Time-time covariance for last passage percolation in half space

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Alessandra Occelli

joint work with P. L. Ferrari

ENS de Lyon, UMPA

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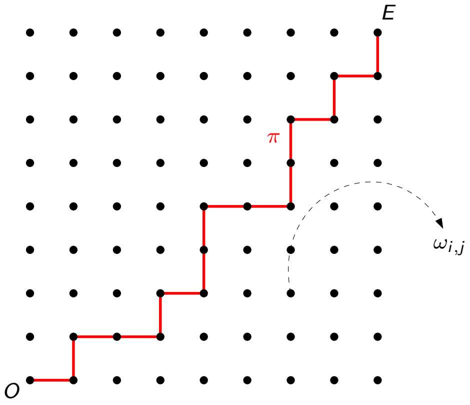
Last passage percolation (LPP)

- O, E point in $\mathbb{Z}^2$
- $\omega_{i,j}$ independent r.v.'s, $i, j \in \mathbb{Z}$
- Directed path π composed of $\rightarrow$ and $\uparrow$ s.t. $\pi(0) = O$ and $\pi(n) = E$
- Last passage time: $L_{O \to E} = \max_{\pi : O \to E} \sum_{1 \le k \le n} \omega_{\pi(k)}$



Half-space last passage percolation

- LPP in the half-quadrant of $\mathbb{Z}^2$

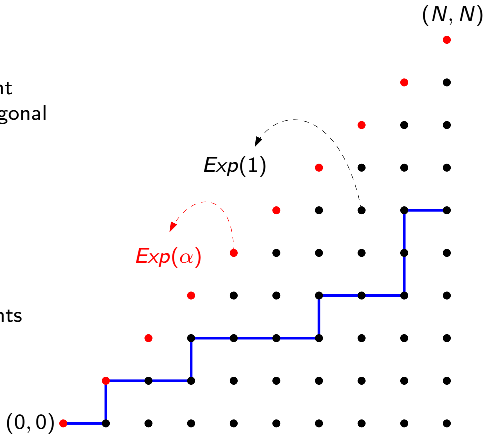
$$\omega_{i,j} \sim \begin{cases} Exp(1), & i \geq j+1 \\ Exp(\alpha), & i = j \end{cases}$$

- Equivalent to LPP on the full quadrant with weights symmetric w.r.t. the diagonal $\omega_{i,j} = \omega_{j,i}$

Hammersley LPP in half-space
Baik–Rains '01
Sasamoto–Imamura '04

Symmetrized LPP with geometric weights
Baik–Rains '01

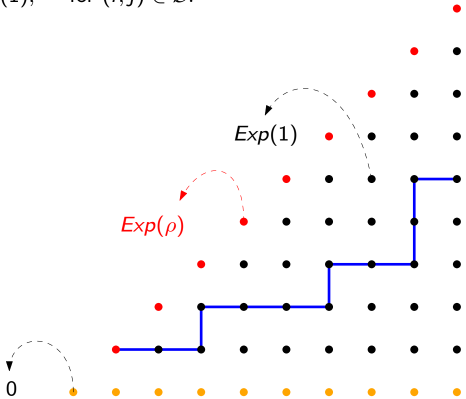
and exponential weights
Baik–Barraquand–Corwin–Suidan '18



Point-to-point LPP in half space

Let L^{pp} be the point-to-point LPP with weights

$$\begin{cases} \omega^{pp}_{0,0} = 0, & \\ \omega^{pp}_{i,i} \sim Exp(\rho), & \text{for } i \in \mathbb{N}, \\ \omega^{pp}_{i,0} = 0, & \text{for } i \in \mathbb{N}, \\ \omega^{pp}_{i,j} \sim Exp(1), & \text{for } (i,j) \in \mathcal{B}. \end{cases}$$



