Conformal bootstrap 2d percolation and LCFT

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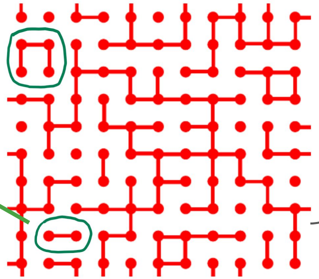
Based on:

- e-Print: 2109.05050, *SciPost Phys. 12 (2022) 100*, w/ Saleur
- e-Print: 2005.07258, *JHEP 12 (2020) 019*, w/ Jacobsen, Saleur
- e-Print: 2002.09071, *JHEP 05 (2020) 156*, w/ Grans-Samuelsson, Jacobsen, Saleur

GGI, Firenze, April 28th 2022

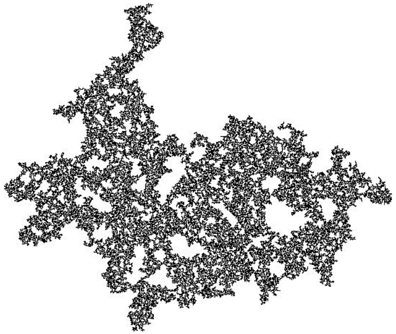
Critical percolation and cluster connectivities

clusters



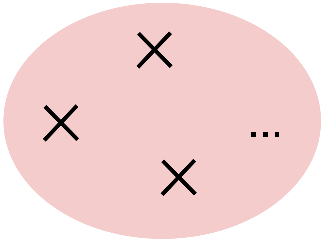
bond
- *close:* p
- *open:* $1-p$

p_c



cluster connectivities

geometrical phase transition

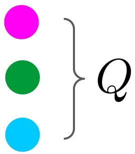


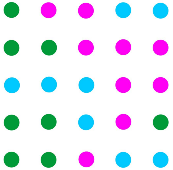
CFT description?

non-unitary conformal field theory

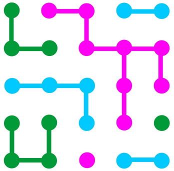
non-locality

Q-state Potts model





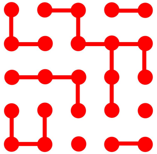
$$Z_{\text{spin}} \longrightarrow Z_{\text{cluster}}$$



$Q \in \mathbb{R}$ [Fortuin, Kasteleyn, 1972]

critical cluster model $0 < Q < 4$ ➡ Potts CFT $-2 < c < 1$

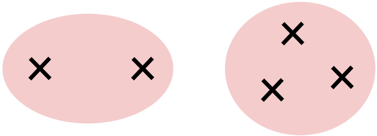
$Q \to 1 \ (c \to 0)$ percolation



Potts spin: order parameter

cluster connectivities ⟷ CFT correlator of spin operator $\Phi_{1/2,0}$

[Delfino, Viti, 2011]

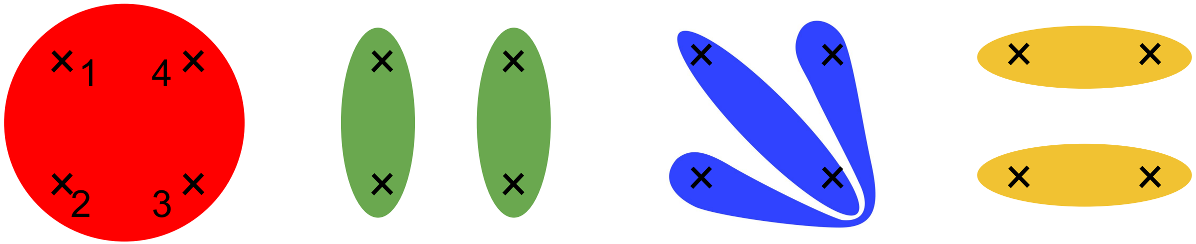


connectivities are understood

[Delfino, Viti, 2010]
[Picco, Santachiara, Viti, Delfino, 2013]
[Ikhlef, Jacobsen, Saleur, 2015]

