Contractions of Integrability Algebras and R-Matrices

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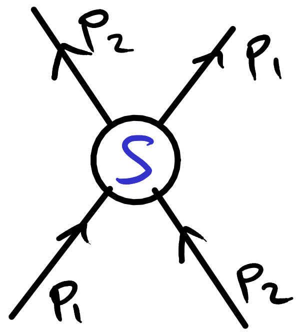
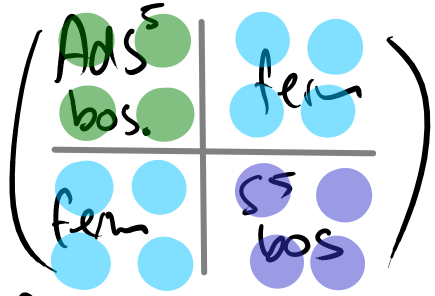
Workshop: Randomness, Integrability and Universality
Galileo Galilei Institute – Arcetri, Firenze
28 April 2022

work with: R. Hecht, B. Hoare [1704.05093]; Egor Im (in progress)

AdS/CFT Worldsheet S-Matrix



- Planar $N=4$ SYM & $AdS^5 \times S^5$ Strings (apparently) integrable.
- Scattering problem on 2D worldsheet factorises.
- 8+8 (bos+ferm) worldsheet excitations (magnons) above vacuum.
- $psu(2|2) \times psu(2|2) \subset psu(2,2|4)$ residual symmetry.



- excitations in bi-fundamental of $psu(2|2) \times psu(2|2)$
- S-matrix structure factorises: $S = \exp(i\,\text{Phase})(R \otimes R)$

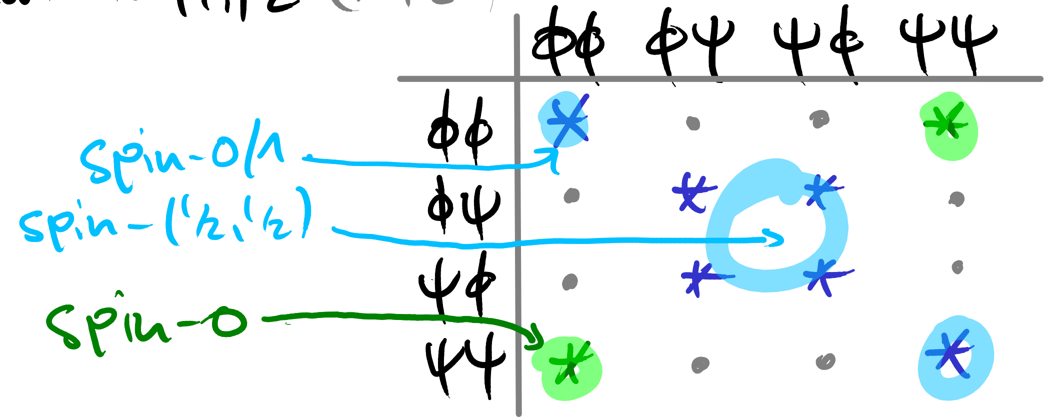
Remarkable R-Matrix

R-matrix for 2+2 (bos+ferm) particle flavours has unusual features:

- two manifest $SU(2)$ symmetries for 2+2 representations (rational/XXX type)
- not plain $psu(2|2)$ R-matrix $R \not\equiv R_{psu(2|2)}$
- not of difference form $R(p_1,p_2) \not\equiv R(u(p_1)-u(p_2))$
- elliptic functional dependence on momenta p_1, p_2 (XYZ?)
- off-diagonal scattering (XYZ?)

  ϕ: bosons $(\tfrac{1}{2},0)$

  ψ: fermions $(0,\tfrac{1}{2})$



- equivalent to Shastry's R-matrix for 1D Hubbard model $R \equiv R_{\text{Shastry}}$
- Quantum Algebra origin of R?!

R-Matrix Symmetry Algebra

R-matrix constructed using psu(2|2) symmetry
with some unconventional (psu(2|2)-specific) assumptions.

