Pulling yourself up by your Bootstraps in Quantum Gravity

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Bootstrap

Find the space of *possible theories* just using consistency: Symmetries and Quantum Mechanics

From Wikipedia: “A self-starting process that is supposed to proceed without external input”
CFTs are quantum field theories invariant under angle-preserving transformations of space (or spacetime).

They are central objects in physics. Mathematically rigid structures.
Conformal Bootstrap

Compile a catalogue of all consistent CFTs in various dimensions

Early 70s: Ferrara Gatto Grillo, Polyakov
bootstrap axioms

Early 80s: Belavin Polyakov Zamolodchikov
exact solution of some 2d CFTs, e.g. 2d Ising

Modern era (since 2008): Rattazzi Rychkov Tonni Vichi
Carve out theory space by the linear functional method
The bootstrap oracle

*Not anything goes* in CFT!

Tentative CFT data are constrained by rigorous bootstrap inequalities

Figure credit: S. Rychkov
The $\lambda$-point experiment

Specific heat $C \sim |T - T_c|^\alpha$ with $\alpha = \frac{2\Delta_s - 3}{3 - \Delta_s}$

Microgravity experiment in space (1992): $8\sigma$ discrepancy with Montecarlo
Our vision: the Bootstrap

Determining Theory Space using Consistency: Symmetries & Quantum Mechanics.

Sharp rigorous predictions without resorting to approximations.

Ultimate goal: compile a complete catalogue of consistent QFTs.

[Wikipedia: Bootstrap is “a self-starting process that is supposed to proceed without external input”

Chester, Landry, Liu, Poland, Simmons-Duffin, Su, Vichi]

[Chester, Landry, Liu, Poland, Simmons-Duffin, Su, Vichi]

Simons Collaboration on The Nonperturbative Bootstrap
Specific to 2d CFTs. Consider the theory at finite temperature $T = 1/\beta$

$$Z(\beta) = \sum_i e^{-\beta E_i} \quad \quad \quad Z(\beta) = Z\left(\frac{1}{\beta}\right)$$

Positivity and crossing: *not anything goes*

$$\sum_n \alpha_n \left(\frac{\partial}{\partial \beta}\right)^{2n+1} \bigg|_{\beta=1} \left[ \sum_i e^{-\beta E_i} - e^{-E_i/\beta} \right] = 0$$

Cleverly constructed *linear functional* constrains possible spectra $\{E_i\}$
Bootstrapping quantum gravity?

*Not anything goes* in quantum gravity!

Give a low-energy effective theory (particle content and interactions), does it admit a UV completion or is part of the *Swampland*?

\[
S = \frac{1}{8\pi G} \int d^D x \sqrt{g} \left( R + \alpha_2 R^2 + \ldots + (\nabla \phi)^2 + \lambda \phi^4 + g_2 (\nabla \phi)^4 + \ldots \right)
\]
Bootstrapping quantum gravity?

Bootstrap methods apply most directly to gravity in Anti de Sitter space

CFTs are holograms for quantum gravity theories in AdS
Black hole thermodynamics

The original hint for holography

\[
\text{Entropy} = \frac{A}{4G\hbar} = \frac{A}{4\ell_{\text{Planck}}^2}
\]

For M87 black hole, \( S \sim 10^{96} \)
What is the simplest theory of 3D gravity?

Unlike in 4D, no propagating bulk gravitons. There are however boundary gravitons.

\[ S = \frac{1}{8\pi G} \int d^3 x \sqrt{g} (R - \frac{1}{\ell^2}) + \ldots? \]

Is there a theory of "pure" gravity? Or do we need strings, supersymmetry, ...?

Black hole entropy \( \mathcal{C} \text{ Planck}^3 / G \), which is very large for weakly coupled gravity

By "pure" gravity we mean just boundary gravitons and black holes

\( c = \frac{3\mathcal{C}}{2G} \) is also the central charge (\# of degrees of freedom) of the dual CFT
Bootstrapping 3D gravity

Look for the theory with the highest possible $E_1$

For given $c$, $E_1 \leq E_{\text{gap}}(c)$, or else we cannot satisfy $Z(\beta) = Z(1/\beta)$

Hellerman
Numerical bounds

Afkhami-Jeddi, Hartman, Tajdini

Two numerical surprises at $c = 4$ and $c = 12$: spectrum and degeneracies converge to integers, for no good apparent reason.

