The Open $U_q(sl(2))$-Invariant Staggered Six-Vertex Model

Sascha Gehrmann

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  - ⇒ Specific choice of inhomogeneities i.e. staggering needed for the six-vertex model.

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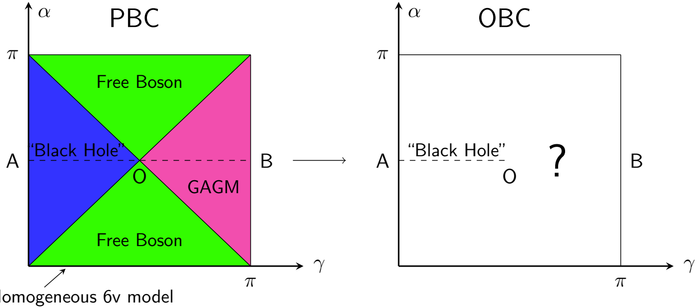
Goal: Explore Parameter Space



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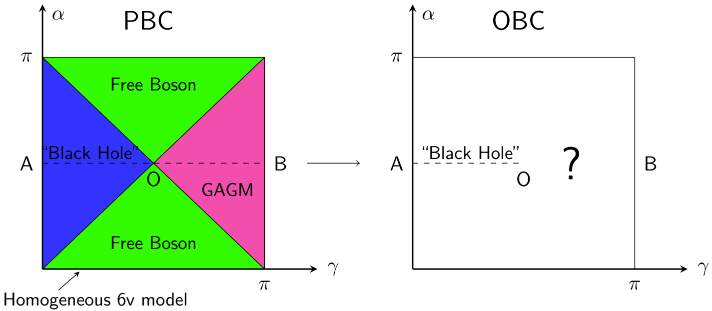

- Thanks to G. Kotousov for PBC graphic!
- Study the *open* case. Little is known expect [Robertson, Jacobsen, Saleur '21].

Overview

1. Recall Open Boundary Conditions and Staggered Models
2. Root Density Approach
3. Finite-Size-Spectrum-Analysis
4. Summary and Open Problems

The Ingredients

- We consider the standard XXZ-R-matrix:

$$R(u) = \begin{pmatrix} \sinh(u+i\gamma) & 0 & 0 & 0 \\ 0 & \sinh(u) & \sinh(i\gamma) & 0 \\ 0 & \sinh(i\gamma) & \sinh(u) & 0 \\ 0 & 0 & 0 & \sinh(u+i\gamma) \end{pmatrix}$$

- And the following matrices :

$$K_-(u) = \begin{pmatrix} e^{u} & 0 \\ 0 & e^{-u} \end{pmatrix}, \qquad K_+(u) = \begin{pmatrix} e^{-u-i\gamma} & 0 \\ 0 & e^{u+i\gamma} \end{pmatrix},$$

Open boundary conditions for symmetric R-matrices

YBE: $R_{23}(v)R_{13}(u)R_{12}(u-v) = R_{12}(u-v)R_{13}(u)R_{23}(v)$

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Reflectionalgebras or BYBE:

- $R_{12}(u-v)\overset{1}{K}_-(u)R_{12}(u+v)\overset{2}{K}_-(v) = \overset{2}{K}_-(v)R_{12}(u+v)\overset{1}{K}_-(u)R_{12}(u-v)$
- $R_{12}(-u+v)\overset{1}{K}{}_+^t(u)R_{12}(-u-v-2i\gamma)\overset{2}{K}{}_+^t(v) = \overset{2}{K}{}_+^t(v)R_{12}(-u-v-2i\gamma)\overset{1}{K}{}_+^t(u)R_{12}(-u+v)$

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$\overset{BYBE+YBE}{\Longrightarrow} \left[\tau^{\mathrm{OBC}}(u), \tau^{\mathrm{OBC}}(v)\right] = 0$

Hamiltonian limit is given by $H = A\, \frac{\mathrm{d}}{\mathrm{d}u}\tau^{\mathrm{OBC}}(u)\Big|_{u=0} + \mathrm{B}$

The Staggered Model

Possibility to include arbitrary inhomogeneities:

- $\tau^{OBC}(u) = \mathrm{tr}_0 \left(\overset{0}{K}_+(u) R_{0L}(u+\delta_L) \cdots R_{01}(u+\delta_1) \overset{0}{K}_-(u) R_{01}(u-\delta_1) \cdots R_{0L}(u-\delta_L) \right)$

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We will focus on the staggering $\pm\frac{i\alpha}{2}$ in horizontal

