Symmetry Resolved Entanglement in Integrable Quantum Field Theory

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Recent works:

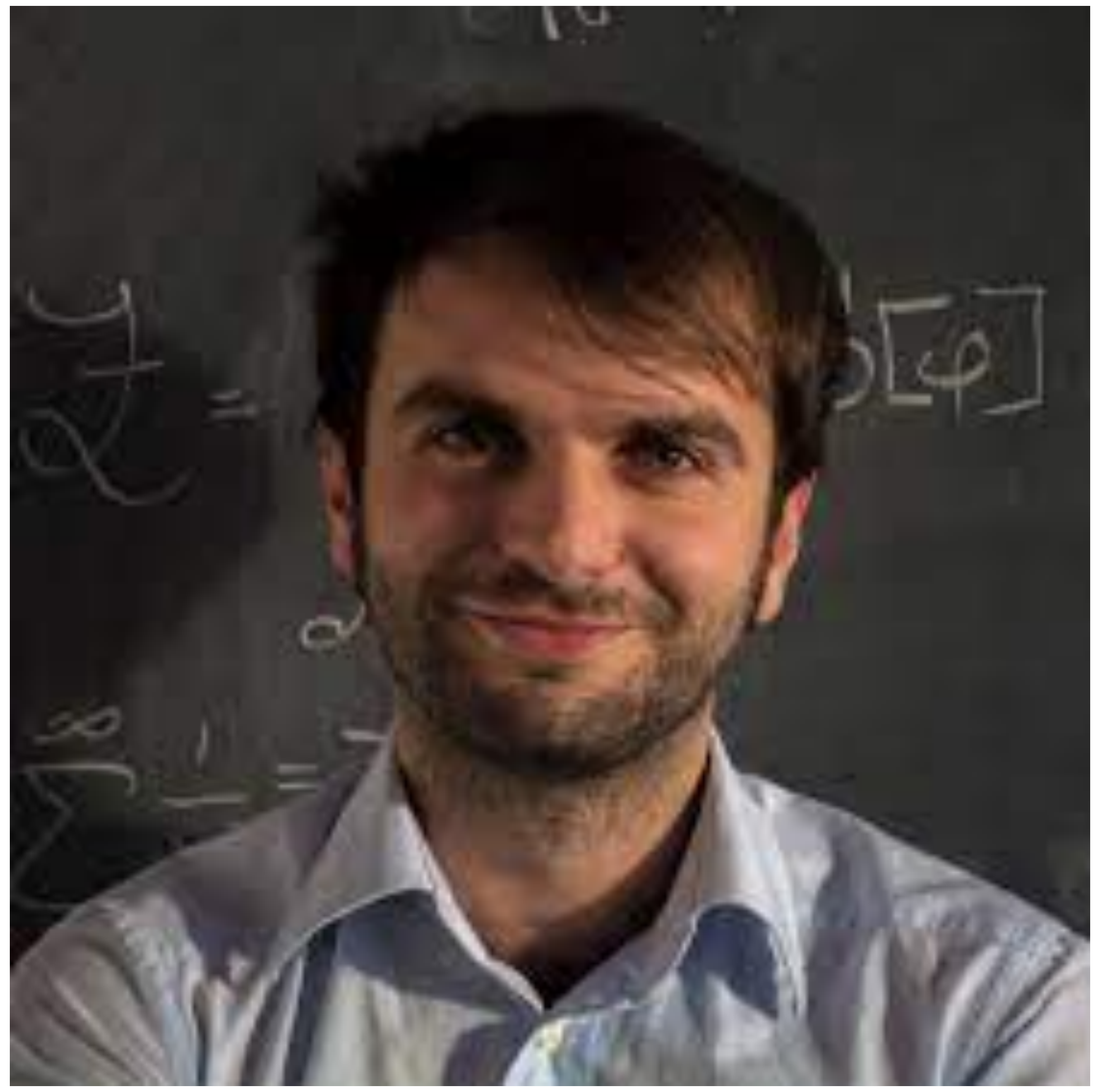
- Symmetry Resolved Entanglement of Excited States in Quantum Field Theory I: Free Theories, Twist Fields and Qubits, L. Capizzi, O.A. Castro-Alvaredo, C. De Fazio, M. Mazzoni and L. Santamaría-Sanz, arXiv:2203.12556