E.g. for $c = 12$:

$$E_{\text{gap}} \leq 2 + 10^{-30} \quad \text{and} \quad n_1 = 196884 = 196883 + 1, \quad Z^{c=12} \rightarrow j(\tau) - 744$$

Modularity and monster symmetry! But why?

From $c \lesssim 2000$, extrapolated asymptotics $E_{\text{gap}}(c) \sim \frac{c}{9.08}$ as $c \rightarrow \infty$.

Today I will focus on the simplest bootstrap constraint for CFT: modularity.
Sphere Packing Problem

What is the densest configuration of identical spheres in $d$ dimensions?

Deep question, with connections to modular forms, number theory, cryptography, ... 

Solved only in $d = 1, 2, 3, 8, 24$.

In $d = 2$, honeycomb lattice (Toth, 1940)
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In $d = 8$, $E_8$ lattice.

In $d=24$, Leech lattice.
Solved a week later with the same technique.
Viazovska built on the Cohn-Elkies approach. A function $f(r)$ with certain positivity and Fourier properties gives an upper bound. Find the best $f(r)$ by linear optimization.
This amounts to a numerical proof in $d = 8, 24$. But an analytic proof was elusive.

Viazovska constructed analytically the “magic function” $f(r)$
Punchline

Cohn-Elkies $f(r) \equiv$ linear functional for the modular bootstrap $Z(\beta) = Z(1/\beta)!

More precisely, the Cohn-Elkies problem is equivalent to the modular bootstrap for $U(1)^c$

<table>
<thead>
<tr>
<th>Sphere packing</th>
<th>Modular bootstrap</th>
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</thead>
<tbody>
<tr>
<td>$d$</td>
<td>$c = \frac{d}{2}$</td>
</tr>
<tr>
<td>density $\rho$</td>
<td>$E_1 = \rho^{2ld}$</td>
</tr>
</tbody>
</table>

Viazovska `16

Mazáč`16
Bootstrapping graviton scattering?

Constrain $2 \rightarrow 2$ amplitude $\mathcal{M}(s, t)$ subject to analyticity, crossing, boundedness
Bounds in a weakly coupled low-energy theory of a scalar + gravity

\[ S = \frac{1}{8\pi G} \int d^D x \sqrt{g} \left( R + (\nabla \phi)^2 + \lambda \phi^4 + g_2 (\nabla^4 \phi^4) + g_3 (\nabla^6 \phi^4) + g_4 (\nabla^8 \phi^4) \ldots \right) \]

\[ g_2 \text{ allowed to be a bit negative thanks to gravitational time delay} \]
A bound in maximal sugra

\[ S = \frac{1}{8\pi G} \int d^D x \sqrt{g} \left( R + g_0 R^4 + \ldots \right) \]

0 \leq g_0 M^6 \leq 3.000. Compatible with type II string, \( g_0 M^6 = 2\zeta(3) \approx 2.40 \)
Bounds from graviton scattering in D=4

Caron-Huot Li Parra-Martinez Simmons-Duffin

(See also Chiang Huang Li Rodina Weng)
Simplest theory of 3D gravity? Still an open question

New insights bootstrap ↔ sphere packing
E.g. improved numerics for large $d$: $\rho \lesssim 2^{-0.6044d}$ Afkhami-Jeddi et al

Is there a more general relation between the constraints on both sides?
Outlook

- In asymptotic Minkowski, beginning of the quantum gravity S-matrix bootstrap
  Must make plausible physical assumptions

- In asymptotic AdS, a corner of the CFT bootstrap
  Fully rigorous

Proof that large $N$ CFTs with large gap have a local AdS dual, with sharp inequalities

Caron-Huot Mazáč LR Simmons-Duffin
MUST WE NOT THEN RENOUNCE THE OBJECT ALTOGETHER, THROW IT TO THE WINDS AND INSTEAD LAY BARE THE PURELY ABSTRACT?

VASILY KANDINSKY 1911