- $\tau^{OBC}(u) = \mathrm{tr}_0 \left(\overset{0}{K}_{+}(u) R_{02L}(u+\frac{i\alpha}{2}) \cdots R_{01}(u-\frac{i\alpha}{2}) \overset{0}{K}_{-}(u) R_{01}(u+\frac{i\alpha}{2}) \cdots R_{02L}(u-\frac{i\alpha}{2}) \right)$

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as well as in the vertical direction via

- $\mathcal{T}(u) = \tau^{OBC}(u+\frac{i\alpha}{2})\tau^{OBC}(u-\frac{i\alpha}{2})$

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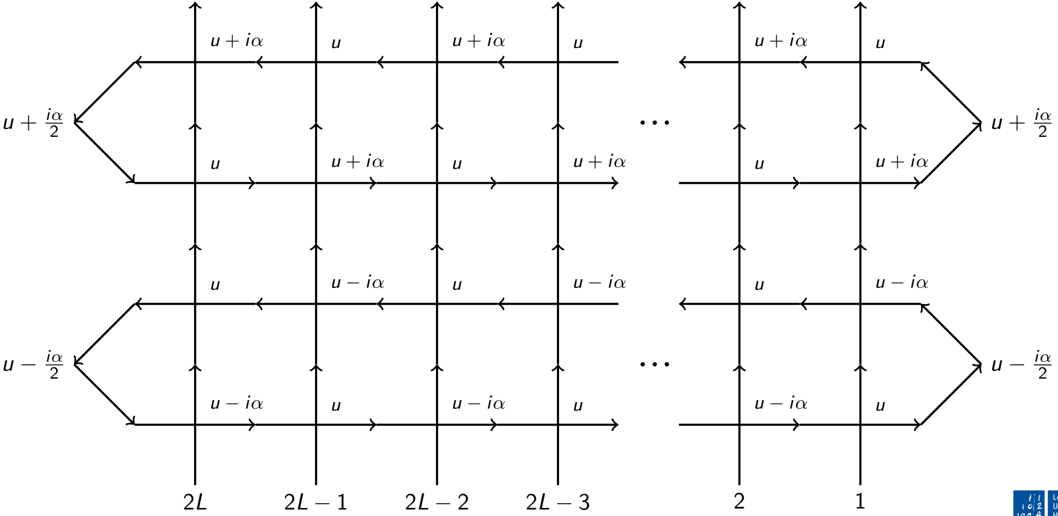
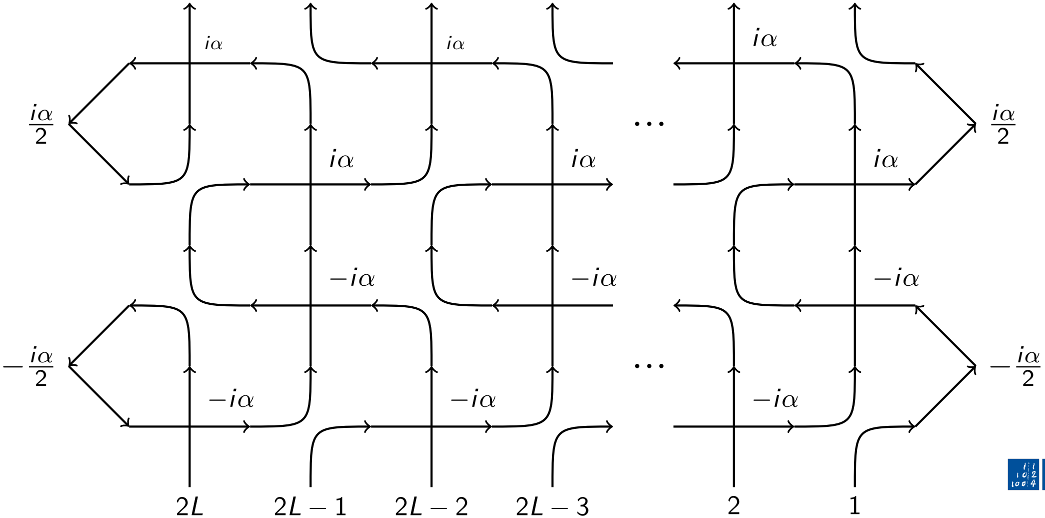
The Staggered Model



The Staggered Model

Evaluated at zero this gives:



