Thermodynamic limits of fishnet graphs with various boundary conditions

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LPENS

Randomness, Integrability and Universality
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Based on works
with L.Dixon, G.Ferrando, V.Kazakov,
D.Kosower, A.Krajenbrink, D.l.Zhong

Motivation

Understand dynamics of planar graphs and relation to sigma models

['t Hooft]
[Polyakov]
[Maldacena]

Primary target: N=4 SYM in 4d

1. String dual - conjecturally - known
2. Theory - conjecturally - integrable (exact methods for planar graphs)

Hard to be rigorous / trace back origin of integrability

But huge progress at "bootstrapping" a solution

Simpler examples: fishnet theories

[Zamolodchikov'80]
[Gurdogan,Kazakov'15]
[many recent studies]

Huge family of planar QFTs without SUSY but with conformal symmetry and integrability in planar limit

Simple "graphical" content: **fishnet graphs** with regular lattice structures

Related to conformal spin chains, sigma models, deformations of SUSY theories

Hope to be rigorous here and draw lessons for more sophisticated theories

4d fishnet theory

A theory for matrix scalar fields with quartic coupling [Gurdogan,Kazakov'15]

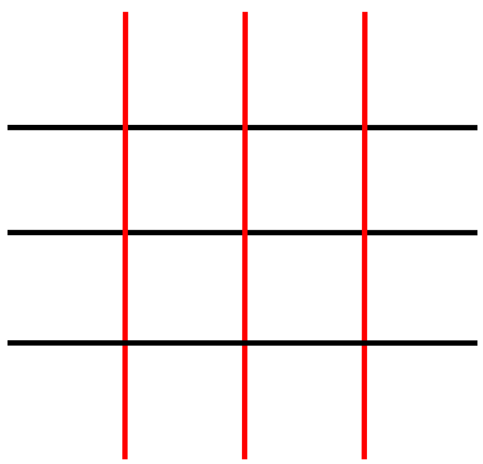
$$\mathcal{L}_{\text{fishnet}} = \text{tr}\ \partial_\mu \phi_1 \partial_\mu \phi_1^\dagger + \partial_\mu \phi_2 \partial_\mu \phi_2^\dagger - (4\pi g)^2 \phi_1 \phi_2 \phi_1^\dagger \phi_2^\dagger$$

Baby version of N = 4 SYM - related to it by certain deformation
[Caetano,Gurdogan,Kazakov'16]

Planar graphs all look the same

Edge: massless propagator in position space $\dfrac{1}{(x-y)^2}$

Vertex: integration point $\displaystyle\int \frac{d^4x}{\pi^2}$



bulk of any graph

Different observables correspond to different BC for external legs [Zamolodchikov'80] [Isaev'03]

Integrability arises from quartic vertex in d=4 [Gromov,Kazakov,Korchemsky,Negro,Sizov'17] [Chicherin,Kazakov,Loebbert,Muller,Zhong'16]

Thermodynamic limit

What about smooth limit and relation to string world-sheet?

Hard to follow string duality all the way from N=4 theory to fishnet theory

Deformation procedure sends the YM coupling to zero (**highly curved AdS background for string)**

Important observation concerning large fishnets [Zamolodchikov'80]

Volume 97B, number 1 PHYSICS LETTERS 17 November 1980

"FISHING-NET" DIAGRAMS AS A COMPLETELY INTEGRABLE SYSTEM

A.B. ZAMOLODCHIKOV
The Academy of Sciences of the USSR, L.D. Landau Institute for Theoretical Physics, Chernogolovka, USSR

Received 29 July 1980

The "fishing-net" planar Feynman diagrams with massless scalar propagators are shown to be equivalent to some completely integrable lattice statistical system. The infinite-volume partition function for this system is computed exactly.

Thermodynamic limit

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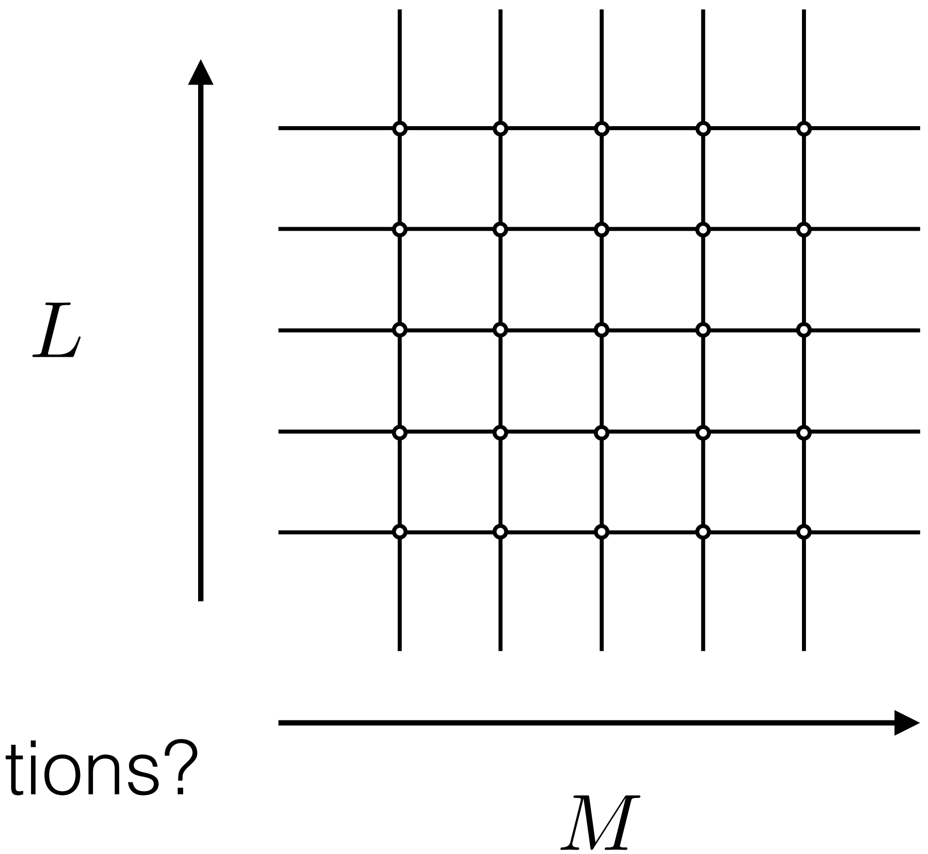
Important observation concerning large fishnets [Zamolodchikov'80]