- Symmetry Resolved Entanglement of Excited States in Quantum Field Theory II: Numerics, Interacting Theories and Higher Dimensions, L. Capizzi, C. De Fazio, M. Mazzoni, L. Santamaría-Sanz and O.A. Castro-Alvaredo (to appear soon)

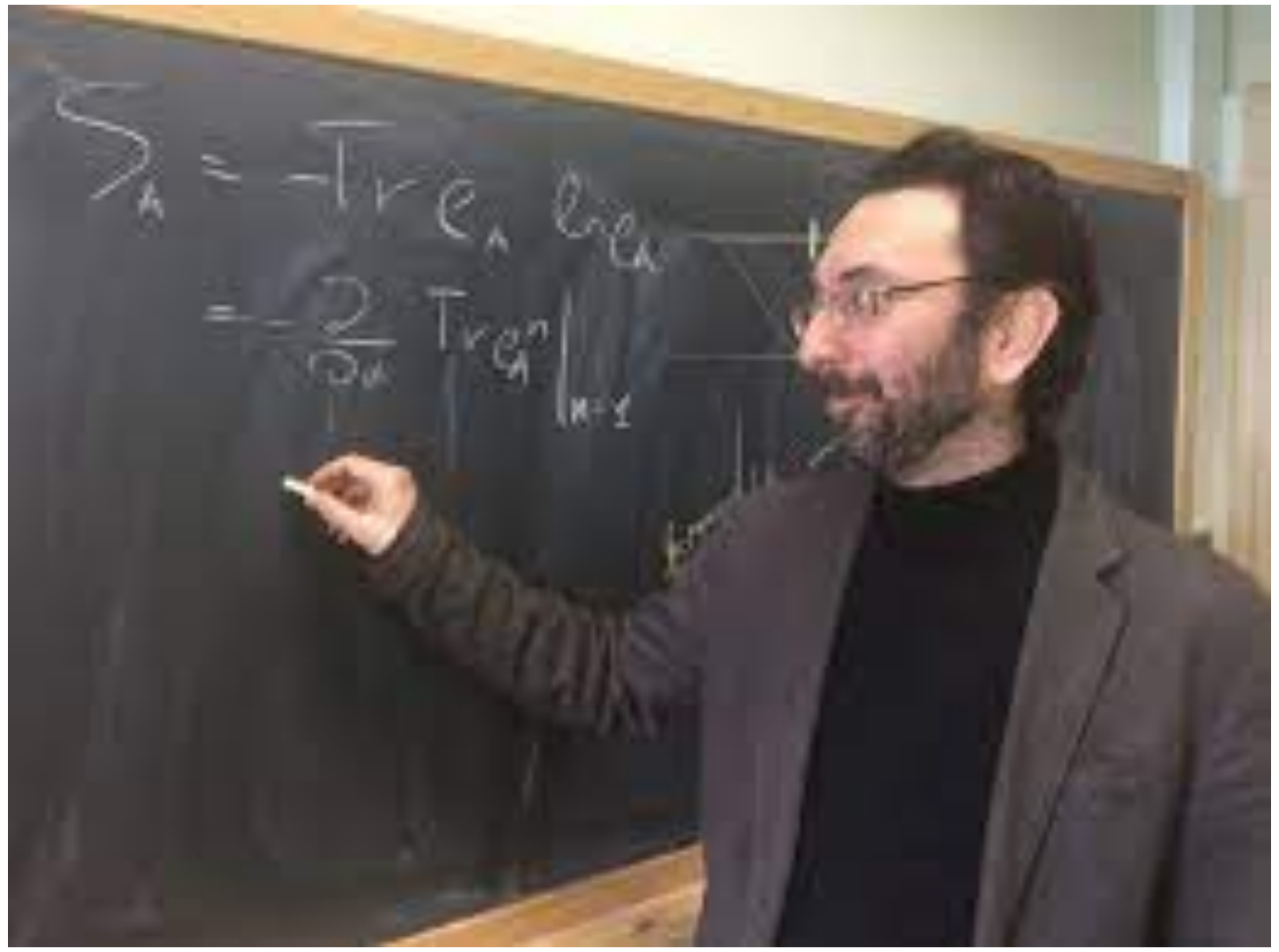
- Entanglement of the 3-State Potts Model via Form Factor Bootstrap: Total and Symmetry Resolved Entropies, L. Capizzi, D. X. Horváth, P. Calabrese and O.A. Castro-Alvaredo, To appear in SciPost, arXiv:2108.10935.