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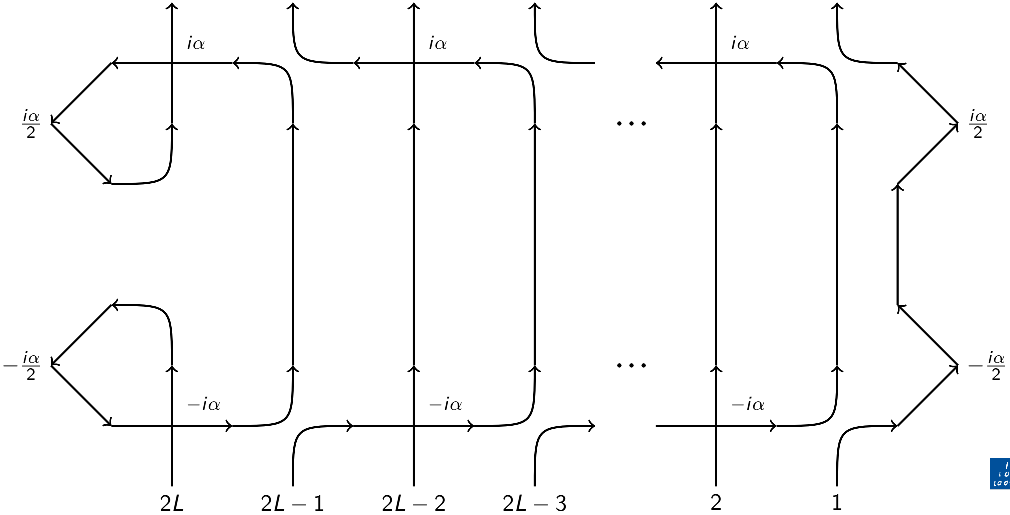

Hamiltonian $H = A\frac{\mathrm{d}}{\mathrm{d}u}\tau(u + \frac{i\alpha}{2})\tau(u - \frac{i\alpha}{2})|_{u=0} + B$

$$
\begin{aligned}
H \propto & 2\sin^2(\gamma)\sum_{j=1}^{2L-1}\cos(\gamma)\sigma_j^z\sigma_{j+1}^z + 2\cos(\alpha)(\sigma_j^+\sigma_{j+1}^- + \sigma_j^-\sigma_{j+1}^+) \\
& + \cos(\gamma)\sin^2(\alpha)\sum_{j=1}^{2L-2}\sigma_j^z\sigma_{j+2}^z + 2(\sigma_j^+\sigma_{j+2}^- + \sigma_j^-\sigma_{j+2}^+) \\
& - \sin(\alpha)\sin(2\gamma)\sum_{j=1}^{2L-2}(-1)^{j+1}\sigma_j^z\sigma_{j+1}^+\sigma_{j+2}^- + (-1)^j\sigma_j^z\sigma_{j+1}^-\sigma_{j+2}^+ + (-1)^{j+1}\sigma_j^+\sigma_{j+1}^-\sigma_{j+2}^z + (-1)^j\sigma_j^-\sigma_{j+1}^+\sigma_{j+2}^z \\
& - \sin(\gamma)\sin(2\alpha)\sum_{j=1}^{2L-2}(-1)^{j+1}\sigma_j^-\sigma_{j+1}^z\sigma_{j+2}^+ + (-1)^j\sigma_j^+\sigma_{j+1}^z\sigma_{j+2}^- \\
& - \cos(\gamma)\sin^2(\alpha)(\sigma_1^z\sigma_2^z + \sigma_{2L-1}^z\sigma_{2L}^z) \\
& - (i\sin(\alpha)\cos(2\gamma) - i\sin(\alpha)e^{2i\alpha})(\sigma_1^+\sigma_2^- + \sigma_{2L-1}^+\sigma_{2L}^-) \\
& - (i\sin(\alpha)\cos(2\gamma) - i\sin(\alpha)e^{-2i\alpha})(\sigma_1^-\sigma_2^+ + \sigma_{2L-1}^-\sigma_{2L}^+) \\
& - 2\sin(\alpha-\gamma)\sin(\alpha+\gamma)\sinh(i\gamma)(\sigma_1^z - \sigma_{2L}^z)
\end{aligned}
$$