Four-point connectivities

non-trivial, probe the spectrum of the CFT



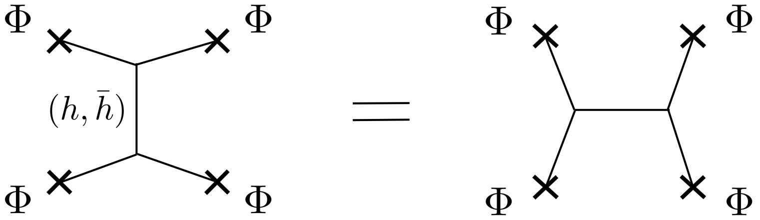
P_{aaaa} P_{aabb} P_{abab} P_{abba}

Potts CFT: $\left\langle \Phi_{1/2,0}\Phi_{1/2,0}\Phi_{1/2,0}\Phi_{1/2,0} \right\rangle$ **determine using CFT**

fractional Kac indices

cannot use BPZ

Conformal bootstrap approach



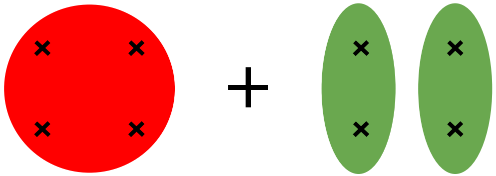
★ amplitudes

$$\langle \Phi_{1/2,0}\Phi_{1/2,0}\Phi_{1/2,0}\Phi_{1/2,0}\rangle = \sum_{(h,\bar{h})\in\text{s-channel}} A(h,\bar{h})\mathcal{F}^{(s)}_{h,\bar{h}} = \sum_{(h,\bar{h})\in\text{t-channel}} A(h,\bar{h})\mathcal{F}^{(t)}_{h,\bar{h}}$$

★ spectrum

conformal block ✓

first attempt to
bootstrap connectivity:

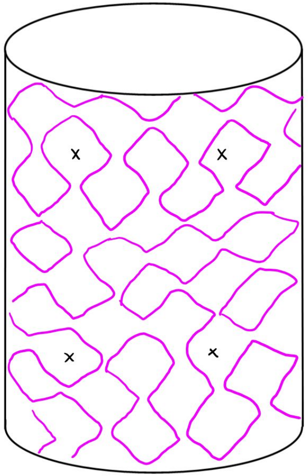


[Picco, Ribault, Santachiara, 2016]

Spectrum of connectivities

[Jacobsen, Saleur, 2018]

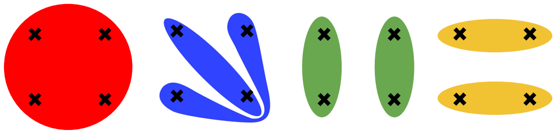
equivalent loop representation



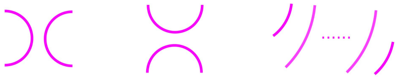
eigenvalues λ_i of transfer matrix

continuum limit

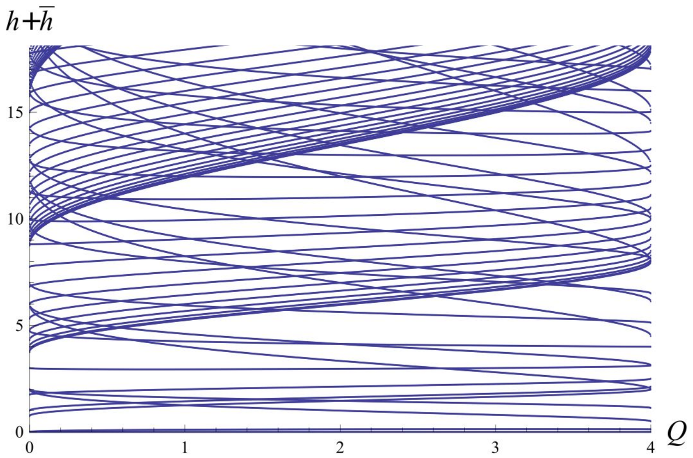
scaling dimensions $(h, \bar{h})$ in the spectrum



irreducible module $\mathcal{W}$ of affine Temperley-Lieb algebra

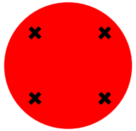
generators 

infinite tower of Virasoro conformal family



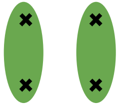
Universal amplitude ratios on the lattice