Thermodynamic scaling $L, M \to \infty$

$$\ln Z_{L\times M} = -LM \ln g_c^2$$

$$g_c = \frac{\Gamma(3/4)}{\sqrt{\pi}\Gamma(5/4)} = 0.7...$$



Critical coupling: graphs become "dense"

Q1: How does that depend on Boundary Conditions?

Q2: 2d Effective Field Theory of large fishnet graphs?

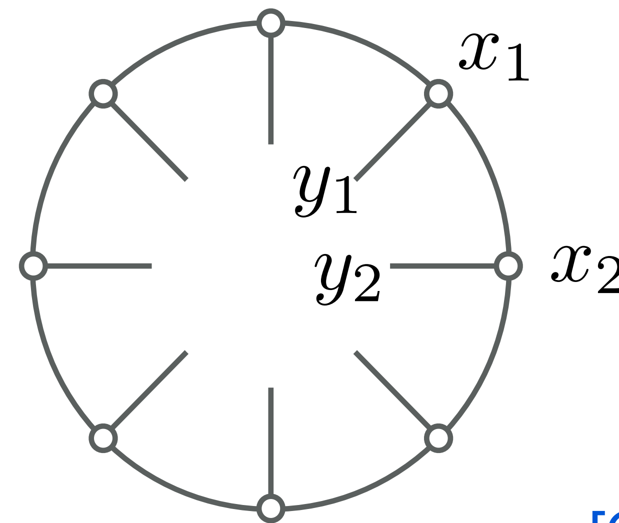
Periodic Fishnets

PBC

Closed-string channel

Graph building operator B_L

Integral operator acting on functions of L points subject to PBC



massless propagator

[Gromov,Kazakov,Korchemsky,Negro,Sizov'17]

$$B_L \cdot \psi(x_1, \ldots, x_L) = \int \prod_{i=1}^{L} \frac{d^4 y_i}{\pi^2 (x_i - x_{i+1})^2 (x_i - y_i)^2} \psi(y_1, \ldots, y_L)$$

with $x_{L+1} = x_1$

Commute with conformal symmetry generators $\in SO(1,5)$

Action on a free scalar field

Lorentz $M_{\mu\nu} = x_u \partial_\nu - x_\nu \partial_\mu$ Translation $P_\mu = \partial_\mu$

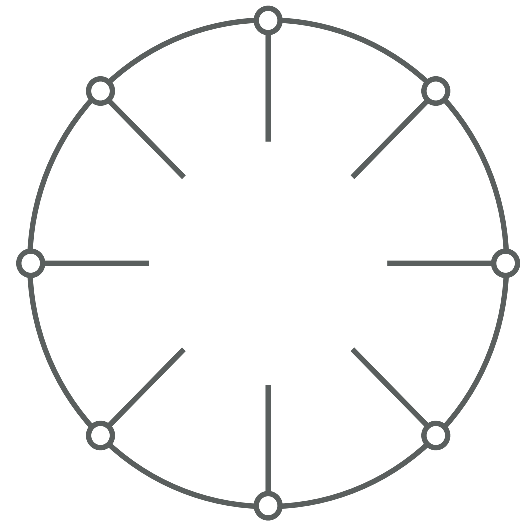
Dilatation $D = (x\partial) + 1$ Conformal boost $K_\mu = 2x_\mu (x\partial) - x^2 \partial_\mu + 2x_\mu$

PBC

Closed-string channel

Graph building operator B_L

Integral operator acting on functions of L points subject to PBC



[Gromov,Kazakov,Korchemsky,Negro,Sizov'17]

$$B_L \cdot \psi(x_1, \ldots, x_L) = \int \prod_{i=1}^{L} \frac{d^4 y_i}{\pi^2 (x_i - x_{i+1})^2 (x_i - y_i)^2} \psi(y_1, \ldots, y_L)$$

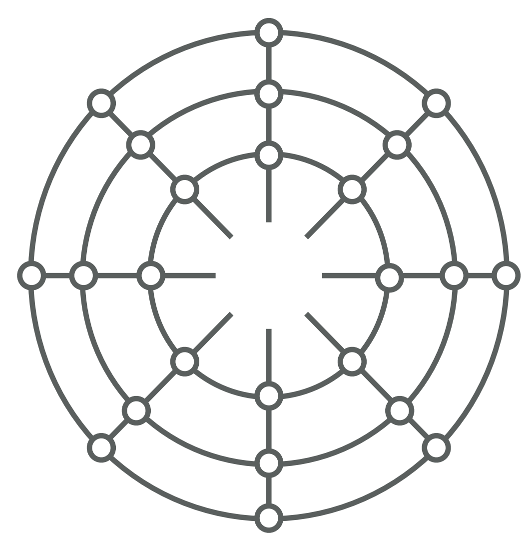
with $x_{L+1} = x_1$

Fishnet graphs have iterative structure and can be generated by iterating B

E.g. Graph with M wheels

$(B_L)^M$

$M \times L$ fishnet



Thermodynamic limit
Study large L and large M

Spectrum

[Gromov,Kazakov,Korchemsky,Negro,Sizov'17]
[Grabner,Gromov,Kazakov,Korchemsky'17]
[Gromov,Sever'19'20]