Hamiltonian $H = A\frac{\mathrm{d}}{\mathrm{d}u}\tau(u+\frac{i\alpha}{2})\tau(u-\frac{i\alpha}{2})|_{u=0} + B$

$$
\begin{aligned}
H =& 2\sin^2(\gamma)\sum_{j=1}^{2L-1}\cos(\gamma)\sigma_j^z\sigma_{j+1}^z + 2\cos(\alpha)(\sigma_j^+\sigma_{j+1}^- + \sigma_j^-\sigma_{j+1}^+)\\
&- \cos(\gamma)\sin^2(\alpha)\sum_{j=1}^{2L-2}\sigma_j^z\sigma_{j+2}^z + 2(\sigma_j^+\sigma_{j+2}^- + \sigma_j^-\sigma_{j+2}^+)\\
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&- \cos(\gamma)\sin^2(\alpha)(\sigma_1^z\sigma_2^z + \sigma_{2L-1}^z\sigma_{2L}^z)\\
&- (i\sin(\alpha)\cos(2\gamma) - i\sin(\alpha)e^{2i\alpha})(\sigma_1^+\sigma_2^- + \sigma_{2L-1}^+\sigma_{2L}^-)\\
&+ (i\sin(\alpha)\cos(2\gamma) - i\sin(\alpha)e^{-2i\alpha})(\sigma_1^-\sigma_2^+ + \sigma_{2L-1}^-\sigma_{2L}^+)\\
&- 2\sin(\alpha-\gamma)\sin(\alpha+\gamma)\sinh(i\gamma)(\sigma_1^z - \sigma_{2L}^z) \qquad \xrightarrow{\alpha\to 0}\sim H_{XXZ}^{\mathrm{OBC}}\ \text{homogeneous}
\end{aligned}
$$

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Hamiltonian $H = A\frac{\mathrm{d}}{\mathrm{d}u}\tau(u+\frac{i\alpha}{2})\tau(u-\frac{i\alpha}{2})|_{u=0} + B$

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H = & 2\sin^2(\gamma)\sum_{j=1}^{2L-1}\cos(\gamma)\sigma_j^z\sigma_{j+1}^z + 2\cos(\alpha)(\sigma_j^+\sigma_{j+1}^- + \sigma_j^-\sigma_{j+1}^+) \\
& - \cos(\gamma)\sin^2(\alpha)\sum_{j=1}^{2L-2}\sigma_j^z\sigma_{j+2}^z + 2(\sigma_j^+\sigma_{j+2}^- + \sigma_j^-\sigma_{j+2}^+) \\
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\end{aligned}
$$

ferromagnetic!

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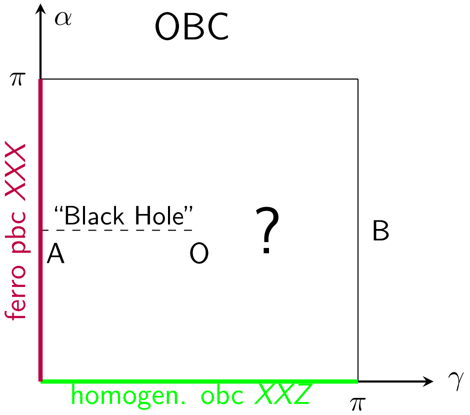
Parameter Space Diagram



Symmetries of the model

- The single transfer matrix commutes [Kulish, Sklyanin '91] with the operators

$$R(Q) = S^z, \qquad R(E^\pm) = X^\pm = \sum_{n=1}^{2L} e^{\pm(-1)^{n+1}\frac{i\alpha}{2}} e^{i\gamma(\frac{1}{2}\sigma_1^z + \ldots + \frac{1}{2}\sigma_{n-1}^z)} \sigma_n^\pm e^{-i\gamma(\frac{1}{2}\sigma_{n+1}^z + \ldots + \frac{1}{2}\sigma_{2L}^z)},$$

which are a representation R of $U_q(\mathfrak{sl}(2))$:

$$\left[Q, E^\pm\right] = \pm E^\pm \qquad \left[E^+, E^-\right] = [2Q]_q\,, \qquad \text{with } [x]_q = \frac{q^x - q^{-x}}{q - q^{-1}}, \quad q = e^{i\gamma}$$

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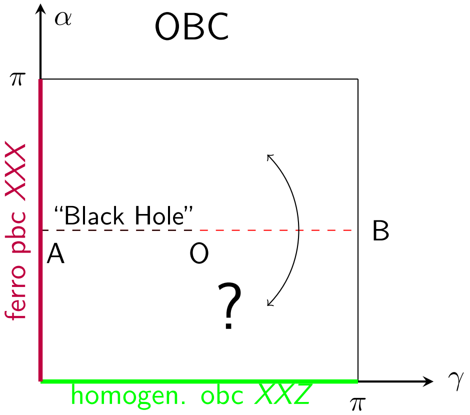
- The spectrum of $\mathcal{T}(u)$ is invariant under the transformation $\mathcal{D} : \alpha \to \pi - \alpha$:

$$\mathcal{D}\left(\mathcal{T}(u)\right) = \left(\prod_{i=1}^{L}\sigma_{2j}^z\right)\mathcal{C}(\alpha)\mathcal{T}(u)\mathcal{C}^{-1}(\alpha)\left(\prod_{i=1}^{L}\sigma_{2j}^z\right),$$

$$\mathcal{C}(\alpha) = \prod_{i=1}^{L} c_{2i-1,2i}(\alpha) \quad \text{with} \quad c_{i,j}(\alpha) = P_{i,j}R_{i,j}(i\alpha)\,.$$