Stationary LPP in half space

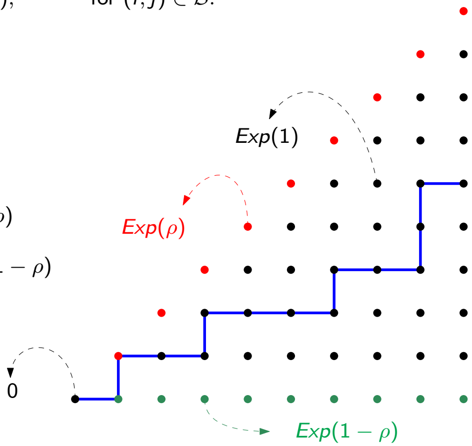
Let $L^{st,\rho}$ be the stationary LPP with weights

$$\begin{cases} \omega^{\rho}_{0,0} = 0, & \\ \omega^{\rho}_{i,i} \sim Exp(\rho), & \text{for } i \in \mathbb{N}, \\ \omega^{\rho}_{i,0} \sim Exp(1-\rho), & \text{for } i \in \mathbb{N}, \\ \omega^{\rho}_{i,j} \sim Exp(1), & \text{for } (i,j) \in \mathcal{B}. \end{cases}$$

Stationary increments

$$L^{st,\rho}_{m,n} - L^{st,\rho}_{m,n-1} \sim Exp(\rho)$$

$$L^{st,\rho}_{m,n} - L^{st,\rho}_{m-1,n} \sim Exp(1-\rho)$$



Scaling limits of half space LPP

Let $\rho = \frac{1}{2} + \alpha = \frac{1}{2} + \delta 2^{-4/3} N^{-1/3}$ and consider the end-points

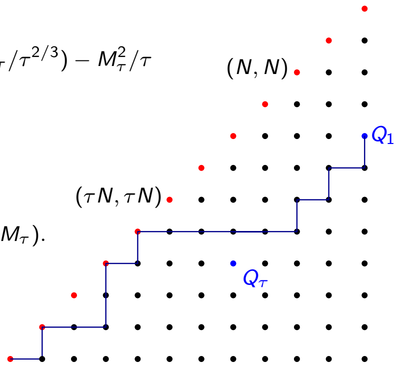
$$Q_1 = (N + M_1 (2N)^{2/3}, N - M_1 (2N)^{2/3}), \quad M_1 = (1-\tau)^{2/3} \tilde{M}_1$$
$$Q_\tau = (\tau N + M_\tau (2N)^{2/3}, \tau N - M_\tau (2N)^{2/3}), \quad M_\tau = (1-\tau)^{2/3} \tilde{M}_\tau$$

We know that (Baik–Barraquand–Corwin–Suidan '18)

$$\mathcal{L}_N^{pp}(M_1, 1) := \frac{L^{pp}(Q_1) - 4N}{2^{4/3} N^{1/3}} \overset{N\to\infty}{\longrightarrow} \mathcal{A}_\delta^{pp}(M_1) - M_1^2,$$
$$\mathcal{L}_N^{pp}(M_\tau, \tau) := \frac{L^{pp}(Q_\tau) - 4\tau N}{2^{4/3} N^{1/3}} \overset{N\to\infty}{\longrightarrow} \tau^{1/3} \mathcal{A}^{pp}_{\delta \tau^{1/3}}(M_\tau/\tau^{2/3}) - M_\tau^2/\tau$$

and (Betea–Ferrari–O. '22)

$$\mathcal{L}_N^{st,\rho}(M_1, 1) := \frac{L^{st,\rho}(Q_1) - 4N}{2^{4/3} N^{1/3}} \overset{N\to\infty}{\longrightarrow} \mathcal{A}_\delta^{st,hs}(M_1),$$
$$\mathcal{L}_N^{st,\rho}(M_\tau, \tau) := \frac{L^{st,\rho}(Q_\tau) - 4\tau N}{2^{4/3} N^{1/3}} \overset{N\to\infty}{\longrightarrow} \tau^{1/3} \mathcal{A}^{st,hs}_{\delta \tau^{1/3}}(M_\tau).$$