- typical psu(2|2) supermatrix structure $\left(\begin{array}{c|c} J_1 & Q \\ \hline \bar{Q} & J_2 \end{array}\right)$

su(2)
odd generators $(\frac{1}{2},\frac{1}{2})$
su(2)

- 3 central elements: $E, C, \bar{C}$

spin-1

part of su(2|2)

- $\{Q, \bar{Q}\} \sim J_1 + J_2 + E$
- $\{Q, Q\} \sim C$
- $\{\bar{Q}, \bar{Q}\} \sim \bar{C}$

spin-0

exceptional central elements

all functionally dependent

$$E = E(p) = \sqrt{1 + \lambda \sin^2(p/2)}$$
$$C = C(p) = 1 - e^{ip}$$
$$\bar{C} = \bar{C}(p) = 1 - e^{-ip}$$

- also some derivation element B (classical limit analysis)

- $[B, Q] \sim +Q + * \bar{Q}$
- $[B, \bar{Q}] \sim -\bar{Q} + * Q$

similar/related to off-diagonal scattering

NB, Hecht, Hoare 17 04.05093

R-Matrix and 3D Poincaré Symmetry

$sl(2) \times su(2) \times su(2) \times 8$ susy

- 3 central elements from $D(2,1;\epsilon) \xrightarrow{\epsilon \to 0} psu(2|2) \ltimes \mathbb{C}^3$
- 3 derivations from other limit $D(2,1;\epsilon) \xrightarrow{\epsilon \to 0} sl(2) \ltimes psu(2|2)$
- central elements + derivations from combined contraction limit

$$sl(2) \times D(2,1;\epsilon) \xrightarrow{\epsilon \to 0} sl(2) \ltimes psu(2|2) \ltimes \mathbb{C}^3$$

$$L^\mu \qquad J_1\, J_2\, Q\, \bar{Q} \qquad P^\mu$$

- algebra structure

$$[L,L] \sim L \qquad [L, Q/\bar{Q}] \sim Q/\bar{Q} \qquad \{Q/\bar{Q}, Q/\bar{Q}\} \sim P$$
$$[L,P] \sim P \qquad [P, Q/\bar{Q}] = 0$$
$$[P,P] = 0$$

$\longrightarrow$ 3D Poincaré with injected supersymmetry extension

- simplified algebra contraction: $so(4) = sl(2) \times sl(2) \longrightarrow sl(2) \ltimes \mathbb{C}^3 = iso(3)$
- further reduction to 2D for R-matrix: $sl(2) \ltimes \mathbb{C}^3 \longrightarrow gl(1) \times \mathbb{C}$

SL(2) Bi-Algebra Conventions (factors of ± 2)

Generators J^0, J^+, J^- span Lie Algebra $sl(2)$ (complexify, not care about reality)

Lie brackets: $[J^0, J^\pm] = \pm J^\pm$; $[J^+, J^-] = -2J^0$

Quadratic invariant $J^2 = -J^0 \otimes J^0 + \frac{1}{2} J^+ \otimes J^- + \frac{1}{2} J^- \otimes J^+$

Bi-Algebra Structure (classical limit of $U_q(sl(2))$)

Classical r-Matrix: $r = -J^0 \otimes J^0 + J^+ \otimes J^-$ (satisfies CYBE)

Co-Bracket:
$$\delta(J^0) = 0$$
$$\delta(J^\pm) = J^0 \wedge J^\pm$$
$\longrightarrow$
$$\begin{cases} \Delta(J^0) = J^0 \otimes 1 + 1 \otimes J^0 \\ \Delta(J^+) = J^+ \otimes 1 + e^{\hbar J^0} \otimes J^+ \\ \Delta(J^-) = J^- \otimes e^{-\hbar J^0} + 1 \otimes J^- \end{cases}$$

3D Poincaré Algebra as a Contraction

Flat space(time) $\mathbb{R}^3$ as zero-curvature limit of $(A)dS^3$

→ Isometry algebra $SO(4)$ contracts to $ISO(3)$ 3D Poincaré/Euclidean

Contraction limit $sl(2) \times sl(2) \to sl(2) \ltimes \mathbb{C}^3$

basis transformation:

(regular at finite ϵ
singular at $\epsilon = 0$)

$$M_1 = \frac{1}{\epsilon} P \qquad M_2 = L - \frac{1}{\epsilon} P \quad \Longleftrightarrow \quad L = M_1 + M_2 \qquad P = \epsilon M_1$$

algebra:

$$[M_1, M_1] \sim M_1 \qquad [M_1, M_2] = 0 \qquad [M_2, M_2] \sim M_2$$

$$\Longleftrightarrow$$

$$[L, L] \sim M_1 + M_2 \sim L$$
$$[L, P] \sim \epsilon M_1 \sim P$$
$$[P, P] \sim \epsilon^2 M_1 \sim \epsilon P \to 0$$

Poincaré/Euclidean algebra

Contraction of Bi-Algebra Structure

Apply Contraction to Bi-Algebra Structure:

$$r_1 = \frac{1}{\epsilon^2}\left(-P^0 \otimes P^0 + P^+ \otimes P^-\right)$$

somewhat boring

$$r_2 = \frac{1}{\epsilon^2}\left(-P^0 \otimes P^0 + P^+ \otimes P^-\right) - \frac{1}{\epsilon}\left(-L^0 \otimes P^0 - P^0 \otimes L^0 + L^+ \otimes P^- + P^+ \otimes L^-\right) + \dots$$

more interesting combination in contraction limit:

$$r = \epsilon r_1 - \epsilon r_2 + \xi \epsilon^2 r_1$$

$$= -L^0 \otimes P^0 - P^0 \otimes L^0 + L^+ \otimes P^- + P^+ \otimes L^- - \xi P^0 \otimes P^0 + \xi P^+ \otimes P^- + O(\epsilon)$$

- satisfies classical Yang-Baxter Equation for arbitrary ξ
- two independent parameters (overall factor $+\xi$); same as $sl(2) \times sl(2)$.

co-brackets: $\delta(L^0) = \delta(P^0) = 0$ (classical) kappa-Poincaré

$$\delta(L^\pm) = L^0 \wedge P^\pm + P^0 \wedge L^\pm + \xi P^0 \wedge P^\pm \qquad \delta(P^\pm) = P^0 \wedge P^\pm$$

Contractions in Quantum Algebra

Can apply contraction to $U_{\hbar\epsilon}(sl(2)) \otimes U_{\hbar\tilde{\epsilon}}(sl(2))$ with $\tilde{\epsilon} = -\epsilon + \xi\epsilon^2$.

Algebra + Co-Algebra have a consistent contraction limit

→ kappa-deformed Poincaré with additional ξ-deformation.

Universal R-Matrix: q-exponential (q-dilog)

$$R_{sl(2)} \sim \exp_{\epsilon\hbar}\left(\epsilon J^+ \otimes J^-\right) \exp\left(-\epsilon J^0 \otimes J^0\right)$$

Combination of two $R_{sl(2)}$ has consistent contraction limit

plain dilog

$$\left.\begin{aligned} R \sim &\exp\left[\xi \, \mathrm{Li}_2(P^+ \otimes P^-) + \xi \log\left(1 - P^+ \otimes P^-\right)\right] \\ &\cdot \exp\left[\left(L^+ \otimes P^- + P^+ \otimes L^-\right) \frac{\log\left(1 - P^+ \otimes P^-\right)}{P^+ \otimes P^-}\right] \\ &\cdot \exp\left[-L^0 \otimes P^0 - P^0 \otimes L^0 - \xi P^0 \otimes P^0\right] \end{aligned}\right\} \text{R-Matrix for 3D kappa-Poincaré}$$

$$\frac{\log(1-x)}{x}$$

Note: • Factorisation $R_{sl(2)} \otimes R_{sl(2)}$ lost! • Can supersymmetrise!

with Egor Im

Reduction of Poincaré Bi-Algebra

Can "reduce" 3+3 $\mathfrak{sl}(2) \ltimes \mathbb{C}^3$ to 1+1 $\mathfrak{gl}(1) \ltimes \mathbb{C}$: L, P