[YH, Grans-Samuelsson, Jacobsen, Saleur, 2020]



$A_{aaaa}(\lambda_i)$

$A_{abab}(\lambda_i)$



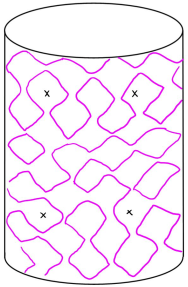
$A_{aabb}(\lambda_i)$

$$\frac{A_{abab}}{A_{aaaa}}(\lambda_i), \quad \frac{A_{aabb}}{A_{aaaa}}(\lambda_i), \quad \dots$$

depend only on Q **and** $\lambda_i \in \mathcal{W}$

do not depend on lattice size

ATL modules



$$\frac{A_{aabb}}{A_{aaaa}}(\mathcal{W}_{2,1}) = \frac{1}{1-Q} \qquad \frac{A_{abab}}{A_{aaaa}}(\mathcal{W}_{4,-1}) = \frac{(Q-1)(Q-4)}{4}$$

$$\frac{A_{abab}}{A_{aaaa}}(\mathcal{W}_{2,1}) = \frac{2-Q}{2} \qquad \frac{A_{aabb}}{A_{aaaa}}(\mathcal{W}_{4,1}) = -\frac{Q^5-7Q^4+15Q^3-10Q^2+4Q-2}{2(Q^2-3Q+1)}$$

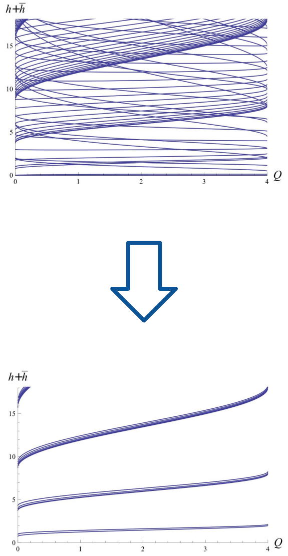
$$\frac{A_{aabb}}{A_{aaaa}}(\mathcal{W}_{4,-1}) = \frac{2-Q}{2} \qquad \frac{A_{abab}}{A_{aaaa}}(\mathcal{W}_{4,1}) = -\frac{(Q^2-4Q+2)(Q^2-3Q-2)}{4}$$

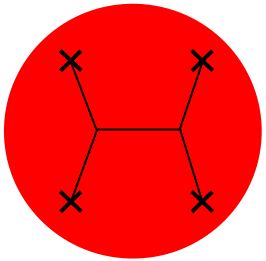
Interchiral conformal blocks

[YH, Jacobsen, Saleur, 2020]

$$\lambda_i \xrightarrow{\text{continuum}} (h,\bar{h})$$

organize the CFT states according to $\mathcal{W}$





P_{aaaa}

$$= \sum_{(h,\bar{h})\in s-\text{channel}} A_{aaaa}(h,\bar{h})\mathcal{F}_{h,\bar{h}}$$

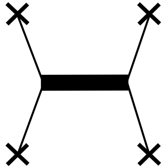
$$= \sum_{\mathcal{W}\in s-\text{channel}} A_{aaaa}(\mathcal{W}) \boxed{\sum_{(h,\bar{h})\in\mathcal{W}} \frac{A_{\cancel{aaaa}}(h,\bar{h})}{A_{\cancel{aaaa}}(\mathcal{W})}\mathcal{F}_{h,\bar{h}}}$$

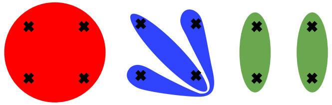
$$= \sum_{\mathcal{W}\in s-\text{channel}} A_{aaaa}(\mathcal{W})\, \mathbb{F}_{\mathcal{W}}$$

interchiral conformal block

interchiral algebra

[Gainutdinov, Read, Saleur, 2012]





...