Time-time covariance

We study the covariance of the process at two times

$$\begin{aligned}\mathrm{Cov}\left(\mathcal{L}_N^*(M_\tau,\tau),\mathcal{L}_N^*(M_1,1)\right) =& \frac{1}{2}\,\mathrm{Var}(\mathcal{L}_N^*(M_1,1)) + \frac{1}{2}\,\mathrm{Var}(\mathcal{L}_N^*(M_\tau,\tau)) \\ &- \frac{1}{2}\mathrm{Var}(\mathcal{L}_N^*(M_\tau,\tau) - \mathcal{L}_N^*(M_1,1))\end{aligned}$$

Previous results in full space

- Two-time and multi-time distribution for geometric LPP

  Johansson '19, Johansson–Rahman '21

- Experimental results on turbulent liquid crystals and numerical simulation of Eden model

  Takeuchi–Sano '12, De Nardis–Le Doussal–Takeuchi '17

- Conjecture on the behavior of the covariance of the limit processes for $\tau \to 1$ (and $\tau \to 0$) based on heuristic arguments for point-to-point, deterministic and stationary random-line-to-point LPPs with points on the diagonal

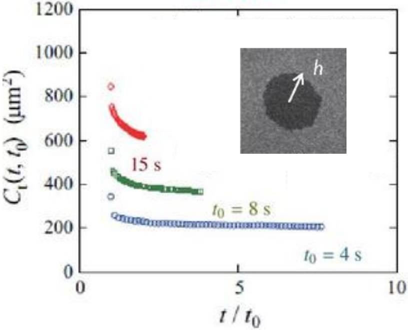
  Ferrari–Spohn '16

Time-time covariance in full space

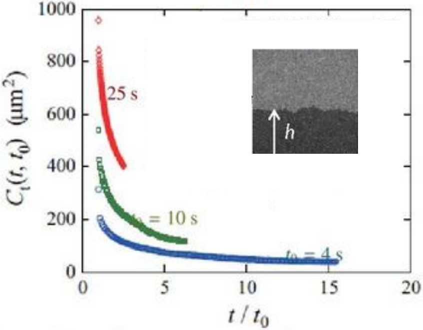
Experiments on turbulent liquid crystals by Takeuchi–Sano '12

$$C_t(t, t_0) = \mathrm{Cov}(h(x,t), h(x,t_0))$$

Droplet



Flat



Time-time covariance in full space

Universal behavior for macroscopically close times

Let $\mathcal{L}_N^*(M_\tau, \tau)$ be the rescaled LPP starting from a point, a deterministic or a random collection of points

Theorem (Ferrari–O. '18)

As $\tau \to 1$*,*

$$\lim_{N\to\infty} \mathrm{Var}(\mathcal{L}_N^*(M_\tau,\tau) - \mathcal{L}_N^*(M_1,1)) = \mathrm{Var}\left(\xi^{stat}((1-\tau)^{-2/3}(M_1 - M_\tau))\right) + \mathcal{O}(1-\tau)^{1-\delta}$$

for any $\delta > 0$*.*

$\xi^{stat}(w)$ is distributed according to the Baik–Rains distribution with parameter w

Time-time covariance in half space

Exact formula for stationary LPP with end-points on the diagonal

Theorem (Ferrari–O. '22)
Let $\rho = \frac{1}{2} + \delta 2^{-4/3} N^{-1/3}$ *and* $M_1 = M_\tau = 0$. *Then,*

$$\lim_{N\to\infty} \mathrm{Cov}(\mathcal{L}_N^{st,\rho}(0,1), \mathcal{L}_N^{st,\rho}(0,\tau)) = \frac{1}{2}\,\mathrm{Var}(\mathcal{A}_\delta^{st,hs}(0)) + \frac{\tau^{2/3}}{2}\,\mathrm{Var}(\mathcal{A}_{\delta\tau^{1/3}}^{st,hs}(0)) - \frac{(1-\tau)^{2/3}}{2}\,\mathrm{Var}(\mathcal{A}_{\delta(1-\tau)^{1/3}}^{st,hs}(0)).$$