Eigenvalue problem

Diagonalize B_L & conformal generators simultaneously

Eigenvalues labelled with scaling dimension Δ and spins

Global symmetry enough to diagonalise L=2 - Bigger lengths requires **integrability**

Key: operator B falls inside a family of commuting conserved charges

Difficulty: Must cope with "unusual" Hilbert space, associated to "wild" infinite dimensional representations of the conformal group (from complementary or principal series)

No Bethe ansatz - must deal with more sophisticated methods
(e.g. SoV for higher-rank models)

Spectrum of local single-trace operators

Solution of $1 = g^{2L} B_L(\Delta)$

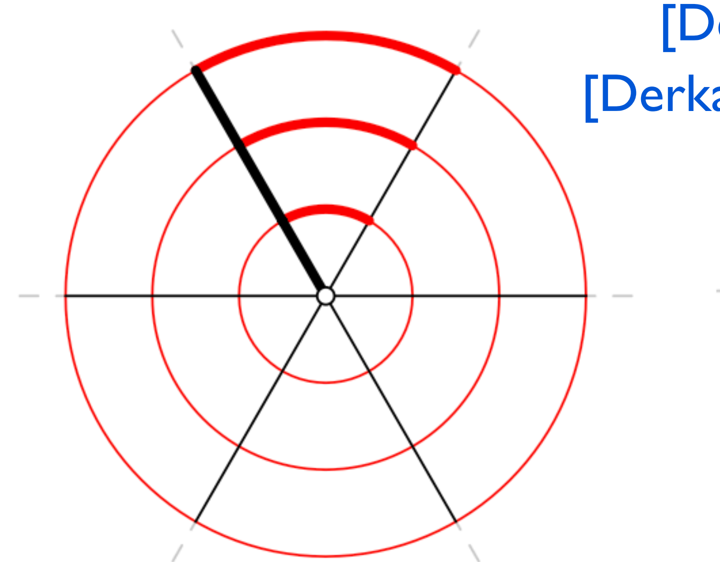
Logic is to solve for arbitrary dimension and then impose above quantization: $\Delta = \Delta_L(g^2)$

TBA strategy

[Derkachov,Kazakov,Olivucci'18]
[BB,Ferrando,Kazakov,Zhong'19]
[Derkachov,Olivucci'19'20'21]
[Derkachov,Ferrando,Olivucci'21]

Open-string channel

Insight from N=4 SYM solution
Switch graph building operator Γ_M



Still an integral operator but open BC

Finite time evolution in 'angular direction'

Symmetries: fix two points (origin and infinity), left with dilatation and Lorentz

(subgroup of residual symmetry of N=4 SYM magnon irreps)

Excitations classified accordingly

Magnon = 3-sphere Harmonics carrying momentum

$$\Gamma_M = \prod_{i=1}^{M} e^{-\epsilon_i}$$

Energy is sum of individual energies of these magnons
Wave functions give us access to their factorised S-matrix
Enough to formulate TBA equations to determine the scaling dimensions

TBA strategy

Goal: calculate grand canonical partition function

$$Z_L(g^2) = \sum_{M=0}^{\infty} g^{2LM} (\Gamma_M)^L$$

Free energy density = scaling dimension $\Delta_L(g^2)$ of local operator $\mathcal{O}_L(x) \sim \mathrm{tr}\, \phi_1(x)^L$

TBA equations

[Yang,Yang'60s]
[Zamolodchikov'90s]

$$\log Y_a(u) = Lh - L\epsilon_a(u) + \sum_b \mathcal{K}_{ab} * \log\left(1 + Y_b(u)\right) + \ldots$$