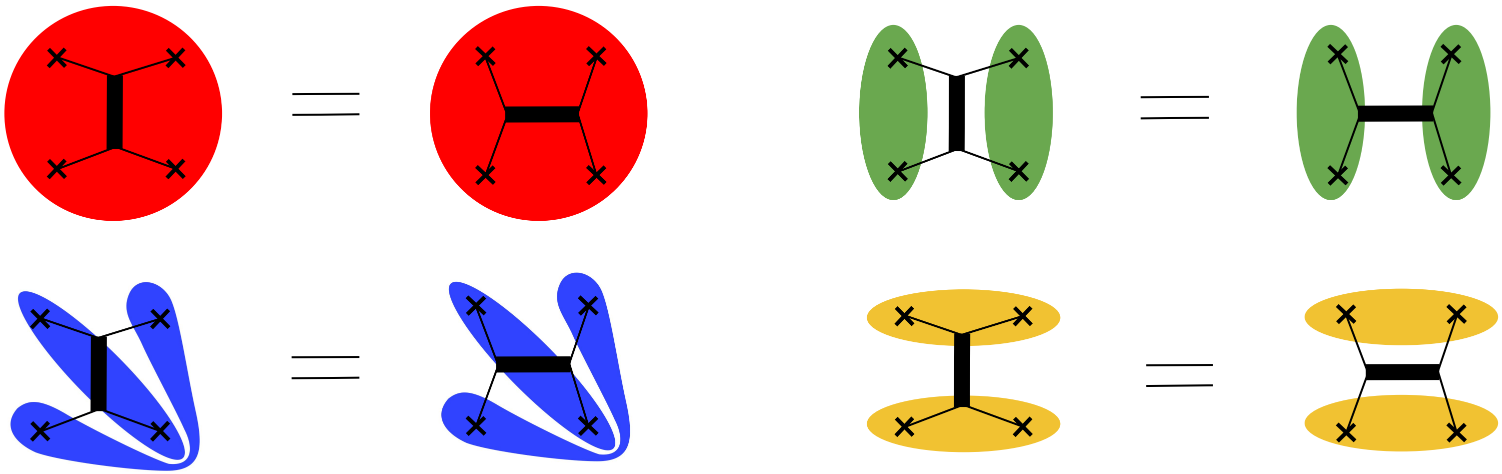
Interchiral conformal bootstrap

[YH, Jacobsen, Saleur, 2020]