To prove it, we derive the identity

$$\max_{v\geq 0}\left\{\sqrt{2}B(v) + 2v\delta + \mathcal{A}_\delta^{pp}(v) - v^2\right\} \stackrel{(d)}{=} \mathcal{A}_\delta^{st,hs}(0).$$

Time-time covariance in half space

Universal behavior for point-to-point LPP as $\tau \to 1$

Theorem (Ferrari–O. '22)
Let $\rho = \frac{1}{2} + \delta 2^{-4/3} N^{-1/3}$ and $M_\tau = 0$. Then, for any $0 < \theta < 1/3$, there exists a constant C such that

$$\lim_{N\to\infty} \left| \mathrm{Var}(\mathcal{L}_N^{pp}(M_1, 1) - \mathcal{L}_N^{pp}(0,\tau)) - \mathrm{Var}(\mathcal{L}_N^{st,\rho}(M_1, 1) - \mathcal{L}_N^{st,\rho}(0,\tau)) \right| \le C(1-\tau)^{1-\theta}$$

as $\tau \to 1$.

As a corollary, when $M_1 = M_\tau = 0$

$$\begin{aligned}\lim_{N\to\infty} \mathrm{Cov}(\mathcal{L}_N^{pp}(0,1), \mathcal{L}_N^{pp}(0,\tau)) =& \frac{1}{2}\mathrm{Var}(\mathcal{A}_\delta^{pp}(0)) + \frac{\tau^{2/3}}{2}\mathrm{Var}(\mathcal{A}_{\delta\tau^{1/3}}^{pp}(0)) \\ &- \frac{(1-\tau)^{2/3}}{2}\mathrm{Var}(\mathcal{A}_{\delta(1-\tau)^{1/3}}^{st,hs}(0)) + \mathcal{O}((1-\tau)^{1-\theta}).\end{aligned}$$

Comparison inequalities

Upper bound on point-to-point LPP

Proposition

Let $\rho_+ \geq \rho$. Consider the stationary LPP with parameter ρ_+ and the point-to-point model with parameter ρ. Then, for all $p \preceq q$,

$$L^{pp}(q) - L^{pp}(p) \leq L^{\rho_+}(q) - L^{\rho_+}(p).$$

Lower bound on point-to-point LPP

Proposition

Let $\rho_- = \frac{1}{2} + (\delta - \kappa)2^{-4/3}N^{-1/3}$ with $\kappa > 0$. Let $p \preceq q$ and define the crossing event

$$\Omega_{cross} = \{\pi^{\rho_-}(q) \cap \pi^{pp}(p) \cap \mathcal{B} \neq \emptyset\}.$$

Under the event Ω_{cross},

$$L^{\rho_-}(q) - L^{\rho_-}(p) \leq L^{pp}(q) - L^{pp}(p).$$

Bound on the crossing event: There exist $C, c > 0$ such that, for all N,

$$\mathbb{P}(\Omega^C_{cross}) \leq Ce^{-c(\kappa-\delta)^3}.$$

Localization of the geodesics

Consider the end-point $Q_1 = (N + M_1(2N)^{2/3}, N - M_1(2N)^{2/3})$ and let

$$I(u) = (\tau N + u(2N)^{2/3}, \tau N - u(2N)^{2/3}), \quad \tau \in (0,1)$$

Define $\tilde{L}$ as $L^{st,\rho}$ but with $\omega_{i,j} = 0$ in the red region

Theorem (Ferrari–O. '22)

Let $\mathcal{L}_M = \{(i,j) | i - j = M(2N)^{2/3}\}$. For all $M \geq M_1 + 9$, with $M = \mathcal{O}(N^{1/3}/\ln(N))$, uniformly for all N large enough,

$$\mathbb{P}\left(\pi^{pp}(Q) \cap \mathcal{L}_M = \emptyset\right) \geq 1 - Ce^{-c(M-M_1)^3}$$

