\right)}$$

Other boundary obtained via

$$\mathcal{K}(x) = \frac{1}{\hat{\mathcal{K}}(x+1)}$$

Relation to stochastic boundary

To derive stochastic boundary conditions for Hamiltonian fix

$$2\frac{\delta}{\gamma} = s - \frac{1}{2}\,, \qquad \alpha = \frac{1}{1-\beta}$$

and compute the logarithmic derivative of the transfer matrix

$$\frac{\partial}{\partial x}\ln T(x)\Big|_{x=0} = \frac{\mathrm{tr}_a\,\mathcal{K}_a'(0)}{\mathrm{tr}_a\,\mathcal{K}_a(0)} + 2\frac{\mathrm{tr}_a\,\mathcal{K}_a(0)\mathcal{H}_{a,1}}{\mathrm{tr}_a\,\mathcal{K}_a(0)} + \frac{\hat{\mathcal{K}}_N'(0)}{\hat{\mathcal{K}}_N(0)} + 2\sum_{k=1}^{N-1}\frac{\partial}{\partial x}\ln\mathcal{R}_{k,k+1}(x)\Big|_{x=0}\,,$$

Full Hamiltonian

$$H = \mathcal{B}_1 + \sum_{i=1}^{N-1}\mathcal{H}_{i,i+1} + \mathcal{B}_N$$

with algebraic expression for boundaries

$$\mathcal{B}_i = e^{-S_-^{[i]}}e^{\rho_i S_+^{[i]}}\Big(\psi(S_0^{[i]}+s)-\psi(2s)\Big)e^{-\rho_i S_+^{[i]}}e^{S_-^{[i]}} \qquad \text{for } i \in \{1, N\}\,.$$

where $\rho_i = \frac{\beta_i}{1-\beta_i}$.

Relation to stochastic boundary

A longer computation shows that we obtain the spin s version of desired boundary terms!

$$\mathcal{B}_i|m_i\rangle = \left(h^{(s)}(m_i) + \sum_{k=1}^{\infty}\frac{\beta_i^k}{k}\right)|m_i\rangle - \sum_{k=1}^{m_i}\frac{1}{k}\frac{\Gamma(m_i+1)\Gamma(m_i-k+2s)}{\Gamma(m_i-k+1)\Gamma(m_i+2s)}|m_i-k\rangle - \sum_{k=1}^{\infty}\frac{\beta_i^k}{k}|m_i+k\rangle,$$

- Process is integrable!
- Derived stochastic boundaries for arbitrary spin s

Steady state of harmonic process with boundaries

Derrida solution

SSEP solved in 1993 using matrix product ansatz [Derrida et al.]

Representation of steady state $H|\mu\rangle = 0$

$$|\mu\rangle = \frac{1}{\langle W|(E+D)^N|V\rangle} \begin{pmatrix} \langle W|E\cdots EEE|V\rangle \\ \langle W|E\cdots EED|V\rangle \\ \langle W|E\cdots EDE|V\rangle \\ \vdots \\ \langle W|D\cdots DDD|V\rangle \end{pmatrix}$$

DEHP algebra

- Bulk relation: $DE - ED = D + E$
- Boundary relations:

$$\langle W|(\alpha E - \gamma D) = \langle W| , \qquad (\beta D - \delta E)|V\rangle = |V\rangle$$

MPA difficult as there are not only two operators E and D

Steady state

Follow alternative route applied for SSEP in [RF '19; RF, Giardina, Kurchan '20], inspired by [Alcaraz,Droz,Henkel,Rittenberg], [Melo,Ribeiro,Martins], [Essler,de Gier], [Crampé,Ragoucy,Vanicat]

1. SSEP generator can be brought to a block triangular form

$$H_\Delta = G^{-1}HG = \begin{pmatrix} -\alpha-\gamma & \Delta \\ 0 & 0 \end{pmatrix}_1 + \sum_{i=1}^{N-1} \omega_{i,i+1} + \begin{pmatrix} -\beta-\delta & 0 \\ 0 & 0 \end{pmatrix}_N$$

   with $\Delta = \frac{(\alpha+\gamma)(\alpha\beta-\gamma\delta)}{\beta+\delta}$ and G only depends on S_a^{tot}.
2. H_Δ is isospectral to diagonal Hamiltonian $H^\circ = H_{\Delta=0}$ with $\Delta = 0$
3. Determine non-local transformation W_Δ s.t.

$$H^\circ = W_\Delta^{-1} H_\Delta W_\Delta$$

4. Obtain closed-form of steady state from pseudovacuum

$$|\Psi\rangle = GW_\Delta|\Omega\rangle$$

Same logic works for non-compact boundary model [Frassek,Giardina '21]