1) **Coupling constant** enters as chemical potential $h = \log g^2$

2) Length **L** of chain acts as inverse temperature

Solution to TBA determines the free energy = scaling dimension

$$\Delta = L - 2 \sum_a \int \frac{du}{2\pi} \log\left(1 + Y_a(u)\right)$$

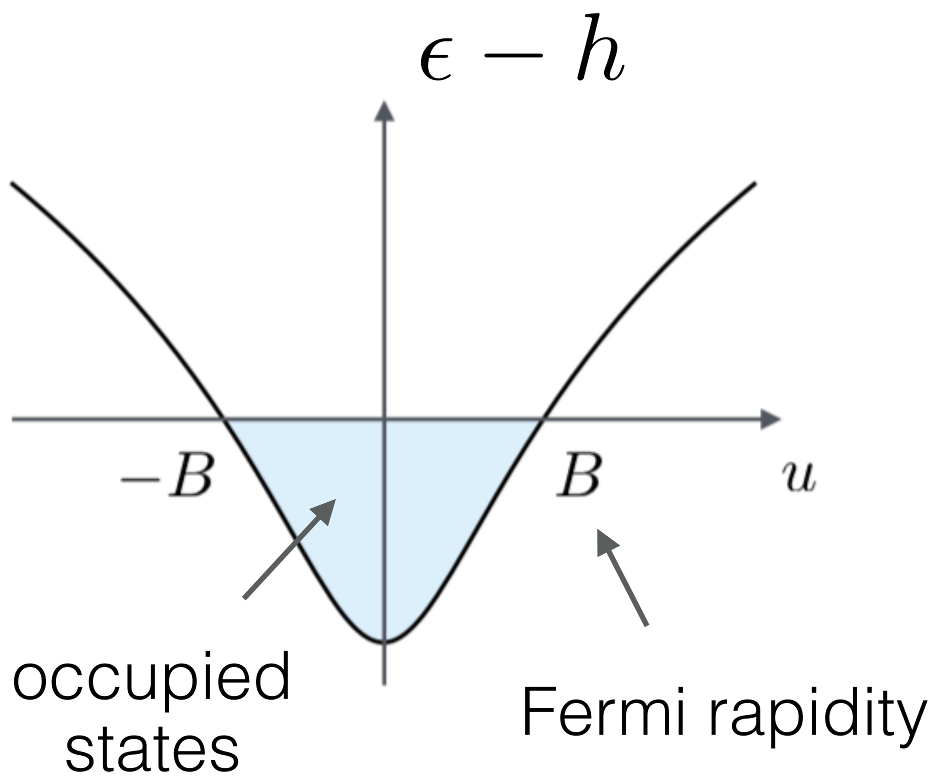
Thermodynamic limit

Thermodynamic limit $L \to \infty$

[BB,Zhong'18]
[BB,Ferrando,Kazakov,Zhong'19]

Lightest modes (s-wave, a=1) dominate in the thermodynamic limit

A **Fermi sea** forms: all states below the Fermi rapidity B are filled



Increasing coupling amounts to increasing B

Dense fishnet limit corresponds to Fermi sea extending across all rapidity axis

The value of the coupling for which it is realized is Zamolodchikov critical coupling g_c

Analysis

General: solve linear integral equation for the distribution of energy levels

$$\chi(u) = C - \epsilon(u) + \int_{-B}^{B} \frac{du}{2\pi} \mathcal{K}(u-v)\chi(v)$$

With $\chi(u = \pm B) = 0$

$$C = \log g^2 - \int_{-B}^{B} \frac{du}{2\pi} k(u)\chi(u) \qquad \text{and} \qquad \Delta/L = 1 - \int_{-B}^{B} \frac{du}{\pi}\chi(u)$$

Critical regime: (maximal density of wheels)

(i) Scaling function vanishes $f = \Delta/L \to 0$

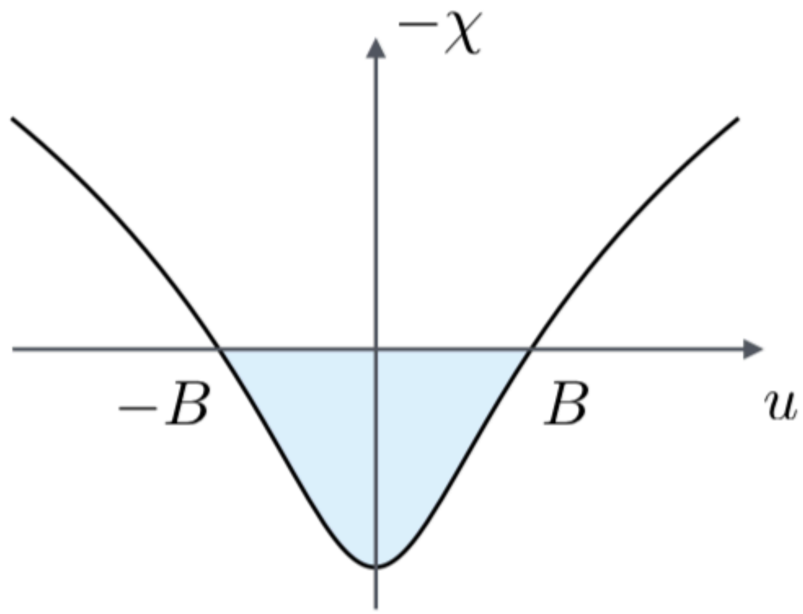
(ii) Chemical potential (coupling) approaches predicted value

$$\chi_{cr} \sim e^{-|\theta|} \quad \Rightarrow \quad C_{cr} = 0 \quad \Rightarrow \quad g_c = \Gamma(3/4)/\sqrt{\pi}\Gamma(5/4)$$

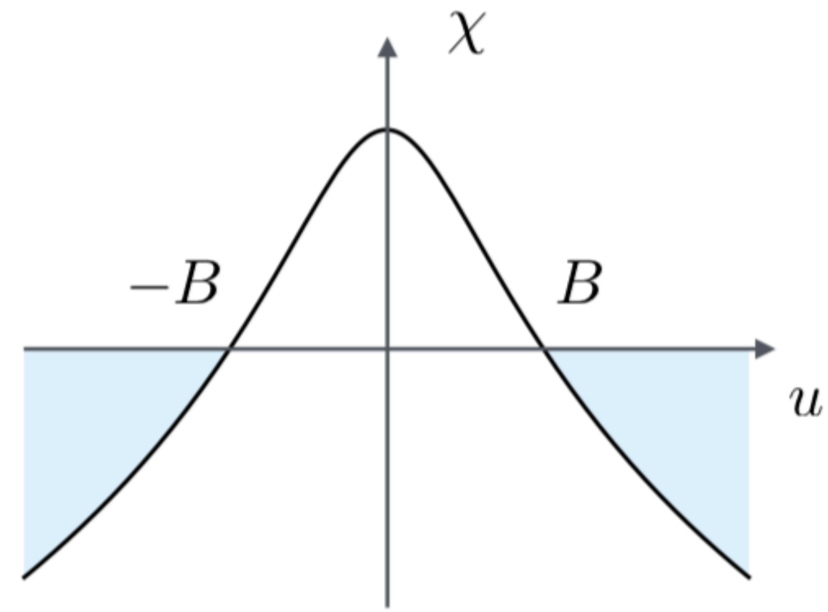
(Zamolodchikov result for 4d regular square lattice)

