Long range spin chains from freezing

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work in progress w/ J. Lamers

see also J. Lamers's talk and
J. Lamers, V. Pasquier, D.S., arXiv:2004.13210

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Long-range interacting integrable models

- Long range integrable deformations of Heisenberg-like models are important for various applications (e.g. AdS/CFT, 2d CFT, condensed matter, stochastic processes, etc)

- Very few cases are completely understood

- The algebraic structure is rather rigid

- Unified framework for nearest-neighbour and long range interaction (e.g: separation of variables for long range spin chains)?

- Here: a new method to obtain the eigenfunctions of the isotropic Haldane-Shastry Hamiltonian which can be potentially extended to various deformations

The isotropic Haldane-Shastry Hamiltonian

[Haldane, 88; Shastry, 88]

N su(2) spins 1/2 on a circle with periodic boundary conditions $z_j \longmapsto \omega^j = \mathrm{e}^{2\pi \mathrm{i} j/N}$

$$\boxed{H_{\mathrm{HS}} = -\sum_{i \neq j} V(z_i, z_j)\; P_{ij}}$$

$$V(z_i, z_j) = \frac{z_i z_j}{(z_i - z_j)^2} = \frac{1}{\sin^2 \pi(i-j)/N}$$

$$P_{jk} = \frac{1}{2}\left(\sigma_j^a \sigma_k^a + 1\right)$$ spin permutation

also solvable for su(n) spins in fundamental representation

Yangian symmetry and CFT limit: [Haldane, Ha, Talstra, Bernard, Pasquier, 92]

algebraic structure: [Bernard, Gaudin, Haldane, Pasquier, 93]

Yangian and spinon description of $su(2)_{k=1}$ CFT: [Bernard, Pasquier, D.S. 94]

A family of long-range isotropic spin chains

particular limit in a family of long-range potentials $V(z)$ with parameter κ

Heisenberg
nearest-neighbour
$\delta_{i,j+1} + \delta_{i,j-1}$

Inozemtsev **[Inozemtsev, 92]**
elliptic
$\wp(z)$

Haldane-Shastry
trigonometric
$\dfrac{(\pi/N)^2}{\sin^2(\pi z/N)}$

hyperbolic
$\dfrac{\kappa^2}{\sinh^2(\kappa\, z)}$

rational
$\dfrac{1}{z^2}$

$$\begin{tikzcd}
\delta_{i,j+1} + \delta_{i,j-1} & \wp(z) \arrow[l, "\kappa \to \infty"'] \arrow[r, "\kappa \to 0"] \arrow[d, "N \to \infty"'] & \dfrac{(\pi/N)^2}{\sin^2(\pi z/N)} \arrow[d, "N \to \infty"'] \\
& \dfrac{\kappa^2}{\sinh^2(\kappa\, z)} \arrow[r, "\kappa \to 0"] & \dfrac{1}{z^2}
\end{tikzcd}$$

$\wp(z)$ Weierstrass elliptic function with periods N and $i\pi/\kappa$

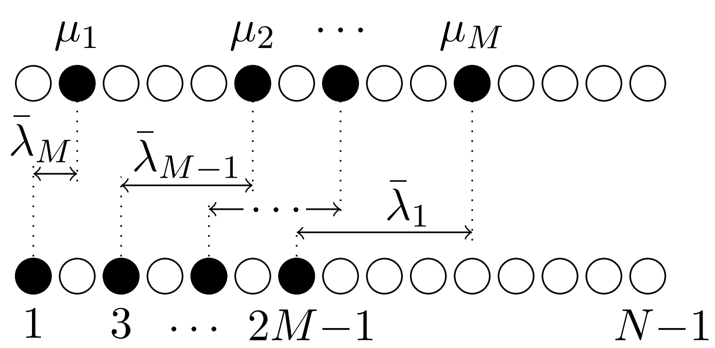
interpolation between the spectra of Heisenberg and Haldane-Shastry models

The spectrum of the Haldane-Shastry Hamiltonian

[Haldane, Ha, Talstra, Bernard, Pasquier, 92]

[Bernard, Gaudin, Haldane, Pasquier, 93]