Transformations for the non-compact model

Local transformation that block triangularises H:

$$G = \prod_{i=1}^{N} e^{-S_-^{[i]}} e^{\rho_N S_+^{[i]}}$$

Non-local transformation that block diagonalises H_Δ:

$$W_\Delta = \sum_{k=0}^{\infty} \Delta^k \frac{Q_+^k}{k!} \frac{\Gamma(2(S_0^{\mathrm{tot}} + s))}{\Gamma(k + 2(S_0^{\mathrm{tot}} + s))}$$

with

$$Q_+ = s\, S_+^{\mathrm{tot}} + \sum_{i=1}^{N} S_+^{[i]} \left(S_0^{[i]} + 2 \sum_{j=i+1}^{N} S_0^{[j]} \right)$$

Q_+ is obtained from the transfer matrix at leading order in spectral parameter

Evaluation of the steady state

Factorial moments

Steady state

$$\langle m|\mu\rangle = \langle m|GW_\Delta|\Omega\rangle = \sum_{n\geq m} F(n)\Big[\prod_{i=1}^{N} \frac{(-1)^{n_i-m_i}}{n_i!}\binom{n_i}{m_i}\frac{\Gamma(2s+n_i)}{\Gamma(2s)}\Big]$$

with factorial moments

$$F(n) = \sum_{k=0}^{|n|} \rho_N^{|n|-k}(\rho_1-\rho_N)^k f_n(k)$$

where

$$f_n(k) = \sum_{|w|=k}\prod_{i=1}^{N}\binom{n_i}{w_i}\prod_{j=1}^{w_i}\frac{2s(N+1-i)-j+\sum_{k=i}^{N} w_k}{2s(N+1)-j+\sum_{k=i}^{N} w_k}\,.$$

Steady state for length N=1

$N = 1$ and $s = 1/2$

$$\langle m_1|\mu\rangle = \frac{(\beta_L - 1)(\beta_R - 1)}{\beta_L - \beta_R}\left(\gamma_{\beta_L}(m_1 + 1) - \gamma_{\beta_R}(m_1 + 1)\right).$$

with $\beta_L = \beta_1$ and $\beta_R = \beta_N$ and

$$\gamma_\beta(n) = \sum_{k=n}^{\infty} \frac{\beta^k}{k}$$

Steady state for length N=2

$N = 2$ and $s = 1/2$

$$\langle m_1, m_2 | \mu \rangle = 2 \frac{(\beta_L - 1)^2 (\beta_R - 1)^2}{(\beta_L - \beta_R)^2} \left(\phi_{\beta_L}(m_1, m_2) - \kappa(m_1, m_2) + \phi_{\beta_R}(m_2, m_1) \right)$$

where

$$\phi_\beta(m_1, m_2) = \frac{1}{2} \gamma_\beta^2 (1 + m_1) - \sum_{k=m_1+1}^{m_2} \frac{1}{k} \gamma_\beta(m_1 + k + 1) + \sum_{k=m_2+1}^{m_1} \frac{1}{k} \gamma_\beta(m_1 + k + 1)$$

and

$$\kappa(m_1, m_2) = \gamma_{\beta_L}(1 + m_1) \gamma_{\beta_R}(1 + m_2) \,.$$

Eigenstates and mapping to equilibrium

- Other eigenstates of H can be obtained from standard Bethe ansatz for H°:

$$|\Psi\rangle = GW_\Delta|\Psi^\circ\rangle$$

- Process can be mapped to equilibrium H^{eq} with $\rho = \rho_1 = \rho_N$ such that

$$H = G_{\rho_N} W_\Delta \underbrace{G_\rho^{-1} H^{eq} G_\rho}_{H^\circ} W_\Delta^{-1} G_{\rho_N}^{-1}$$

Observed macroscopically in [Tailleur, Kurchan, Lecomte '07]

Conclusion & Outlook

Conclusion & Outlook

Conclusion

- Interesting connections between high energy physics, quantum groups, statistical mechanics and probability theory
- QISM is powerful tool to study integrable stochastic processes

Work in progress

- Boundary K-matrices for non-compact XXZ
- W_Δ for ASEP? Interesting works by [Nichols,Rittenberg,de Gier]
- Role of Baxter Q-operator and relation to [Lazarescu, Pasquier]
- Generalisation to $\mathfrak{su}_q(n,1)$ and relation to stochastic R-matrix [Kuniba,Mangazeev,Maruyama,Okado]

Implications for AdS/CFT? [Olivucci,Vieira '21]

Thank you!

References

arXiv:1904.01048 *“Non-compact quantum spin chains as integrable stochastic particle processes”*
with C. Giardinà and J. Kurchan

arXiv:1904.02191 *“The non-compact XXZ spin chain as stochastic particle process”*

arXiv:1910.13163 *“Eigenstates of triangularisable open XXX spin chains and closed-form solutions for the steady state of the open SSEP”*

arXiv:2004.12796 *“Duality and hidden equilibrium in transport models“*
with C. Giardinà and J. Kurchan

arXiv:2107.01720 *“Exact solution of an integrable non-equilibrium particle systems“*
with C. Giardinà

arXiv:2205.xxxxx *“An integrable heat conduction model“*
with C. Franceschini and C. Giardinà