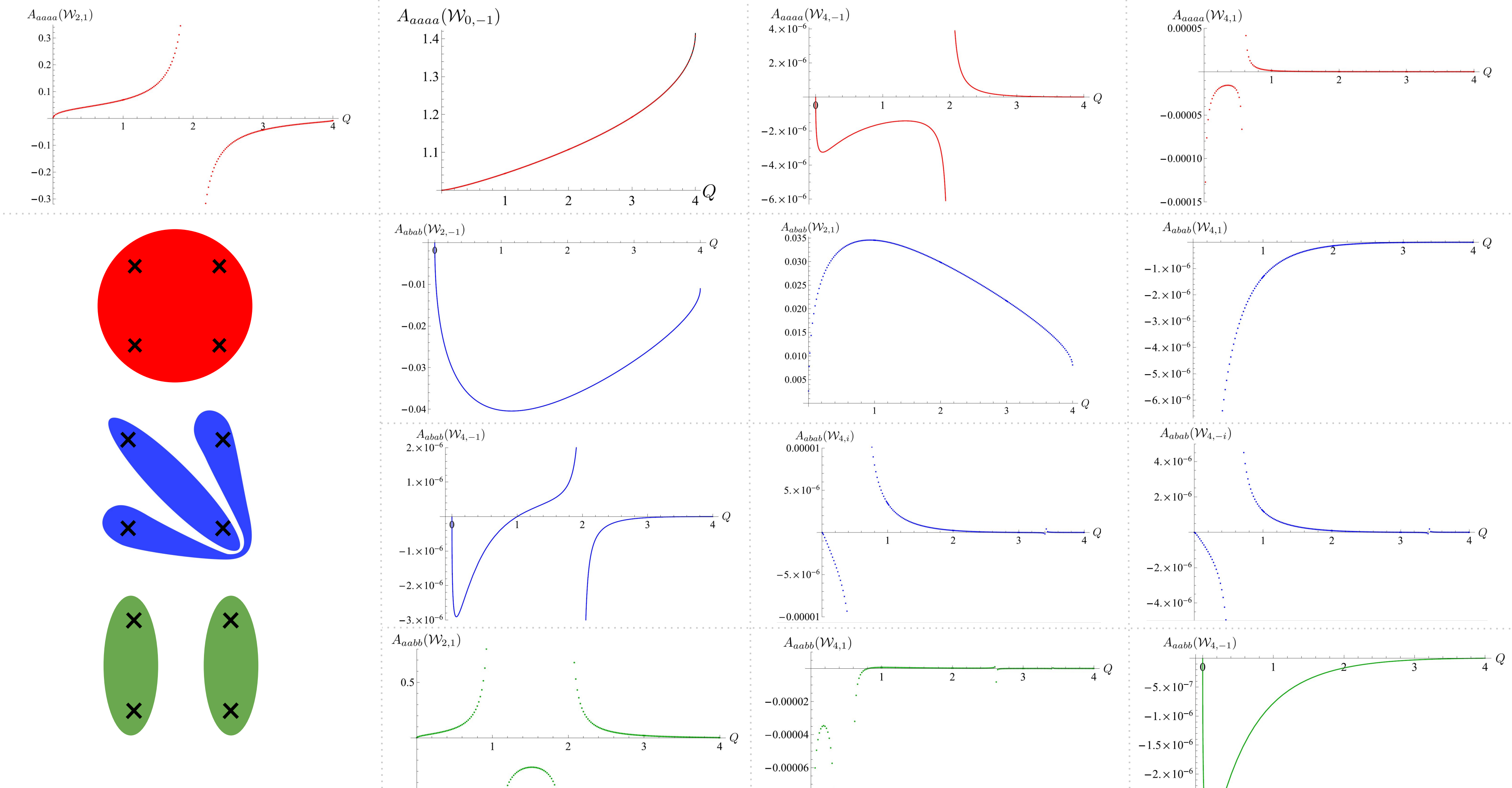
construct $\mathbb{F}_{\mathcal{W}}$: degeneracy $\longrightarrow$ recursion