The model is **Yangian symmetric** (huge degeneracy) and the spectrum is encoded by **motifs**:



M magnon motif

$\bar{\lambda}_m = \mu_{M-m+1} - 2(M-m) - 1$ $\qquad \mu_{m+1} > \mu_m + 1$

"vacuum" M magnon motif

$$E(\mu) - E_0 = \sum_{m=1}^{M} \varepsilon(\mu_m) = \sum_{m=1}^{M} \mu_m (N - \mu_m)$$

each motif corresponds to a Yangian representation

bound states in the XXX model evolve into descendants of Haldane-Shastry highest weight states when $\kappa \to 0$

The hyperbolic limit of the Inozemtsev model

the nice algebraic structure (Yangian symmetry) is lost for the elliptic deformation, but some interesting features are retrieved for infinite length $N \to \infty$

the potential becomes
$$\lim_{N\to\infty} \mathcal{P}_{N,\pi/\kappa}(z) = \kappa^2 \left(\frac{1}{\sinh^2 \kappa z} + \frac{1}{3} \right)$$

and it can be obtained from the Haldane-Shastry potential by the analytical continuation

$$\omega = e^{2\pi i/N} \to t = e^{-2\kappa}$$

higher Hamiltonians obtained by the same analytical continuation

used to match the first few orders of the long-range deformation of the Heisenberg model appearing in AdS/CFT **[D.S., Staudacher, 03, D.S. 12]**

up to exponential corrections of order e^{-N} the spectrum is obtained from the **asymptotic Bethe Ansatz** equations

$$\exp\left(ip_j N\right) = \exp\left(i \sum_{\substack{k=1 \\ k\neq j}}^{M} \chi(p_j, p_k) \right)$$

The hyperbolic limit of the Inozemtsev model

[Inozemtsev, 92]

asymptotic Bethe Ansatz equations: $$e^{ip_jN} = \prod_{k;k\neq j} \frac{\varphi(p_j)-\varphi(p_k)+i}{\varphi(p_j)-\varphi(p_k)-i}$$

κ large
$$\begin{aligned}\varphi(p) &= \frac{1}{2}\cot\frac{p}{2} + \frac{1}{2}\sum_{n>0}\left[\cot(\frac{p}{2}-i\kappa n)+\cot(\frac{p}{2}+i\kappa n)\right] \\ &= \frac{1}{2}\cot\frac{p}{2} + 2\sum_{n>0}\frac{t^n\sin p}{(1-t^n)^2+4t^n\sin^2(p/2)}\end{aligned}$$

κ small
$$\varphi(p) = \frac{\pi}{2\kappa}\coth\frac{\pi p}{2\kappa} + \frac{\pi}{2\kappa}\sum_{m>0}\left[\coth\frac{\pi(p-2\pi m)}{2\kappa}+\coth\frac{\pi(p+2\pi m)}{2\kappa}\right] - \frac{p}{2\kappa}$$

$\kappa \to 0$
$$\varphi(p) \simeq \frac{\pi - p}{2\kappa}$$

$$p_jN = \lim_{\kappa\to 0}\frac{1}{i}\sum_{j\neq l}\ln\frac{p_j-p_l-2i\kappa}{p_j-p_l+2i\kappa} + 2\pi I_j = 2\pi\sum_{j\neq l}\theta(p_j-p_l)+2\pi I_j$$

solution: $p_j = 2\pi m_j/N$ with $m_{j+1}-m_j \geq 2$ (motifs)

The hyperbolic limit of the Inozemtsev model

asymptotic Bethe Ansatz equations:
$$e^{ip_jN} = \prod_{k;k\neq j} \frac{\varphi(p_j) - \varphi(p_k) + i}{\varphi(p_j) - \varphi(p_k) - i}$$

the rapidity $\varphi(p)$ is quasi-periodic $\varphi(p+2\pi) = \varphi(p)$ and $\varphi(p+2i\kappa) = \varphi(p) - i$

the XXX model has bound states; when κ diminishes their rapidities approach the real axis