Parameter Space Diagram



BAE and Energies

- Solve the system via algebraic Bethe-Ansatz [Kulish, Sklyanin '91] with BAE

$$\left(\frac{\sinh\left(v_m - \frac{i\alpha}{2} + \frac{i\gamma}{2}\right)\sinh\left(v_m + \frac{i\alpha}{2} + \frac{i\gamma}{2}\right)}{\sinh\left(v_m + \frac{i\alpha}{2} - \frac{i\gamma}{2}\right)\sinh\left(v_m - \frac{i\alpha}{2} - \frac{i\gamma}{2}\right)}\right)^{2L} = \prod_{k=1,\neq m}^{M} \frac{\sinh(v_m - v_k + i\gamma)}{\sinh(v_m - v_k - i\gamma)} \frac{\sinh(v_m + v_k + i\gamma)}{\sinh(v_m + v_k - i\gamma)},$$

and single transfermatrix eigenvalues:

$$\Lambda(u) \propto \frac{\sinh(2u + 2i\gamma)}{\sinh(2u + i\gamma)} \left(\sinh(u + \frac{i\alpha}{2} + i\gamma)\sinh(u - \frac{i\alpha}{2} + i\gamma)\right)^{2L} \prod_{m=1}^{M} \frac{\sinh(u - v_m - \frac{i\gamma}{2})\sinh(u + v_m - \frac{i\gamma}{2})}{\sinh(u - v_m + \frac{i\gamma}{2})\sinh(u + v_m + \frac{i\gamma}{2})}$$
$$+ \frac{\sinh(2u)}{\sinh(2u + i\gamma)} \left(\sinh(u + \frac{i\alpha}{2})\sinh(u - \frac{i\alpha}{2})\right)^{2L} \prod_{m=1}^{M} \frac{\sinh(u - v_m + \frac{3i\gamma}{2})\sinh(u + v_m + \frac{3i\gamma}{2})}{\sinh(u - v_m + \frac{i\gamma}{2})\sinh(u + v_m + \frac{i\gamma}{2})} \tag{1}$$

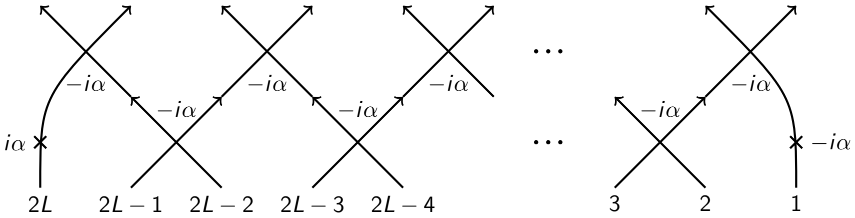
and energies

$$E = \sum_{j=1}^{M} \epsilon(v_j) = \sum_{j=1}^{M} \frac{4\sin(\gamma)(\cos(\alpha)\cosh(2v_j) - \cos(\gamma))}{(\cosh(2v_j) - \cos(\alpha - \gamma))(\cosh(2v_j) - \cos(\alpha + \gamma))}$$

- Note BAE invariant under $v_m \to -v_m$ and energies under $\alpha \to \pi - \alpha$

Quasi Momentum

$$K = \log\left(\frac{\tau\left(u-\frac{i\alpha}{2}\right)}{\tau\left(u+\frac{i\alpha}{2}\right)}\right)\Bigg|_{u=0} \qquad \mathcal{K} = \sum_{i=1}^{M} \underbrace{2\log\left[\frac{\cosh(2v_i)-\cos(\alpha+\gamma)}{\cosh(2v_i)-\cos(\alpha-\gamma)}\right]}_{k_0(v_i)} + C,$$



Densities and Integration Boundaries

- The Bethe-Roots describing the low energy physics in the parameter range $\gamma < \alpha < \pi - \gamma$ are:

$$v_j^x = x_j \qquad v_j^y = y_j + \frac{i\pi}{2} \qquad x_j, y_j \in \mathbb{R}$$