technique in Liouville bootstrap

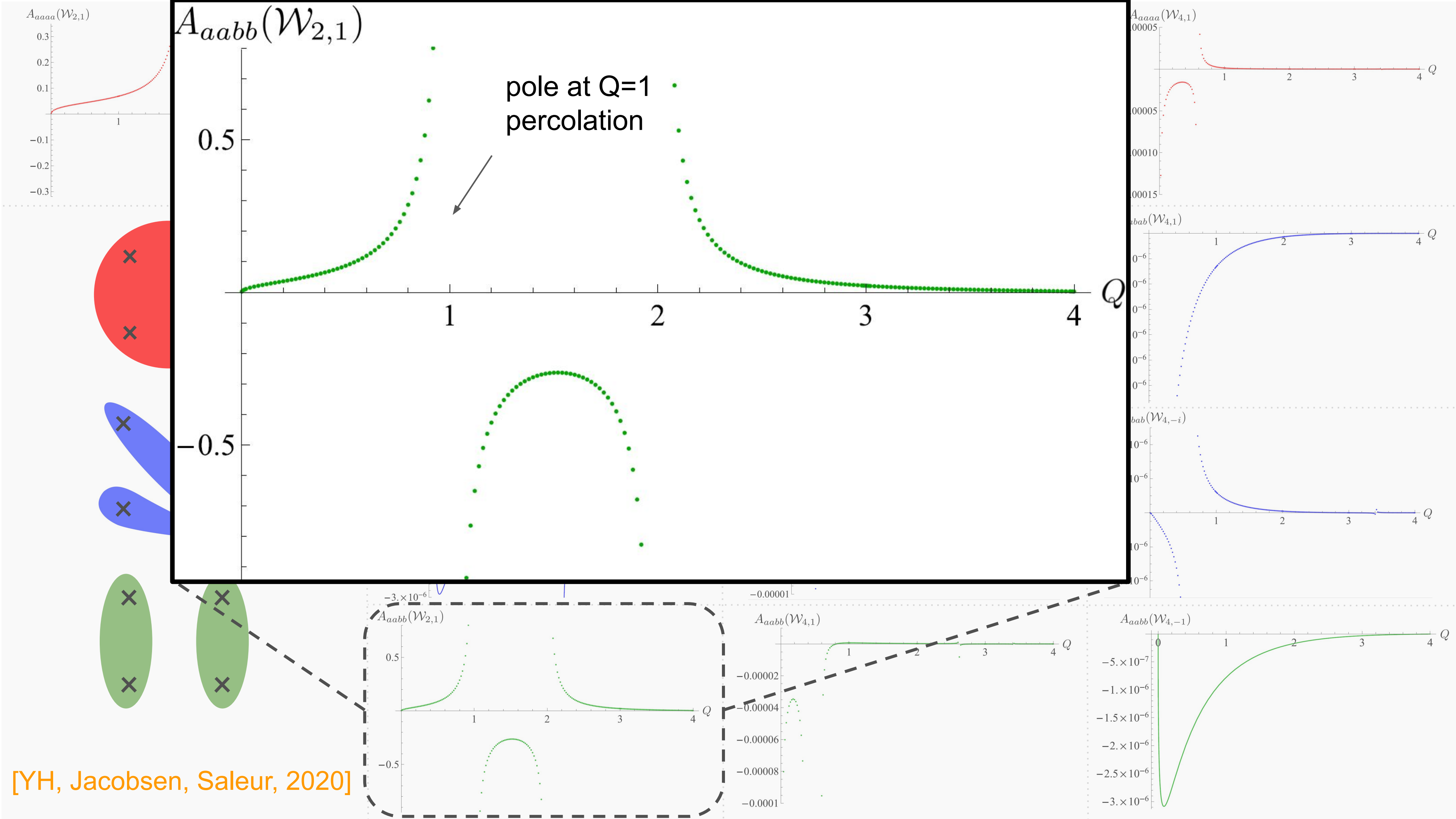
[Zamolodchikov^2, 1995] [Teschner, 1995]
[Estienne, Ikhlef, 2015] [Migliaccio, Ribault, 2017]



solve $A_{aaaa}(\mathcal{W}),\ \ A_{abab}(\mathcal{W}),\ \ A_{aabb}(\mathcal{W}),\ \ A_{abba}(\mathcal{W})$



[YH, Jacobsen, Saleur, 2020]