Duality transformation

Particle-hole transformation (similar to map between ferro and anti-ferro)



Fermi sea of magnons



Dual sea

Dual equation

1) dualize kernel

$$K = -\frac{\mathcal{K}}{1-\mathcal{K}*} = -\mathcal{K} - \mathcal{K} * \mathcal{K} - \dots$$

2) act on both sides of the equation with $1 - K*$

Sigma model picture

Dual equation: $\chi(\theta) = E(\theta) + \int\limits_{\theta^2 > B^2} \frac{d\theta'}{2\pi} K(\theta - \theta')\chi(\theta')$

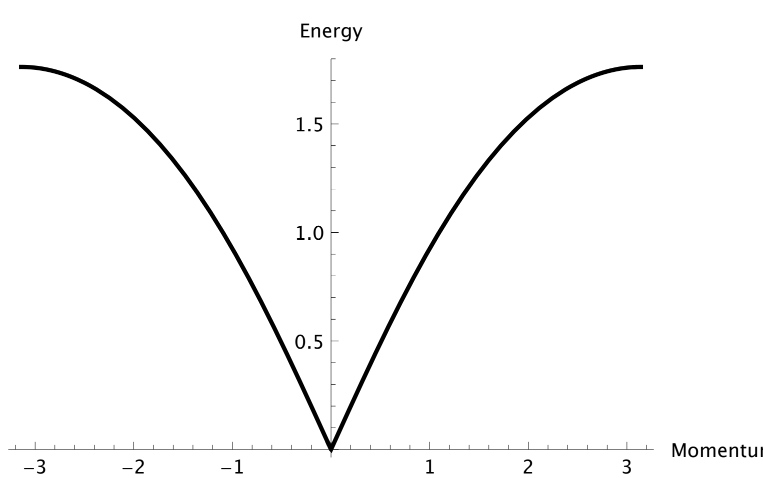
Dual energy: $\log g^2 = \log g_c^2 + \int\limits_{\theta^2 > B^2} \frac{d\theta}{2\pi} P'(\theta)\chi(\theta)$

No chemical potential $\chi(\theta) \sim -2\rho \log\theta$ $\rho = \Delta/L =$ charge density

1) **Kernel**: $K = -i\partial_\theta \log S_{O(6)}$ [Zamolodchikov&Zamolodchikov'78]

Particles scatter as in 2d O(6) non-linear sigma model

2) **Dispersion relation**: $\sinh^2(\frac{1}{2}E) = \sin^2(\frac{1}{2}P)$



But they are gapless (unlike in O(6) model)

Interpretation: sigma model on pseudo-sphere
Dual = integrable lattice version of AdS_5 sigma model

Graph building operator

Duality exchanges energy and chemical potential

Dual energy $= L \log g^2$

It describes eigenvalues of the closed string operator B_L

$$-\log B_L = L \log g_c^2 + H_{AdS}$$

Hamiltonian of 2d sigma model up to vacuum energy density $\log g_c^2$

Dual TBA formula calculates energy levels of H in finite volume L

$$\log Y_1 = LE - K_{O(6)} * \log\left(1 + 1/Y_1\right) + \ldots$$

$$H = -\int \frac{d\theta}{2\pi} P'(\theta) \log\left(1 + 1/Y_1\right)$$

Good description (sigma model is weakly coupled) when length is large and energy is small