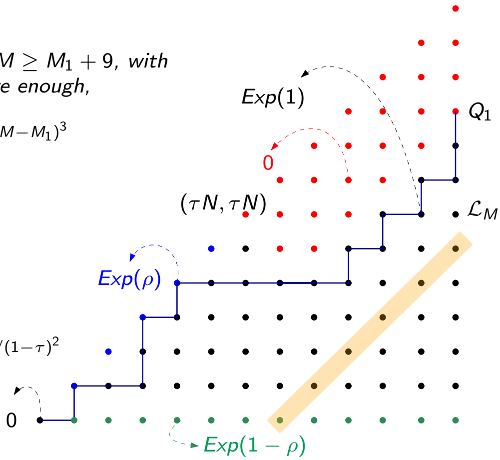
This follows from

Proposition (Ferrari–O. '22)

There exist constants $C, c > 0$ such that

$$\mathbb{P}(\pi^{pp}(Q_1) \prec I(M)) \geq \mathbb{P}(\pi^{\rho}(Q_1) \prec I(M))$$
$$\geq \mathbb{P}(\tilde{\pi}(Q_1) \prec I(M)) \geq 1 - Ce^{-c(M-M_1)^3/(1-\tau)^2}$$

uniformly for all N large enough.



Proof: first order correction of the covariance

Theorem (Ferrari–O. '22)
Let $\rho = \frac{1}{2} + \delta 2^{-4/3} N^{-1/3}$ and $M_\tau = 0$. Then, for any $0 < \theta < 1/3$, there exists a constant C such that

$$\lim_{N\to\infty} \left| \mathrm{Var}(\mathcal{L}_N^{pp}(M_1,1) - \mathcal{L}_N^{pp}(0,\tau)) - \mathrm{Var}(\mathcal{L}_N^{st,\rho}(M_1,1) - \mathcal{L}_N^{st,\rho}(0,\tau)) \right| \leq C(1-\tau)^{1-\theta}$$

as $\tau \to 1$

- Observe that, as $\tau \to 1$,

$$\lim_{N\to\infty} \mathbb{E}(|\mathcal{L}_N^{st,\rho}(M_1,1) - \mathcal{L}_N^{st,\rho}(0,\tau)|) = \mathcal{O}((1-\tau)^{1/3})$$
$$\lim_{N\to\infty} \mathrm{Var}(\mathcal{L}_N^{st,\rho}(M_1,1) - \mathcal{L}_N^{st,\rho}(0,\tau)) = \mathcal{O}((1-\tau)^{2/3})$$

Let $I(u) = (\tau N + u(2N)^{2/3}, \tau N - u(2N)^{2/3})$ and define

$$X_N = \mathcal{L}_N^{pp}(M_1,1) - \mathcal{L}_N^{pp}(0,\tau) = \max_{u\geq 0}[\mathcal{L}_N^{pp}(u,\tau) + \mathcal{L}_N^{\rho}(u,\tau;M_1,1) - \mathcal{L}_N^{pp}(0,\tau))],$$

where

$$\mathcal{L}_N^{\rho}(u,\tau;M_1,1) = \frac{L^{pp;\rho}(I(u),Q_1) - 4(1-\tau)N}{2^{4/3}N^{1/3}}.$$

Define $Y_N^\rho = \mathcal{L}_N^{st,\rho}(M_1,1) - \mathcal{L}_N^{st,\rho}(0,\tau)$ analogously

Proof: first order correction of the covariance

We need to estimate

$$\mathrm{Var}(X_N) - \mathrm{Var}(Y_N^\rho)$$

1. **LOCALIZATION**

Define the random variables

$$X_{N,M} = \max_{0 \le u \le M} [\mathcal{L}_N^{pp}(u,\tau) + \mathcal{L}_N^\rho(u,\tau; M_1, 1) - \mathcal{L}_N^{pp}(0,\tau)],$$
$$X_{N,M^C} = \max_{u > M} [\mathcal{L}_N^{pp}(u,\tau) + \mathcal{L}_N^\rho(u,\tau; M_1, 1) - \mathcal{L}_N^{pp}(0,\tau)]$$

and similarly $Y^\rho_{N,M}$, Y^ρ_{N,M^c} for $\mathcal{L}_N^{st,\rho}$. Then, $X_N = \max\{X_{N,M}, X_{N,M^C}\}$.