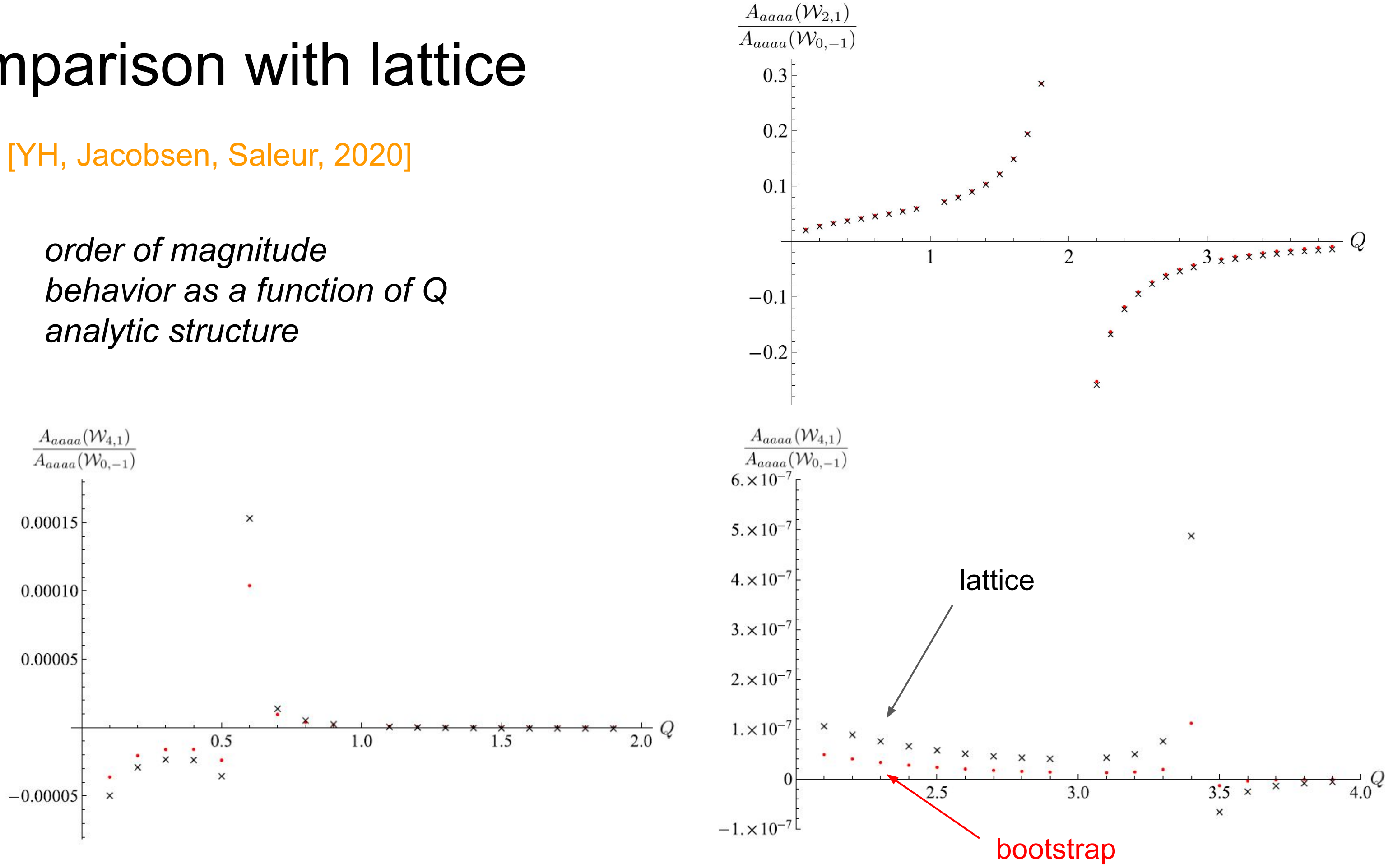


[YH, Jacobsen, Saleur, 2020]

Comparison with lattice

[YH, Jacobsen, Saleur, 2020]

- *order of magnitude*
- *behavior as a function of Q*
- *analytic structure*



“Renormalized” Liouville recursion

[YH, Jacobsen, Saleur, 2020]

in Liouville
(and its non-diagonal generalization)

degenerate
$\Phi_{1,2}, \Phi_{2,1}$

→ analytic bootstrap solution

[Zamolodchikov^2, 1995] [Teschner, 1995]
[Estienne, Ikhlef, 2015] [Migliaccio, Ribault, 2017]