With Δ entering in the asymptotics of Y function

Open Fishnets

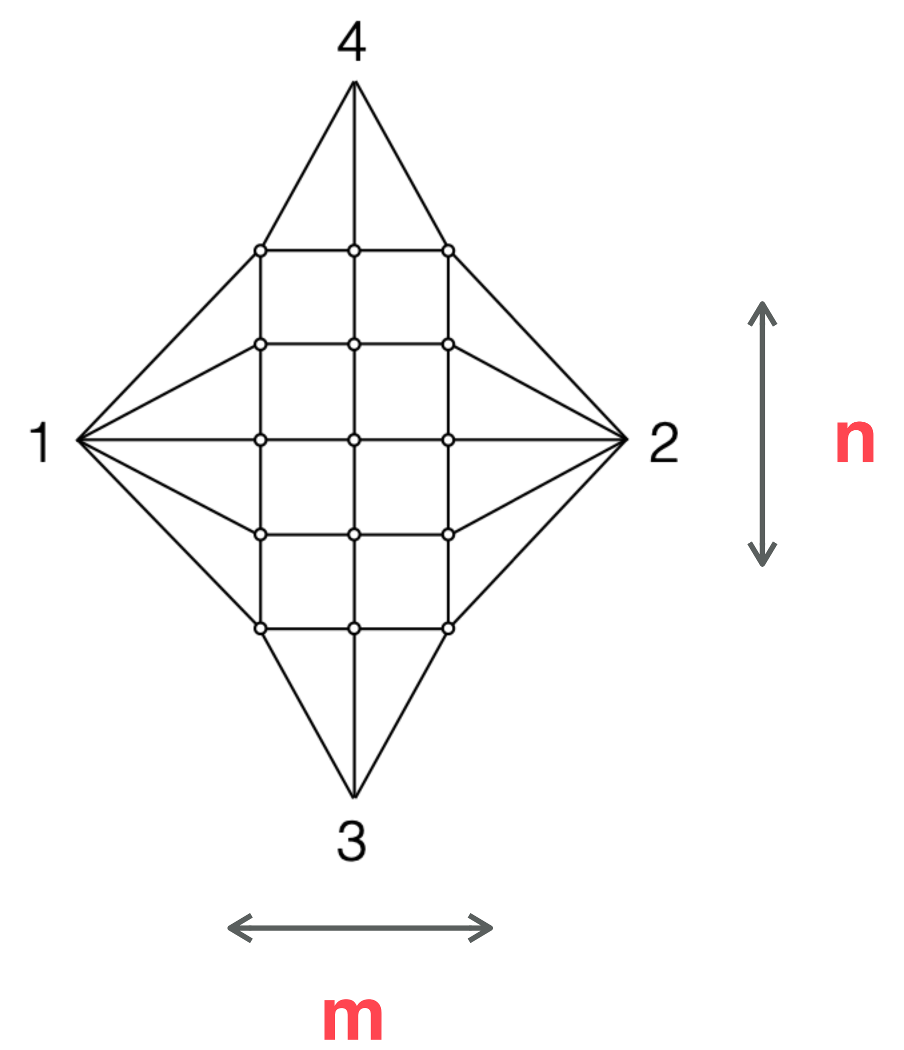
Open Boundary Conditions

4pt function

Attach **n** horizontal lines at spacetime point 1 and 2

Attach **m** vertical lines at spacetime point 3 and 4

*Function of 2 integers (**m**&**n**)
*Function of 2 cross ratios



$$u = \frac{x_{14}^2 x_{23}^2}{x_{12}^2 x_{34}^2} = \frac{z\bar{z}}{(1-z)(1-\bar{z})}$$

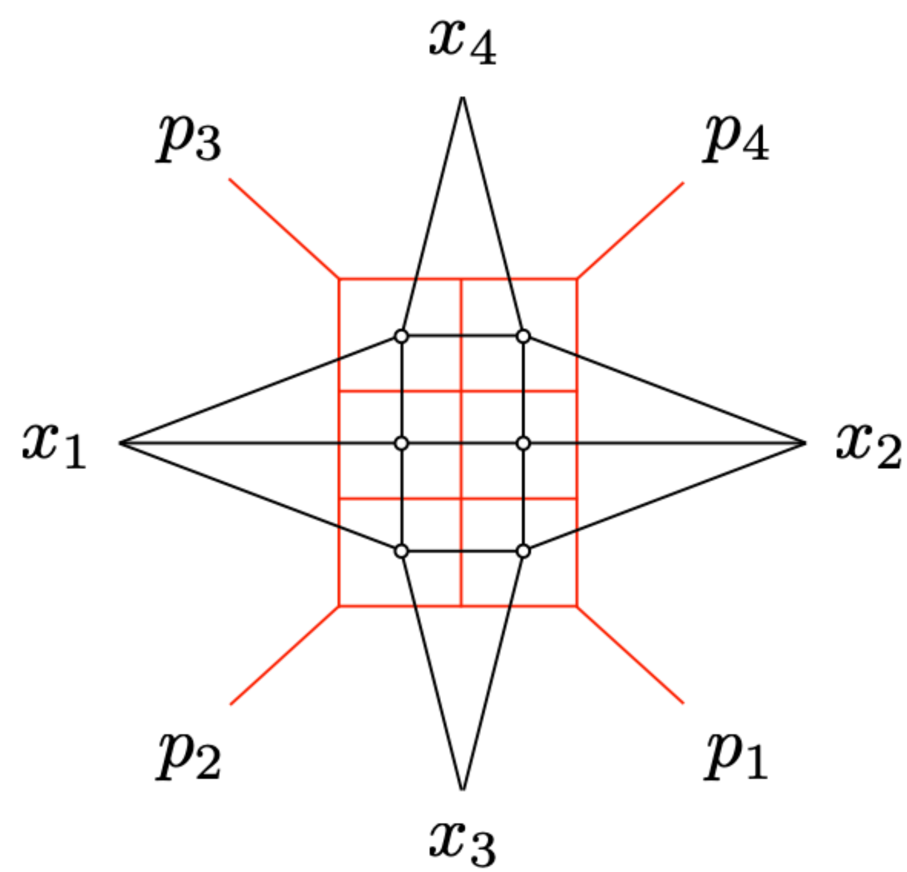
$$v = \frac{x_{13}^2 x_{24}^2}{x_{12}^2 x_{34}^2} = \frac{1}{(1-z)(1-\bar{z})}$$

Shortcut

1) Analyticity

[BB,Dixon'17]

Dual graph = Amplitude (subject to stringent analytical constraints, such as Steinmann relations for double discontinuity)