M=2 example: two types of bound states **[Dittrich, Inozemtsev, 96; Klabbers, Lamers, 20]**

type I bound states: $p_1 = \frac{P}{2} + iq\,, \qquad p_2 = \frac{P}{2} - iq$

transition at $q \simeq \kappa$ at small κ $P_{\rm cr} \simeq 5\kappa.$

type II bound states: $p_1 = \tilde{p}_1 + i\kappa\,, \qquad p_2 = \tilde{p}_2 - i\kappa$

the momentum with smaller real part vanishes at $\kappa \to 0$ and the state becomes a descendant

The solution of the Haldane Shastry Hamiltonian

to solve the Haldane-Shastry model it is useful to solve first the spin Calogero-Sutherland model **[Bernard, Gaudin, Haldane, Pasquier, 93]**

$$H_{B,F} = \sum_{j=1}^{N} (z_j \partial_j)^2 + \sum_{j \neq k} \beta(\beta \mp P_{jk}) \frac{z_j z_k}{(z_j - z_k)(z_k - z_j)}$$

when $\beta \to \infty$ the positions of the particles **freeze** at their equilibrium positions and the Hamiltonian becomes that of Haldane-Shastry **[Polychronakos, 93]**

$$z_j \longmapsto \omega^j = \mathrm{e}^{2\pi \mathrm{i} j/N}$$

the model is solved using the **degenerate double affine Hecke algebra** (DDAHA) generated by

$$\{z_1, z_2, \ldots, z_N\} \ , \ \text{and} \ \{d_1, d_2, \ldots, d_N\}$$

$$[z_i, z_j] = [d_i, d_j] = 0$$

Dunkl operators:

$$d_j = z_j \partial_j + \beta \sum_{k>j} \frac{z_j}{z_j - z_k} K_{jk} - \beta \sum_{k<j} \frac{z_k}{z_k - z_j} K_{jk}$$

$$K_{ij}\, z_i = z_j\, K_{ij}$$

coordinate permutation

$$K_{i,i+1}\, d_k = \begin{cases} d_k\, K_{i,i+1}, & k \neq i, i+1 \ , \\ d_{i+1}\, K_{i,i+1} - \beta, & k = i \ , \\ d_i\, K_{i,i+1} + \beta & k = i+1 \ . \end{cases}$$

The solution of the Haldane Shastry Hamiltonian

the spin Calogero-Sutherland model:

$$H_{B,F} = \sum_{j=1}^{N} (z_j \partial_j)^2 + \sum_{j \neq k} \beta(\beta \mp P_{jk}) \frac{z_j z_k}{(z_j - z_k)(z_k - z_j)}$$

is diagonalised on functions completely (anti)symmetric by permutations of spins and coordinates

$$\Psi = \prod_{i<j} (z_i - z_j)^{\beta} \; \widetilde{\Psi} \qquad\qquad K_{ij} P_{ij} \widetilde{\Psi} = \pm \widetilde{\Psi}$$

$$\widetilde{\Psi} = \sum_{i_1 < i_2 < \ldots < i_M} (-1)^{\sigma_I} \; \Psi(z_{\{i_1, i_2, \ldots, i_M\}}, \bar{z}_{\{i_1, i_2, \ldots, i_M\}}) \; |i_1, i_2, \ldots, i_M\rangle\rangle$$

$$|i_1, \cdots, i_M\rangle\rangle := \sigma^-_{i_1} \cdots \sigma^-_{i_M} \, |\uparrow \cdots \uparrow\rangle$$

partially (anti)symmetric in both groups of variables

$$z_{\{i_1, i_2, \ldots, i_M\}} \equiv \{z_{i_1}, z_{i_2} \ldots z_{i_M}\}$$

$$\bar{z}_{\{i_1, i_2, \ldots, i_M\}} = \{z_1, z_2 \ldots z_N\} \backslash \{z_{i_1}, z_{i_2} \ldots z_{i_M}\}$$

on this space of functions we can define a projection $\quad \pi_{B,F}(\ldots K_{ij}) = \pm \pi_{B,F}(\ldots P_{ij})$

$$H_{B,F} = \pi_{B,F} \left(\sum_{j=1}^{N} d_j^2 \right)$$

The algebraic structure of the Haldane Shastry Hamiltonian

algebraic structure of the spin Calogero-Sutherland model:

[Bernard, Gaudin, Haldane, Pasquier, 93; Uglov, 97]

$$T_a(u) \equiv \pi_F(\widehat{T}_a(u)) \; , \qquad \widehat{T}_a(u) = \prod_{j=1}^{N} \left(1 + \frac{\beta P_{ja}}{u + d_j}\right)$$

$$\pi_F(\ldots K_{ij}) = -\pi_F(\ldots P_{ij})$$

the long range nature of the interaction is contained in the dynamical inhomogeneities d_j

the integrals of motion are generated by the quantum determinant

$$\text{qDet } T_a(u) = \pi_F\left(\prod_{i=1}^{N} \frac{u + d_i + \beta}{u + d_i}\right) \qquad H_k = \pi_F\left(\sum_{i=1}^{N} d_i^k\right)$$

and they commute with the elements of $T_a(u)$ (Yangian symmetry)

Basis of antisymmetric states (wedges)

[Uglov, 97]

su(n) chain in fundamental representation: at each site we use the basis

$$u_k = z^{\bar{k}}\,|\underline{k}\rangle\,, \quad k \in \mathbb{Z}\,, \quad \text{with } \bar{k}, \underline{k} \text{ determined by} \quad k = \underline{k} + \bar{k}\,n\,, \quad 0 \le \underline{k} \le n-1$$

coordinate   su(n) spin

$$\begin{array}{rlcrlcrl}
\vdots & & & & & & & \\
u_{2n} = z^2 & |0\rangle & & \vdots & & & \vdots & \\
u_n = z & |0\rangle & & u_{n+1} = z & |1\rangle & \cdots & u_{2n-1} = z & |n-1\rangle \\
u_0 = & |0\rangle & & u_1 = & |1\rangle & \cdots & u_{n-1} = & |n-1\rangle \\
u_{-n} = z^{-1} & |0\rangle & & u_{1-n} = z^{-1} & |1\rangle & \cdots & u_{-1} = z^{-1} & |n-1\rangle \\
\vdots & & & \vdots & & & \vdots &
\end{array}$$

wedges (Slater determinants):

$$\hat{u}_{\boldsymbol{k}} \equiv u_{k_1} \wedge \ldots \wedge u_{k_N} \equiv \sum_{\sigma \in S_N} \mathrm{sgn}(\sigma)\, u_{k_{\sigma(1)}} \otimes \ldots \otimes u_{k_{\sigma(N)}}$$

$k_1 > \ldots > k_N$; at most n consecutive $\bar{k}$'s can be equal

Basis of antisymmetric states (wedges)

On the wedge basis the action of the Calogero-Sutherland Hamiltonian is triangular:

$$H_{\beta,n}\ \hat{u}_k = E_k\ \hat{u}_k + 2\beta \sum_{1\le i<j\le N} h_{ij}\ \hat{u}_k$$

$$E_k = \sum_{j=1}^{N} \bar{k}_j^2 + \beta \sum_{j=1}^{N}(N+1-2j)\ \bar{k}_j + \beta^2 \frac{N(N^2-1)}{12}$$

and

$$\begin{aligned} & h_{ij}\ u_{k_1} \wedge \ldots \wedge u_{k_i} \wedge \ldots \wedge u_{k_j} \wedge \ldots \wedge u_{k_N} \\ &= \sum_{r=1}^{\bar{k}_i-\bar{k}_j-1} (\bar{k}_i - \bar{k}_j - r)\ u_{k_1} \wedge \ldots \wedge u_{k_i - nr} \wedge \ldots \wedge u_{k_j+nr} \wedge \ldots \wedge u_{k_N} \end{aligned}$$

squeezing

freezing keeps only the part linear in β in the Hamiltonian and evaluates the positions at N-th roots of unity $\mathrm{ev}_\omega\ z_j \equiv \omega^j$

$$H_{HS} = \mathrm{ev}_\omega \sum_{i\neq j} \frac{z_i z_j}{z_{ij} z_{ji}} (1+P_{ij}) \qquad E_{HS,k} = \sum_{j=1}^{N}(N+1-2j)\ \bar{k}_j$$