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- Logarithmic Bethe Equations and Euler-Maclaurin yields the integral equations:

$$\rho^x(x) = \sigma_0^x(x) + \frac{1}{L}\tau_0^x(x) + \frac{1}{24L^2}\eta_0^x + \int_{-\infty}^{\infty} \mathrm{d}x' K_0\left(x - x'\right)\rho^x(x') + \int_{-\infty}^{\infty} \mathrm{d}x' K_1\left(x - x'\right)\rho^y(x')$$

$$\rho^y(y) = \sigma_0^y(y) + \frac{1}{L}\tau_0^y(y) + \frac{1}{24L^2}\eta_0^y + \int_{-\infty}^{\infty} \mathrm{d}x' K_1\left(y - x'\right)\rho^x(x') + \int_{-\infty}^{\infty} \mathrm{d}x' K_0\left(y - x'\right)\rho^y(x')$$

where

$$K_0(x) = \frac{1}{2\pi}\phi'(x,\gamma), \quad K_1(x) = -\frac{1}{2\pi}\psi'(x,\gamma)$$

$$\phi(x,y) = 2\arctan\left(\tanh(x)\cot(y)\right), \quad \psi(x,y) = 2\arctan\left(\tanh(x)\tan(y)\right)$$

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Results for the densities

- For the bulk contribution we recover the results for the PBC:

$$\sigma^x(x) = \frac{2\sin(\frac{\pi(\alpha-\gamma)}{\pi-2\gamma})}{\pi - 2\gamma} \frac{1}{\cosh(\frac{2\pi y}{\pi-2\gamma}) - \cos(\frac{\pi(\alpha-\gamma)}{\pi-2\gamma})}$$

$$\sigma^y(y) = \frac{2\sin(\frac{\pi(\alpha-\gamma)}{\pi-2\gamma})}{\pi - 2\gamma} \frac{1}{\cosh(\frac{2\pi y}{\pi-2\gamma}) + \cos(\frac{\pi(\alpha-\gamma)}{\pi-2\gamma})},$$

- For the self dual case $\alpha = \frac{\pi}{2}$ both densities are equal
- Densities are vanishing for $\alpha = \gamma$ and $\alpha = \pi - \gamma$.
- The surface contribution is the same for both roots:

$$\tau^i(x) = \frac{1}{4\pi} \int_{-\infty}^{\infty} d\omega e^{i\omega x} \frac{\sinh(\frac{3\gamma-\pi}{4}\omega)}{\sinh(\frac{\gamma\omega}{4}) \cosh(\frac{2\gamma-\pi}{4}\omega)}$$

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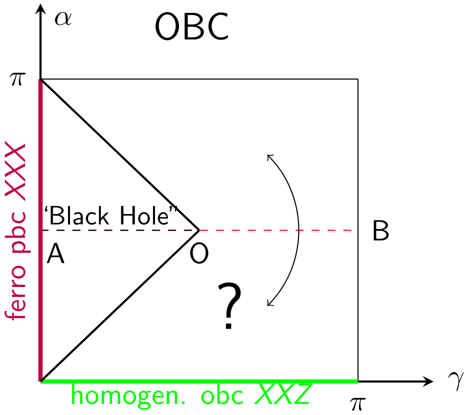
Parameter Space Diagram



Number of Bethe Roots:

$$\frac{2M_{GS}^{0}+1}{L} = 2\cdot\frac{\pi-\alpha-\gamma}{\pi-2\gamma} + \frac{1}{L}\left(\frac{3}{2}-\frac{\pi}{2\gamma}\right) + \mathcal{O}\Big(\frac{1}{L^2}\Big)\,,$$

$$\frac{2M_{GS}^{\frac{\pi}{2}}+1}{L} = 2\cdot\frac{\alpha-\gamma}{\pi-2\gamma} + \frac{1}{L}\left(\frac{3}{2}-\frac{\pi}{2\gamma}\right) + \mathcal{O}\Big(\frac{1}{L^2}\Big)\,.$$

$$\Longrightarrow S^{GS} = \left[-\frac{1}{2}+\frac{\pi}{2\gamma}\right],$$

where the brackets indicate the rounding. Inverting this relation we obtain a range of anisotropies γ for which the ground state is realized in the sector with spin S^{GS}:

$$\frac{\pi}{2S^{GS}+2} < \gamma < \frac{\pi}{2S^{GS}}\,.$$

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Thermodynamics Quantities and CFT