Dual momenta

$$p_1 = x_2 - x_3$$
$$p_2 = x_3 - x_1$$
$$\ldots$$

2) Integrability

Determinant Hankel matrix

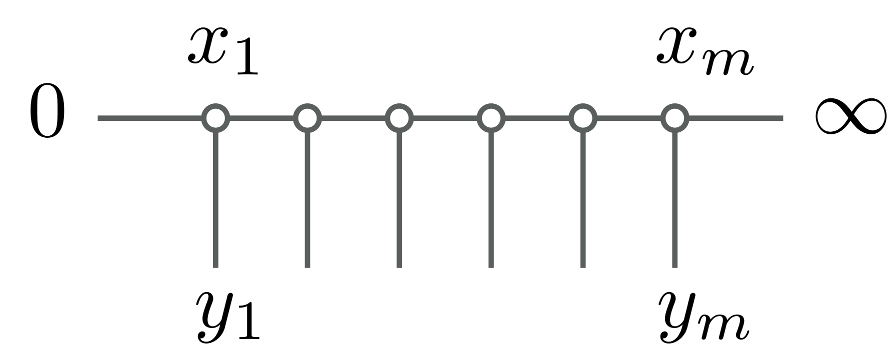
$$I_{m,n} = \frac{1}{\mathcal{N}} \det_{1\leqslant i,j\leqslant m} M_{i+j+n-m-1}$$

Entries given by Ladders (m=1, n=p) $\qquad M_p = p!(p-1)!\ L_p(z,\bar{z})$

with $\quad L_p(z,\bar{z}) = \sum_{j=p}^{2p} \frac{j![-\ln(z\bar{z})]^{2p-j}}{p!(j-p)!(2p-j)!}[\mathrm{Li}_j(z) - \mathrm{Li}_j(\bar{z})]$ $\qquad$ [Usyukina,Davydychev'93]

Rigorous approach

Factorization
Using open graph building operator



$$\Gamma_m \cdot \psi(x_1, \ldots, x_m) = \int \prod_{i=1}^{m} \frac{dy_i}{\pi^2 (x_{i-1} - x_i)^2 (x_i - y_i)^2} \psi(y_1, \ldots, y_m)$$

with $x_0 = 0$

Wave functions ψ_m can be constructed exactly:
(states labelled by rapidities and Harmonics on 3-sphere)

[Derkachov,Kazakov,Olivucci'18]
[BB,Ferrando,Kazakov,Zhong'19]
[Derkachov,Olivucci'19]
[Fleury,Komatsu'16]
[BB,Dixon'17]

Completeness -> concise matrix-model like representation for correlator

$$I_{m,n} = \langle \mathrm{out} | (\Gamma_m)^n | \mathrm{in} \rangle = \int_{\mathcal{H}_m} d\mu(\psi_m) \lambda(\psi_m)^n$$

Integral representation

Massaging the result yields integral representation

$$I_{m,n} = \frac{1}{2^m m! \mathcal{N}} \prod_{i=1}^{m} \int_{|\sigma|}^{\infty} \frac{dx_i x_i (x_i^2 - \sigma^2)^{n-m}}{\cosh \frac{1}{2}(x_i + \varphi) \cosh \frac{1}{2}(x_i - \varphi)} \prod_{i<j}^{m} (x_i^2 - x_j^2)^2$$

With cross ratios parametrized as $z = -e^{\sigma + \varphi}$ $\bar{z} = -e^{\sigma - \varphi}$

Determinant follows from standard manipulation of Vandermonde interaction

Nice form for thermodynamic (large mn) limit

Thermodynamic limit

Large mn : Electrostatic equilibrium in potential $V \approx \frac{m}{n-m} \log x^2 + \frac{|x|}{m}$

Roots (x's) scale large, cross ratios (z,zb) go away

log (repulsive) linear (confining)

Upon rescaling, get well-known singular equation for density

$$0 = \frac{1}{x} - 2\pi + \int_a^b \frac{4x dy \rho(y)}{x^2 - y^2}$$

Seen before in matrix models, classical limit of Bethe equations of SL(2) spin chain, classical string in AdS, …

Solution given in terms of elliptic integrals [Beisert,Minahan,Staudacher,Zarembo'03]

Goal : calculate free energy density and compare with PBC

Tedious but doable - must use several tricks (differential equations)

Solution I

Free energy density $F = \lim_{m,n \to \infty} \frac{\ln I_{m,n}}{mn}$ [BB,Dixon,Kosower,Krajenbrink,Zhong'21]

Surprisingly (or not) F depends on the ratio of lengths of the fishnet

$$k = n/m \quad \in (1, \infty)$$

Close parametric solution in terms of elliptic integrals E&K

$$F = \ln \pi^2 + k \ln \frac{1+\sqrt{1-q}}{2} + \frac{(k-1)^2}{2k} \ln K(q) + \frac{1}{k} \ln \frac{1-\sqrt{1-q}}{2} - \frac{(k+1)^2}{2k} \ln E(q)$$