Wedges in the frozen limit

When the coordinates are evaluated at roots of unity only a finite number of eigenstates survive;

we use a collection of identities:

$$\mathrm{ev}_\omega \sum_{j\neq k} \frac{z_j z_k}{z_{jk} z_{kj}} = \frac{N(N^2-1)}{12}$$

ground state energy of the CS model

$$\mathrm{ev}_\omega \; u_{nN+k} \simeq \mathrm{ev}_\omega \; u_k$$

maximum degree of polynomial in each variable should be N-1

symmetric sums: $p_k(z) \equiv \sum_{j=1}^{N} z_j^k$

and elementary symmetric functions: $e_k(z) \equiv \sum_{1\leq i_1 < \ldots < i_k \leq N} z_{i_1} \ldots z_{i_k}$

$$\mathrm{ev}_\omega \; p_k(z) = N \; \delta_{k,0 \mod N}$$

$$\mathrm{ev}_\omega \; e_k(z) = \delta_{k,0} + (-1)^{N-1} \delta_{k,N}$$

or $\quad \mathrm{ev}_\omega \left(p_k(z_\alpha) + p_k(z_{\bar\alpha})\right) = N\delta_{n,0}, \qquad k < N$

Wedges in the frozen limit

specialise to su(2) for simplicity, and split the integers $\overline{k}_j$ into two groups depending on the value of $\underline{k}_j = 0, 1$

$$\overline{k}_1^{(0)} > \ldots > \overline{k}_{N-M}^{(0)} \ , \qquad \overline{k}_1^{(1)} > \ldots > \overline{k}_M^{(1)}$$

and define the partitions $\lambda, \ \nu$ with parts

$$\lambda_i = \bar{k}_i^{(1)} - M + i \ , \qquad \nu_i = \bar{k}_i^{(0)} - N + M + i$$

Schur polynomials

the wedges can be written as

$$\widehat{u}_k = \sum_{\alpha} (-1)^{|\alpha|} \prod_{j<k\in\alpha} (z_j - z_k) \ S_\lambda(z_\alpha) \ \prod_{j<k\in\bar{\alpha}} (z_j - z_k) \ S_\nu(z_{\bar{\alpha}}) \ |\alpha\rangle\rangle$$

$$z_\alpha \equiv \{z_{i_1}, \ldots, z_{i_M}\} \ , \qquad \text{for} \quad \alpha = \{i_1, \ldots, i_M\} \ ,$$
$$1 \leq i_1 < \ldots < i_M \leq M \ , \qquad |\alpha| = \textstyle\sum_{k\in\alpha} k$$

upon evaluation: $$W_{\bar{\alpha}} \equiv \prod_{j<k\in\bar{\alpha}} (\omega^j - \omega^k) = C_N \prod_{\alpha} \omega^j \ \ W_\alpha$$

and $$\mathrm{ev}_\omega \ S_\nu(z_{\bar{\alpha}}) = j_\nu \ \mathrm{ev}_\omega \ S_{\nu'}(z_\alpha)$$

Wedges in the frozen limit

upon freezing: $$\widehat{u}_k = \text{const} \sum_\alpha \prod_{j<k\in\alpha} (z_j - z_k)^2 \prod_{j\in\alpha} z_j \; S_\lambda(z_\alpha) \; S_{\nu'}(z_\alpha) \; |\alpha\rangle\rangle$$

since $S_\lambda(z_\alpha) \; S_{\nu'}(z_\alpha) = S_{\lambda+\nu'}(z_\alpha) + \text{lower}$ one can assume $\nu = \emptyset$

hence the integers $\overline{k}_j^{(0)}$ are compactly packed and not affected by squeezing

$$\overline{k}_j^{(0)} = N - M - j$$

then the rank of the wedges can be reduced by 1

$$\widehat{u}_k \;\to\; \widehat{u}_{\widetilde{k}} \equiv u_{\widetilde{k}_1} \wedge \ldots \wedge u_{\widetilde{k}_M} \qquad \widetilde{k}_j \equiv \overline{k}_j^{(1)} + \frac{M+1}{2}\,, \qquad j = 1, \ldots, M$$