Proposition

For all $\tilde{M} > 0$, set $M = (1-\tau)^{2/3}\tilde{M}$. Then, uniformly in N,

$$\mathrm{Var}(X_N) = \mathrm{Var}(X_{N,M}) + \mathcal{O}(e^{-c\tilde{M}})$$
$$\mathrm{Var}(Y_N^\rho) = \mathrm{Var}(Y_{N,M}^\rho) + \mathcal{O}(e^{-c\tilde{M}})$$

and

$$\mathbb{E}(X_N) = \mathbb{E}(X_{N,M}) + \mathcal{O}(e^{-c\tilde{M}})$$
$$\mathbb{E}(Y_N^\rho) = \mathbb{E}(Y_{N,M}^\rho) + \mathcal{O}(e^{-c\tilde{M}})$$

Proof: first order correction of the covariance

We need to estimate

$$\mathrm{Var}(X_N) - \mathrm{Var}(Y_N^\rho)$$

1. **LOCALIZATION**

Define the random variables

$$X_{N,M} = \max_{0 \le u \le M} [\mathcal{L}_N^{pp}(u,\tau) + \mathcal{L}_N^{\rho}(u,\tau; M_1, 1) - \mathcal{L}_N^{pp}(0,\tau)],$$
$$X_{N,M^C} = \max_{u > M} [\mathcal{L}_N^{pp}(u,\tau) + \mathcal{L}_N^{\rho}(u,\tau; M_1, 1) - \mathcal{L}_N^{pp}(0,\tau)]$$

and similarly $Y_{N,M}^\rho$, Y_{N,M^c}^ρ for $\mathcal{L}_N^{st,\rho}$. Then, $X_N = \max\{X_{N,M}, X_{N,M^C}\}$.

Key ingredients:

- Bound on the localization of the geodesic

$$\mathbb{P}(X_{N,M} < X_{N,M^C}) = \mathbb{P}(\pi^{pp}(Q_1) \not\subset I(M)) \le Ce^{-cM^3/(1-\tau)^2} = Ce^{-c\tilde{M}^3}.$$

- $X_{N,M} \ge \mathcal{L}_N^{pp}(I(0)) + \mathcal{L}_N^{\rho}(I(0), Q_1) - \mathcal{L}_N^{pp}(I(M_\tau))$, where all the random variables have (at least) exponential upper and lower tails

Proof: first order correction of the covariance

2. COMPARISON WITH THE STATIONARY CASE

Let $\rho_+ = \rho = \frac{1}{2} + \delta 2^{-4/3} N^{-1/3}$ and $\rho_- = \frac{1}{2} + (\delta - \kappa) 2^{-4/3} N^{-1/3}$.

- For all $0 \leq u_1 < u_2 \leq M$,

$$L^{\rho_-}(I(u_2)) - L^{\rho_-}(I(u_1)) \leq L^{pp}(I(u_2)) - L^{pp}(I(u_1)) \leq L^{\rho}(I(u_2)) - L^{\rho}(I(u_1)),$$

  on the event

$$\Omega_{cross} = \{\pi^{\rho_-}(I(u_2)) \cap \pi^{pp}(I(u_1)) \cap \mathcal{B} \neq \emptyset\}$$

- We decompose

$$X_{N,M} = X_{N,M} \mathbb{1}_{\Omega_{cross}} + X_{N,M} \mathbb{1}_{\Omega_{cross}^C}$$

