Large deviations for Gibbs ensembles of the classical Toda chain

with Alice Guionnet

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Introduction

Toda model: N particles on $\mathbb{R}$, with positions $(t \mapsto q_j(t))_{1\leqslant j\leqslant N}$ and speeds $(t \mapsto p_j(t))_{1\leqslant j\leqslant N}$, such that

$$\frac{d}{dt}q_j = p_j \text{ and } \frac{d}{dt}p_j = e^{-r_{j-1}} - e^{-r_j},$$

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The sum $\sum_{j=1}^{N} r_j$ is constant to zero

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this system can be put in the form

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$\Rightarrow$ Eigenvalues of L_N independent of time
$\Rightarrow$ For all $V : \mathbb{R} \to \mathbb{R}$, $\frac{1}{N}\sum_{i=1}^{N} V(\lambda_i) =: \mathrm{Tr}(V(L_N))$ is constant.

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H. Spohn, 2019 : Generalized Gibbs ensembles for the Toda chain :

$$\mathrm{d}\mathbb{T}_N^{(V,P)} = \frac{1}{Z} \exp\{-\mathrm{Tr}(V(L_N)\} \prod_{i=1}^{N} e^{-Pr_i} \mathrm{d}r_i \mathrm{d}p_i.$$

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Taking the initial condition with respect to this measure, L_N becomes a random matrix independent of time.
Properties of its spectrum ? Convergence of

$$\hat{\mu}_N := \frac{1}{N}\sum_{i=1}^{N}\delta_{\lambda_i} \ ?$$

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$\Rightarrow$ Here : $V = x^2/2 +$ continuous bounded.

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For all $A \in \mathcal{B}(\mathcal{X})$,

$$-\inf_{\mathring{A}} \mathrm{J} \leqslant \liminf_N \frac{1}{N} \log \mathbb{P}(X_N \in A) \leqslant \limsup_N \frac{1}{N} \log \mathbb{P}(X_N \in A) \leqslant -\inf_{\overline{A}} \mathrm{J}.$$

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"roughly",

$$\mathbb{P}(X_N \simeq x) \simeq e^{-N\mathrm{J}(x)}.$$

Key property

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Suppose J has compact level sets : $J^{-1}(]-\infty, a])$.
If J has a unique minimizer x_0, then (X_N) converges almost surely towards x_0.

Strategy : Show that $(\hat{\mu}_N)$ satisfies a large deviation principle, and show the uniqueness of minimizer.

Other key property

If $(X_N)_N$ satisfies an LDP with rate function J and if Y_N has distribution

$$d\mathbb{P}_{Y_N} = \frac{1}{Z} e^{-Nf(x)} d\mathbb{P}_{X_N}(x),$$

(with $f \in \mathcal{C}_b^0$),

Other key property

If $(X_N)_N$ satisfies an LDP with rate function J and if Y_N has distribution
$$d\mathbb{P}_{Y_N} = \frac{1}{Z} e^{-Nf(x)} d\mathbb{P}_{X_N}(x),$$
(with $f \in \mathcal{C}_b^0$),
Then (Y_N) satisfies an LDP with rate function $\tilde{J}(x) = J(x) + f(x) - \inf_x\{J(x) + f(x)\}$.

Beta-ensembles

A matrix A_N of size $N \times N$ is in the β-ensemble with potential V if the joint law of its (unordered) eigenvalues is given by

$$d\mathbb{P}_N(\lambda_1, \dots, \lambda_N) = \frac{1}{Z_N^{V,\beta}} \prod_{1 \leqslant i < j \leqslant N} |\lambda_i - \lambda_j|^\beta e^{-\sum_{i=1}^N V(\lambda_i)} \mathrm{d}\lambda_1 \dots \mathrm{d}\lambda_N.$$

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Widely studied ensemble

Garcia-Zelada (2018) : For β of order $\frac{1}{N}$, under this measure, $(\hat{\mu}_N)_N$ satisfies an LDP with nice, explicit rate function.

Tridiagonal representation

Theorem (Dumitriu, Edelman - 2002)

Let $N \geqslant 1$, and $\beta > 0$. The matrix T_N^β (independent entries up to symmetry) given by

$$T_N^\beta(i,i) \sim N(0,1)$$

and

$$T_N^\beta(i,i+1) = T_N(i+1,i) \sim \frac{1}{\sqrt{2}}\chi_{(N-i)\beta},\ 1 \leqslant i \leqslant N-1,$$

is in the β ensemble with potential $V = \frac{1}{2}x^2$.

Matrix representation of β-ensemble

$$\begin{pmatrix} \mathcal{N}(0,1) & \frac{1}{\sqrt{2}}\chi_{(N-1)\beta} & & & \\ \frac{1}{\sqrt{2}}\chi_{(N-1)\beta} & \mathcal{N}(0,1) & \frac{1}{\sqrt{2}}\chi_{(N-2)\beta} & & \\ & \ddots & \ddots & \ddots & \\ & & \frac{1}{\sqrt{2}}\chi_{2\beta} & \mathcal{N}(0,1) & \frac{1}{\sqrt{2}}\chi_{\beta} \\ & & & \frac{1}{\sqrt{2}}\chi_{\beta} & \mathcal{N}(0,1) \end{pmatrix}$$

Key remark

Toda, potential $V = \frac{x^2}{2}$, pressure $P > 0$:

$$L_N(P) = \begin{pmatrix} \mathcal{N}(0,1) & \frac{1}{\sqrt{2}}\chi_{2P} & & & \frac{1}{\sqrt{2}}\chi_{2P} \\ \frac{1}{\sqrt{2}}\chi_{2P} & \mathcal{N}(0,1) & \frac{1}{\sqrt{2}}\chi_{2P} & & \\ & \ddots & \ddots & \ddots & \\ & & \frac{1}{\sqrt{2}}\chi_{2P} & \mathcal{N}(0,1) & \frac{1}{\sqrt{2}}\chi_{2P} \\ \frac{1}{\sqrt{2}}\chi_{2P} & & & \frac{1}{\sqrt{2}}\chi_{2P} & \mathcal{N}(0,1) \end{pmatrix}$$

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Link with Toda

Case $V = \frac{1}{2}x^2$. $N = kM + r$ Consider

$$\begin{pmatrix} L_k^{(M)} & & & \\ & \ddots & & \\ & & L_k^{(1)} & \\ & & & 0 \end{pmatrix},$$

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$L_k^{(j)} = L_k(jP/M)$ Link between the LDP for Toda and for β ensemble.
General theory of large deviations : We deduce a link between those LDP for $V = \frac{1}{2}x^2 +$ bounded continuous

Conclusion

Show uniqueness of minimizer $\mu_{Toda}^{(V,P)}$ of the Toda rate function, with

$$\mu_{\text{Toda}}^{(V,P)} = \partial_P \left(P \mu_{\beta\text{-ens.}}^{(V,P)} \right),$$

Conclusion

Show uniqueness of minimizer $\mu_{Toda}^{(V,P)}$ of the Toda rate function, with

$$\mu_{\text{Toda}}^{(V,P)} = \partial_P \left(P \mu_{\beta\text{-ens.}}^{(V,P)} \right),$$

i.e for $f \in \mathcal{C}_b$

$$\int_{\mathbb{R}} f \mathrm{d}\mu_{\text{Toda}}^{(V,P)} = \partial_P \left(\int_{\mathbb{R}} f \mathrm{d}\mu_{\beta\text{-ens.}}^{(V,P)} \right).$$

Thank you !