on the new wedge basis the HS acts like a dynamical CS with parameter $\beta = 1$

$$H_{HS}\; \hat{u}_{\tilde{k}} = \left(\sum_{j=1}^{M} \tilde{k}_j^2 + \sum_{j=1}^{M} (M+1-2j)\tilde{k}_j - N \sum_{j=1}^{M} \tilde{k}_j + \frac{M(M^2-1)}{12} + E_{HS}^F \right) \; \hat{u}_{\tilde{k}}$$
$$+\, 2 \sum_{r=1}^{\tilde{k}_i - \tilde{k}_j - 1} (\bar{k}_i - \bar{k}_j - r) \; u_{\tilde{k}_1} \wedge \ldots \wedge u_{\tilde{k}_i - r} \wedge \ldots \wedge u_{\tilde{k}_j + r} \wedge \ldots \wedge u_{\tilde{k}_M} \,.$$

Eigenfunctions of the Haldane Shastry Hamiltonian

recap: $$\widetilde{\Psi} = \sum_{i_1<i_2<\ldots<i_M} \Psi(z_{\{i_1,i_2,\ldots,i_M\}}, \bar{z}_{\{i_1,i_2,\ldots,i_M\}})\ |i_1, i_2, \ldots, i_M\rangle\rangle$$

$$\Psi(z_{\{i_1,i_2,\ldots,i_M\}}, \bar{z}_{\{i_1,i_2,\ldots,i_M\}}) \longrightarrow \psi(z_{i_1}, z_{i_2}, \ldots, z_{i_M})$$ using $$\sum_{j=1}^{N} z_j^n = N\,\delta_{n,0 \bmod N}$$

$$\psi_\lambda(z_1, z_2, \ldots, z_M) = \prod_{m<n} (z_m - z_n)^2\ P_\lambda^{\beta=2}(z_1, z_2, \ldots, z_M)$$

[BGHP, 93;

Bernard, Pasquier, D.S., 93]

symmetric Jack polynomials labelled by partitions $N - 2M + 1 \geq \lambda_1 \geq \ldots \geq \lambda_M \geq 1$

this construction works for the highest weight states

when extra magnons are excited $\nu \neq \emptyset$ and the maximum degree of $\psi(z)$, $D_{\max} = 2M + 1 + \nu_1' + \lambda_1$, reaches N; the corresponding states are Yangian descendants (no closed form)

Comments and outlook

similar result at higher rank: su(n) HS Hamiltonian equivalent to dynamical su(n-1) CS Hamiltonian with $\beta = 1$

wedges are well-adapted for the infinite length, low-energy limit (CFT); the antiferromagnetic state is particularly simple in this language

we expect a similar description for the q-deformed Haldane-Shastry Hamiltonian **[Lamers, Pasquier, D.S., 20]** in terms of q-wedges **[Kashiwara, Miwa, Stern, 95]**

$$\widetilde{\Psi}_\lambda(z_1, \cdots, z_M) := \left(\prod_{m<n}^{M} (\mathsf{q}\, z_m - \mathsf{q}^{-1} z_n)\, (\mathsf{q}^{-1} z_m - \mathsf{q}\, z_n) \right) P^\star_\lambda(z_1, \cdots, z_M)$$

$P^\star_\lambda$ is a Macdonald polynomial with parameters $q^* = (t^*)^{1/2} = \mathsf{q}$

q roots of unity, e.g. q=i? (the spectrum is extremely degenerate)

Comments and outlook

Separation of variables: in the traditional approach à la Sklyanin one needs the transfer matrix/ B operator to generate the spectrum/eigenvectors

$\longrightarrow$ define a hybrid model starting from the CS monodromy matrix by taking as generating function of the twisted transfer matrix

$$t_x(u) = xA(u) + x^{-1}D(u)$$

for zero/infinite twist the eigenvectors are the same as for CS;
what happens upon freezing?

the eigenvalues of the separated variables are given by the eigenvalues of the Dunkl operators; they can and do occur at degenerate values $d_j \sim d_{j+1} + \beta$

[in progress with G. Ferrando, J. Lamers, F. Levkovich-Maslyuk]