  (and similarly for $Y^{\rho}_{N,M}$)

- We have

$$Y^{\rho_-}_{N,M} \mathbb{1}_{\Omega_{cross}} \leq X_{N,M} \mathbb{1}_{\Omega_{cross}} \leq Y^{\rho}_{N,M} \mathbb{1}_{\Omega_{cross}}$$

  and

$$\mathbb{P}(Y^{\rho}_{N,M} > s) - \mathbb{P}(\Omega^C_{cross}) \leq \mathbb{P}(X_{N,M} > s) \leq \mathbb{P}(Y^{\rho_-}_{N,M} > s) + \mathbb{P}(\Omega^C_{cross})$$
$$\mathbb{P}(Y^{\rho_-}_{N,M} \leq s) - \mathbb{P}(\Omega^C_{cross}) \leq \mathbb{P}(X_{N,M} \leq s) \leq \mathbb{P}(Y^{\rho}_{N,M} \leq s) + \mathbb{P}(\Omega^C_{cross})$$

Proof: first order correction of the covariance

❸ COUPLING BETWEEN STATIONARY LPPs

- We have

$$\mathcal{L}_N^{st,\rho_-}(u,\tau) - \mathcal{L}_N^{st,\rho_-}(0,\tau) = \frac{1}{2^{4/3}N^{1/3}} \sum_{i=1}^{u(2N)^{2/3}} (\tilde{X}_i - \tilde{Y}_i)$$

where

$$\tilde{X}_i \sim Exp(1-\rho_-), \quad \tilde{Y}_i \sim Exp(\rho_-)$$

are independent random variables, and

$$\mathcal{L}_N^{st,\rho}(u,\tau) - \mathcal{L}_N^{st,\rho}(0,\tau) = \frac{1}{2^{4/3}N^{1/3}} \sum_{i=1}^{u(2N)^{2/3}} (X_i - Y_i)$$

where

$$X_i \sim Exp(1-\rho), \qquad Y_i \sim Exp(\rho)$$

are independent random variables

Proof: first order correction of the covariance

❸ COUPLING BETWEEN STATIONARY LPPs

- With the coupling $\omega^{\rho-}_{i,i} \geq \omega^{\rho}_{i,i}$, $\omega^{\rho-}_{i,0} \leq \omega^{\rho}_{i,0}$

$$\tilde{X}_i - \tilde{Y}_i \leq X_i - Y_i$$

- Thus,

$$\mathcal{L}_N^{st,\rho-}(u,\tau) - \mathcal{L}_N^{st,\rho-}(0,\tau) \overset{(d)}{=} \mathcal{L}_N^{st,\rho}(u,\tau) - \mathcal{L}_N^{st,\rho}(0,\tau) - R(u), \quad (*)$$

with

$$R(u) = \frac{1}{2^{4/3}N^{1/3}} \sum_{i=1}^{u(2N)^{2/3}} (P_i + Q_i),$$

where P_i and Q_i are independent and have explicit laws and $\mathbb{E}[R(u)] = 2u\kappa + \mathcal{O}(u\kappa^3 N^{-2/3})$

- The terms on the r.h.s of $(*)$ are not independent!
But $R(u)$ goes to 0 as $N \to \infty$

First order correction of the covariance

4. **CONCLUSION**

- Putting together the localization result and the previous estimates and taking $\kappa = \tilde{M} = 1/(1-\tau)^{\theta/2}$, with $0 < \theta < 1/3$,

$$|\operatorname{Var}(X_N) - \operatorname{Var}(Y_N^\rho)| \leq C(1-\tau)^{2/3-\theta}\mathbb{E}(|Y_N^\rho|)$$

  as $\tau \to 1$

- Observing that $\mathbb{E}(|Y_N^\rho|) = \mathcal{O}((1-\tau)^{1/3})$ and taking $N \to \infty$, the proof is completed

Thank you
for your attention!