where

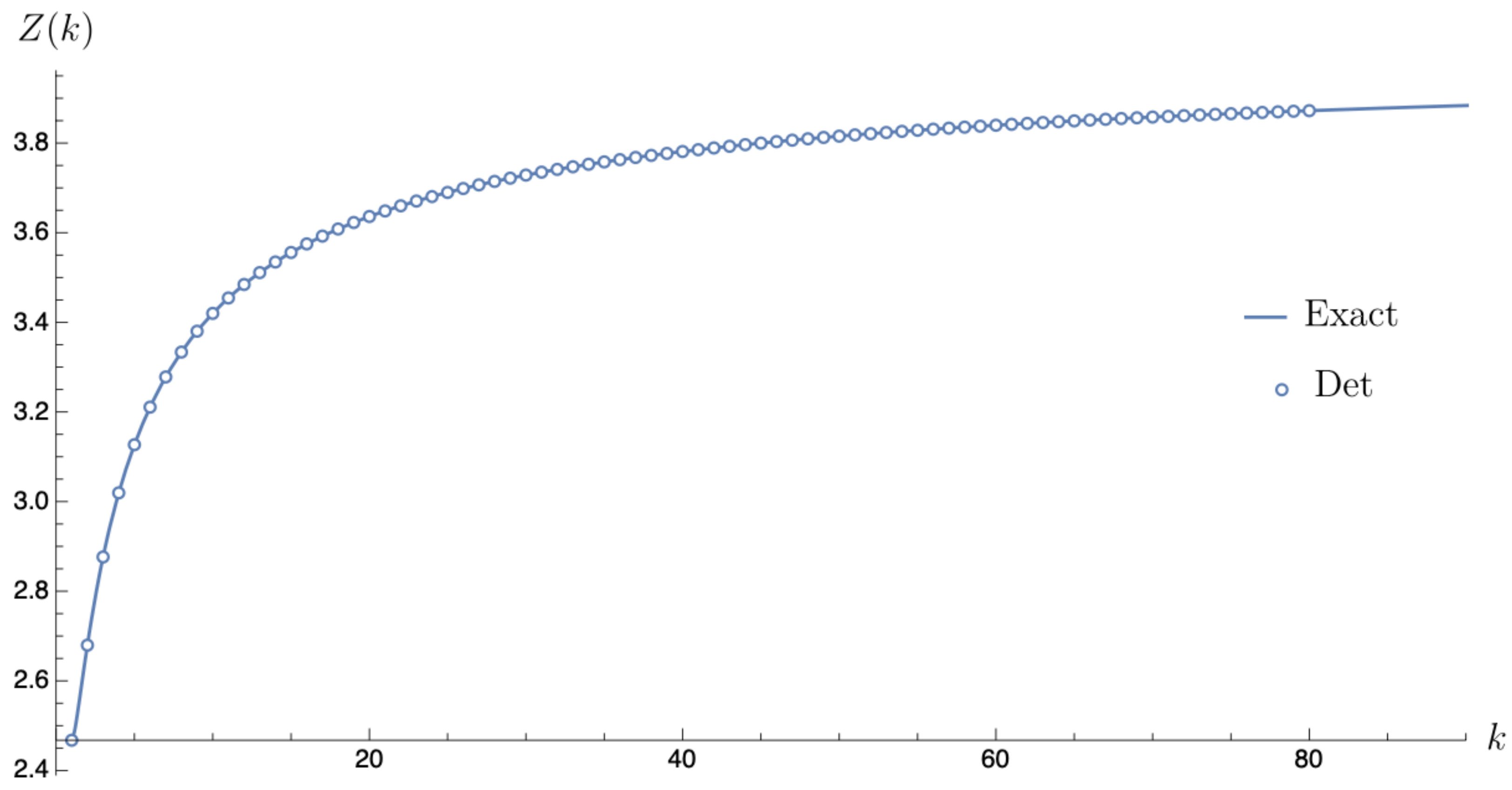
$$k = \frac{E(q) + \sqrt{1-q}K(q)}{E(q) - \sqrt{1-q}K(q)}$$

Solution II

Explicit expressions may be found in particular limits $k \to 1 \text{ or } \infty$

Numerical plot for other values
(Compare well to direct extrapolation of the determinant)

$$Z = e^F$$



But no matter how we look at it (analytically or numerically)
the result appears everywhere different from the free energy for PBC

Thermodynamic puzzle

Should the thermodynamic limit not be universal (same for all BCs)?
Presumably, for local enough systems and regular enough BCs

Similar phenomena have been seen before and many 2d lattice models exhibit strong sensitivity to BCs

E.g.: 6-vertex model shows different thermodynamic behaviours for PBC or DWBC (Domain Wall BC)

Interpretation: two different macroscopic phases close and far from boundaries

Analogy mathematically convincing
(PBC -> TBA vs DWBC -> Determinant)

Physically, operators (corners) carry large dimensions, possibly invalidating the low energy (sigma model) description there

Can we still have a continuum description across the entire graphs?

Connection with fish-chain model in AdS?

[Gromov,Sever'19]

Other limits

Refine scalings such as to capture some dependence on cross ratios in the thermodynamic limit; in particular **short-distance** limit $|\sigma|, m, n \to \infty$

$$0 = \frac{x}{x^2 - \xi^2} - 2\pi + \int_a^b \frac{4x dy \rho(y)}{x^2 - y^2} \qquad \xi = |\sigma|/m$$

Deformed potential - here again answer very different from PBC

Remark: equation same as for a folded string spinning in $AdS_3 \times S^1$

[Kazakov,Zarembo'04]
[Casteill,Kristjansen'07]

Mathematical coincidence ….
…. or hint at a world-sheet description in the "frozen phase"?

Recent studies introduce stochastic methods to describe behaviours of certain Feynman graphs / planar correlators in short distance limits

[Olivucci,Vieira'21]

It may shed light on the distribution of free energy across large fishnets and detect the various phases, as in Aztec diamonds / vertex models analysis

Conclusion

Planar graphs form nice arena for connecting several
aspects of integrability (e.g. spin chains and sigma models)

Also connected to lattice integrable models - especially fishnet graphs

Show strong sensitivity on BCs in thermodynamic limit

How universal is it? What’s the general behaviour for general graphs?

What does it imply for dual AdS sigma model description?

THANK YOU!