Potts: only $\Phi_{2,1}$ degenerate (energy operator)

$$\frac{A_{aaaa}(\mathcal{W}_{4,-1})}{A_{aaaa}(\mathcal{W}_{0,-1})} = \frac{(Q-2)(Q^2-4Q+2)}{Q(Q-3)^2} \frac{A^L(\mathcal{W}_{4,-1})}{A^L(\mathcal{W}_{0,-1})}$$

$$\frac{A_{abab}(\mathcal{W}_{4,-1})}{A_{abab}(\mathcal{W}_{2,-1})} = \frac{(Q-1)(Q-4)(Q^2-4Q+2)}{2Q(Q-3)^2} \frac{A^L(\mathcal{W}_{4,-1})}{A^L(\mathcal{W}_{2,-1})}$$

$$\frac{A_{aaaa}(\mathcal{W}_{4,1})}{A_{aaaa}(\mathcal{W}_{2,1})} = \frac{(Q-2)^2}{(Q-1)^2(Q^2-4Q+2)} \frac{A^L(\mathcal{W}_{4,1})}{A^L(\mathcal{W}_{2,1})}$$

Liouville recursion
IF $\Phi_{1,2}$ is degenerate

dressed by factors -- rational functions of Q

analytic solution?

“$c \to 0$ catastrophe”

$Q \to 1$ **percolation** → $c = 0$ ← $O(n \to 0)$ loop model **polymers**

OPE of order parameter
$$\Phi_{1/2,0}(z,\bar z)\Phi_{1/2,0}(0,0) \sim 1 + \boxed{\frac{2h_{1/2,0}}{c}}(z^2 T + \bar z^2 \bar T) + \dots$$

logarithmic partner t

$$\begin{aligned} \langle t(z,\bar z)t(0,0)\rangle &= \frac{\theta - 2b\ln|z|^2}{z^4} \\ \langle T(z)t(0,0)\rangle &= \frac{b}{z^4} \\ \langle T(z)T(0)\rangle &= 0 \end{aligned}$$

b-number

boundary (chiral): $b_{\text{percolation}} = -\frac{5}{8} \quad b_{\text{polymers}} = \frac{5}{6}$

[Gurarie, Ludwig 1999]

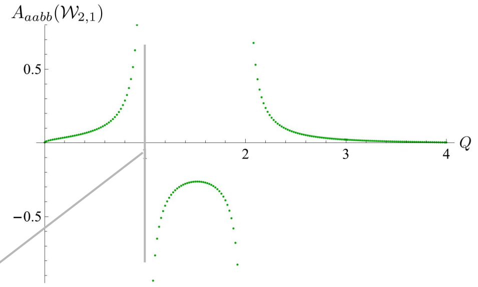
bulk (non-chiral): $b_{\text{percolation}} = b_{\text{polymers}} = -5$

lattice measurement

[Vasseur, Gainutdinov, Jacobsen, Saleur, 2011]

Logarithmic OPE

[YH, Saleur, 2021]



a generic c OPE in Potts or O(n):

$$\Phi_{1/2,0}(z,\bar{z})\Phi_{1/2,0}(0,0) \sim 1 + \frac{2h_{1/2,0}}{c}\left(z^2T + \bar{z}^2\bar{T}\right) + \frac{4h_{1/2,0}}{c^2}(z\bar{z})^2T\bar{T} + (z\bar{z})^{h_{1,2}}\mathcal{A}\left(\bar{z}^2\bar{X} + z^2X + \dots\right) + \dots$$

$c \rightarrow 0$ limit finite

log OPE at c=0:

$$\Phi_{1/2,0}(z,\bar{z})\Phi_{1/2,0}(0,0) \sim 1 + z^2\frac{h_{1/2,0}}{b}\left(t + T\ln(z\bar{z})\right) + \bar{z}^2\frac{h_{1/2,0}}{b}\left(\bar{t} + \bar{T}\ln(z\bar{z})\right) + (z\bar{z})^2\frac{h_{1/2,0}^2}{a_0}\left(\frac{1}{2}\ln^2(z\bar{z})T\bar{T} + \dots\right) + \dots$$

$$\boxed{b = -5}$$

$$\boxed{a_0 = -\frac{25}{48}}$$

identical for percolation and polymers

Summary

- conformal bootstrap approach + lattice algebra

  non-unitary CFT of percolation

  numerically determined four-point cluster connectivities

- “renormalized” Liouville recursion → analytic solution?
- c=0 logarithmic CFT:

  interesting connections between percolation and polymers CFTs

  more to study on geometrical quantities

Thank you!