Using the densities we obtain:

$$e_\infty = -2\int_{-\infty}^{\infty} \mathrm{d}\omega \frac{\sinh(\frac{\gamma\omega}{2})\left(\sinh\left(\frac{\pi\omega}{2}-\frac{\omega\gamma}{2}\right)\cosh(\frac{\omega\pi}{2}-\alpha\omega)-\sinh(\frac{\gamma\omega}{2})\right)}{\sinh(\frac{\omega\pi}{2})\sinh((\frac{\pi-2\gamma}{2})\omega)},$$

$$v_F = \frac{2\pi}{\pi-2\gamma} \qquad f_\infty = \cdots \qquad k_\infty = \cdots \qquad k_s = \cdots \qquad \mathcal{K}_{thermo} = Lk_\infty + k_s + \mathcal{O}(\frac{1}{L})$$

Relation to CFT:

$$\frac{L}{\pi v_F}(E(L) - Le_\infty - f_\infty) = \underbrace{-\frac{c}{24} + h_n + d}_{h_{eff}} \tag{2}$$

Taking Small Excitations into Account

- Expand energy around the ground state energy in the limit $L \to \infty$ gives

$$h_{eff} = \left(-\frac{1}{12} + \frac{\gamma}{4\pi} \left(2S^Z + 1 - \frac{\pi}{\gamma} \right)^2 + \frac{1}{4} \frac{(dN - dN_{GS})^2}{\tilde{Z}_D^2} + n_{ph} \right),$$

where

$$dN = M^0 - M^{\frac{\pi}{2}}, \quad \tilde{Z}_D = \lim_{\omega \to 0} \left(1 - \int_{-\infty}^{\infty} \mathrm{d}x \, \mathrm{e}^{i\omega x} \left(K_0(x) - K_1(x) \right) \right)^{-1}.$$

- Penultimate term vanishes formally since $\tilde{Z}_D = \infty$. Numerics shows that the decrease is actually $\propto \frac{1}{\log(L)}$

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Defining a Quantum number

- Idea: Bring this logarithmic correction under control by the quasi momentum operator as done in the (quasi)-periodic case [Frahm, Seel '14, Ikhlef Jacobsen, Saleur'12, Bazhanov, Kotousov, Lukyanov '21] :

$$s = \frac{\pi - 2\gamma}{4\pi\gamma} \left(\mathcal{K} - \mathcal{K}_{Thermo} \right)$$

- Using this variable we obtain:

$$h_{\text{eff}} = -\frac{1}{12} + \frac{\gamma}{4\pi} \left(2S + 1 - \frac{\pi}{\gamma} \right)^2 + \frac{\gamma s^2}{\pi - 2\gamma} + n_{ph}$$

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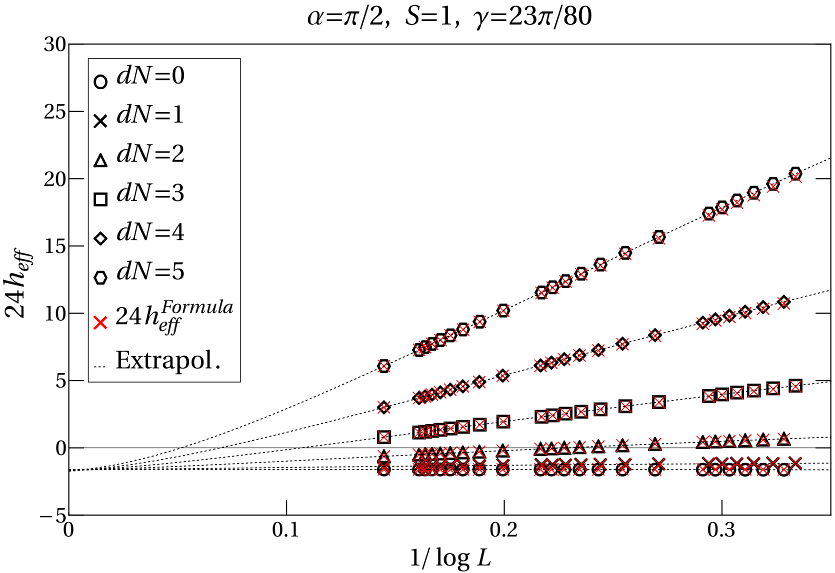
Numeric Results