- boring : $[L, P] = 0$, $\delta(L) = \delta(P) = 0$, $r \sim L \otimes P$
- r-matrix / bi-algebra structure reduces "nicely"
- can be extended by $\mathfrak{psu}(2|2)$ supersymmetry to non-standard $\mathfrak{gl}(1) \ltimes \mathfrak{psu}(2|2) \ltimes \mathbb{C}$

First Step: Twist r-matrix / co-algebra

$$r = r_0 - L^0 \wedge P^0 = -2 L^0 \otimes P^0 + L^+ \otimes P^- + P^+ \otimes L^- - \xi P^0 \otimes P^0 + \xi P^+ \otimes P^-$$

valid twist: $\delta_0(L^0) = \delta_0(P^0) = [L^0, P^0] = 0$

co-algebra: $\delta(P^0) = \delta(P^+) = 0$, $\delta(P^-) = 2 P^0 \wedge P^-$,

$\delta(L^0) = 0$, $\delta(L^-) = 2 L^0 \wedge P^- + 2 P^0 \wedge L^- + \xi P^0 \wedge P^-$, $\delta(L^+) = \xi P^0 \wedge P^+$

Reduction of Poincaré Bi-Algebra (cont)

Second Step: Subalgebra of Rotations

- Define 1D subalgebra spanned by $L := L^0 + \beta^- L^+$
- algebra
$$\left.\begin{aligned} [L, P^+] &= P^+ \\ [L, P^0] &= -\beta^- P^+ \\ [L, P^-] &= -P^- - 2\beta^- P^0 \end{aligned}\right\}$$
new ideal spanned by:
$$P^+, \quad P^- + 2\beta^- P^0$$

Third Step: Divide out Ideal of Momenta

- set $P^+ = 0$, $P^0 =: P$, $P^- = -2\beta^- P \Rightarrow [L, P] = 0$
- r-matrix reduces "nicely" to $r = -2 L \otimes P - \xi\, P \otimes P$

$$r = -2(\underbrace{L^0 + \beta^- L^+}_{\text{sub-alg}}) \otimes P^0 + \underbrace{L^+}_{\text{non-sub}} \otimes (\underbrace{P^- + 2\beta^- P^0}_{\text{ideal}}) + P^+ \otimes \underbrace{L^-}_{\text{non-sub}} - \xi \underbrace{P^0 \otimes P^0}_{\text{sub-alg}} + \xi\, P^+ \otimes \underbrace{P^-}_{\text{ideal}}$$

- CYBE follows; r-matrix reduces on ideal subspace of modules...

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Towards AdS/CFT Worldsheet S-Matrix

Work in Progress:

- Extend to Supersymmetry
- Extend to Loop/Affine Algebras (rational/trigonometric)
- Lift to Quantum (Affine) Algebra
- Apply to Representations (Poincaré: Fields / Momentum Space)
- Adjust Reduction for Loop Algebras (interpretation of reduction?)

$\Rightarrow$ Find Worldsheet S-Matrix and corresponding Quantum Algebra

- as contraction/reduction of Quantum Affine $D(2,1;\epsilon)$
- understand affine structures, Lorentz boost
- algebraic insights into 3D momentum space for AdS/CFT magnons?

Reduction of Poincaré Loop Algebra

rational : twisted

$$r = \frac{L\otimes P + P\otimes L + \xi\, P\otimes P}{u_1 - u_2} - L^0 \wedge P^+$$

sub-algebra : $L := L^0 - \frac{1}{2} u^{-1} L^+ - \frac{1}{2} u^{-1} L^-$.

non-homogeneous loop level!

ideal quotient: $P^0 =: P,\quad P^+ = u^{-1} P,\quad P^- = u^{-1} P$.

r-matrix reduces; CYBE follows:

$$r = -\frac{1}{u_1 - u_2}\left[\frac{u_1}{u_2} L\otimes P + \frac{u_2}{u_1} P\otimes L + \left(1 - \frac{1}{u_1 u_2}\right) \xi\, P\otimes P\right]$$

- extra dependence on loop level
- not of difference form
- analogous reduction for trigonometric r-matrix
- can be supersymmetrised and quantised ...