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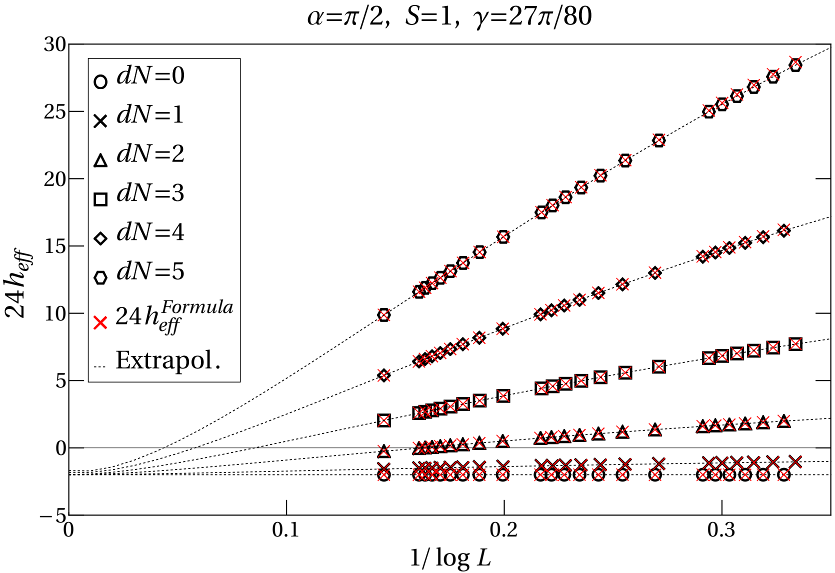

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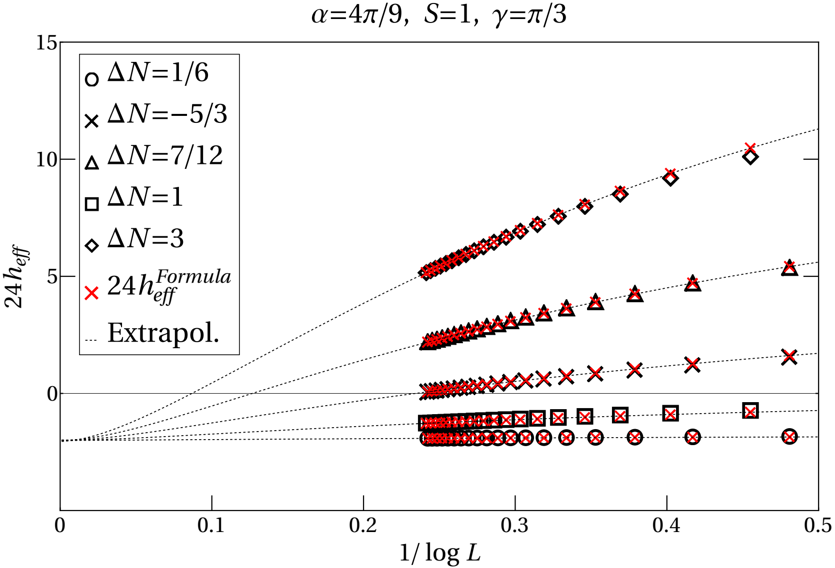

Identification with the Black Hole CFT Continuous Part

The BH CFT has a continuous part of spectrum which has following central charge and scaling dimensions [Ribault, Schomerus '04] :

$$c_{BH} = 2 + \frac{6}{(k-2)}\,, \quad h_{BH} = \frac{(n+wk)^2}{4k} - \frac{J(J-1)}{k-2} \quad \text{with} \quad J = \frac{1}{2} + i\tilde{s}, \quad \tilde{s} \in \mathbb{R}_0^+\,, \tag{3}$$

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$$k = \frac{\pi}{\gamma}, \qquad n = -2S - 1, \qquad w = 1, \qquad \left(J - \frac{1}{2}\right)^2 = (is)^2, \qquad d = n_{ph}.$$

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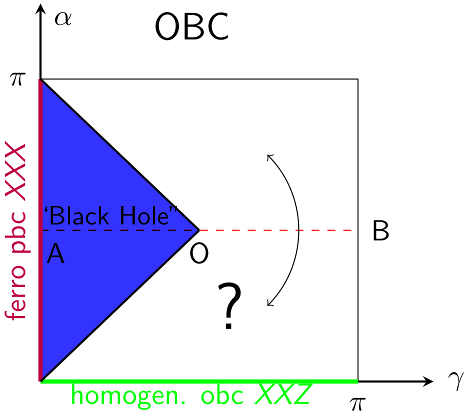

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- Continuous part (and discrete part) can be described by the real (and imaginary) eigenvalues of quasi momentum operator.

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Thank you!