- Branch Point Twist Field Form Factors in the sine-Gordon Model II: Composite Twist Fields and Symmetry Resolved Entanglement, D.X. Horváth, P. Calabrese and O.A. Castro-Alvaredo, SciPost Phys. 12, 088 (2022), arXiv:2105.13982


David


Luca


Cecilia


Pasquale

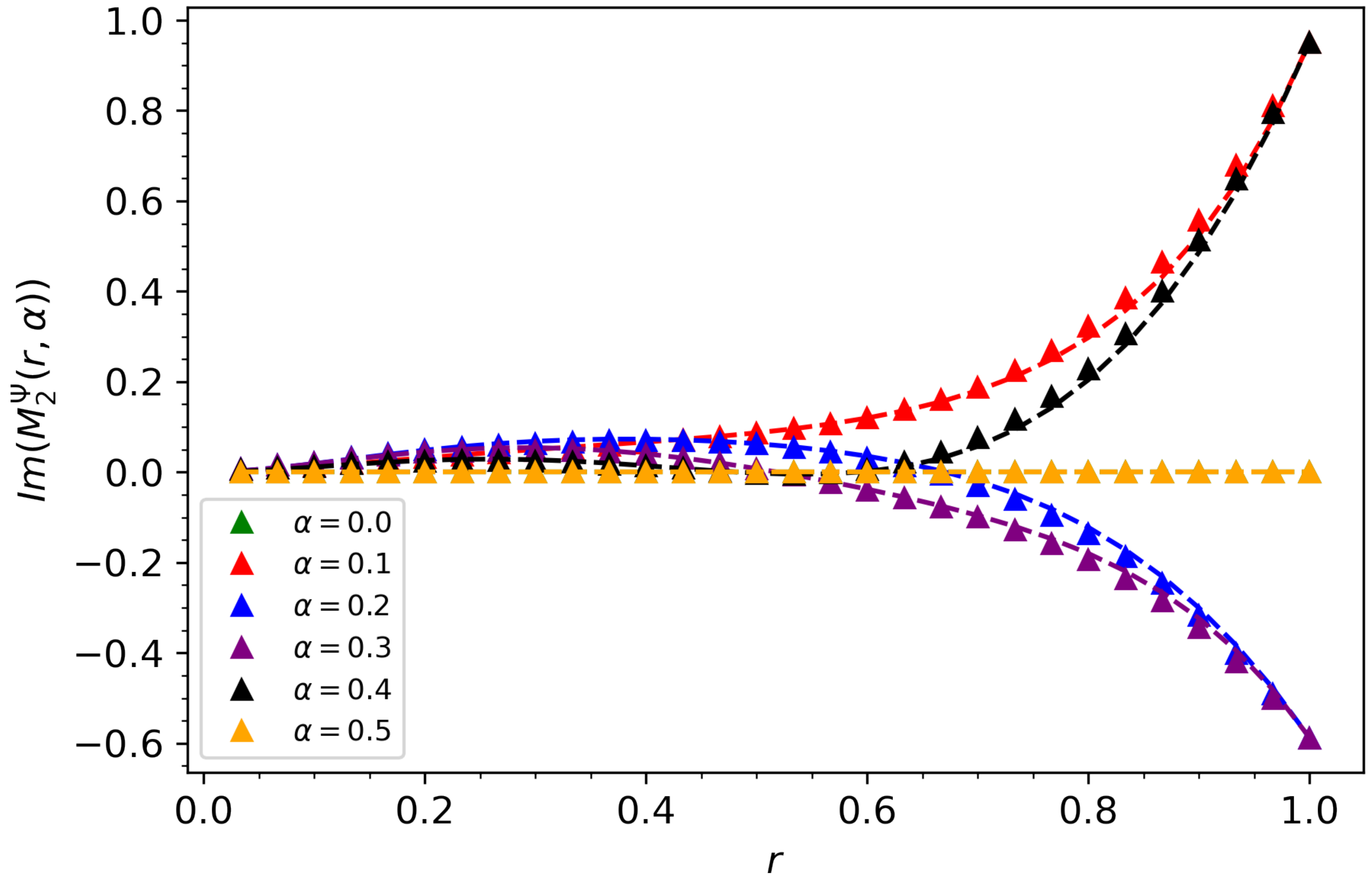
Michele

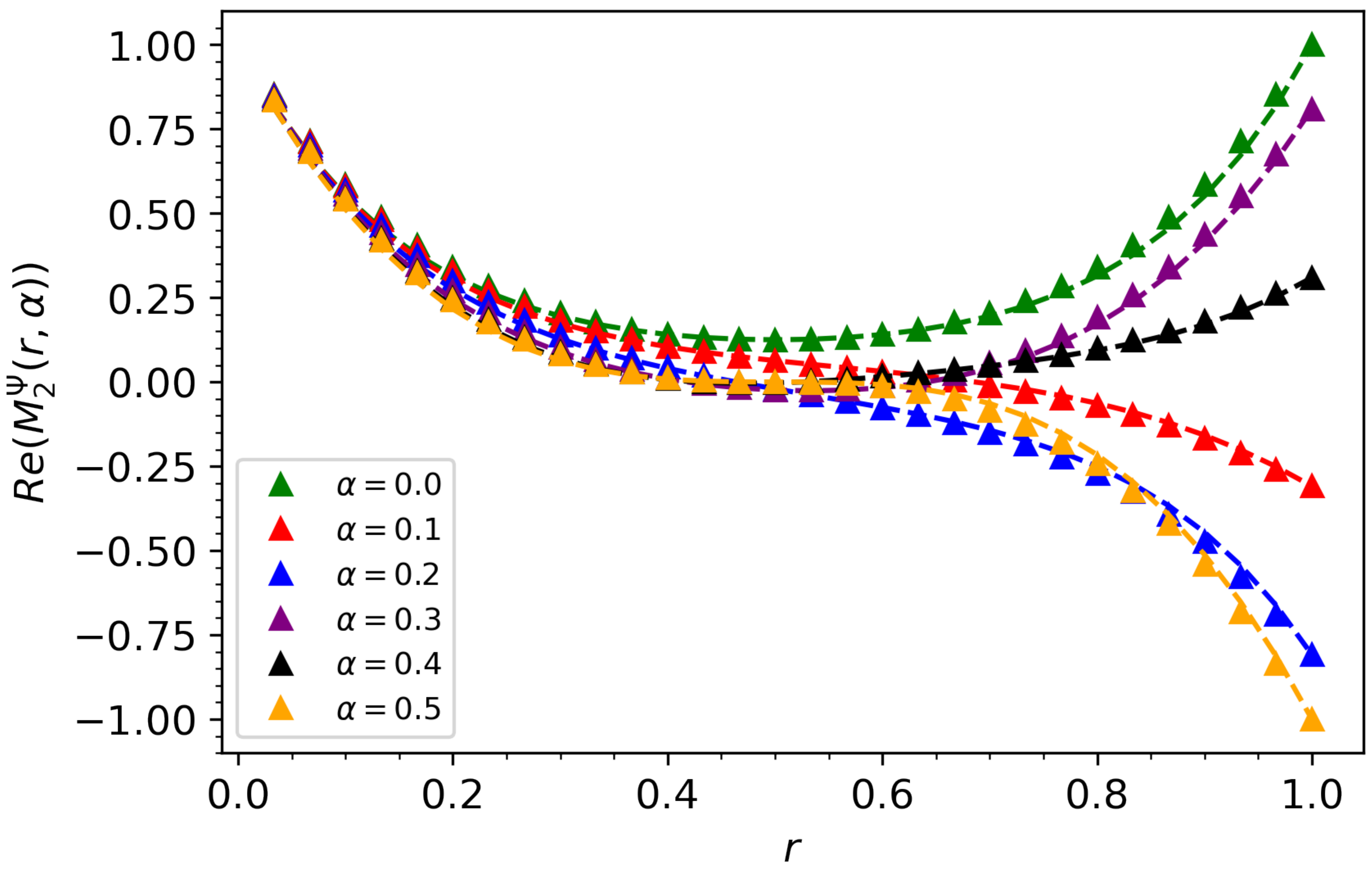


Lucía

Plan of the Talk:

- Introduction: what is it and why the interest?
- How can it be computed?
- Some of my contributions, with a focus on excited states





Introduction

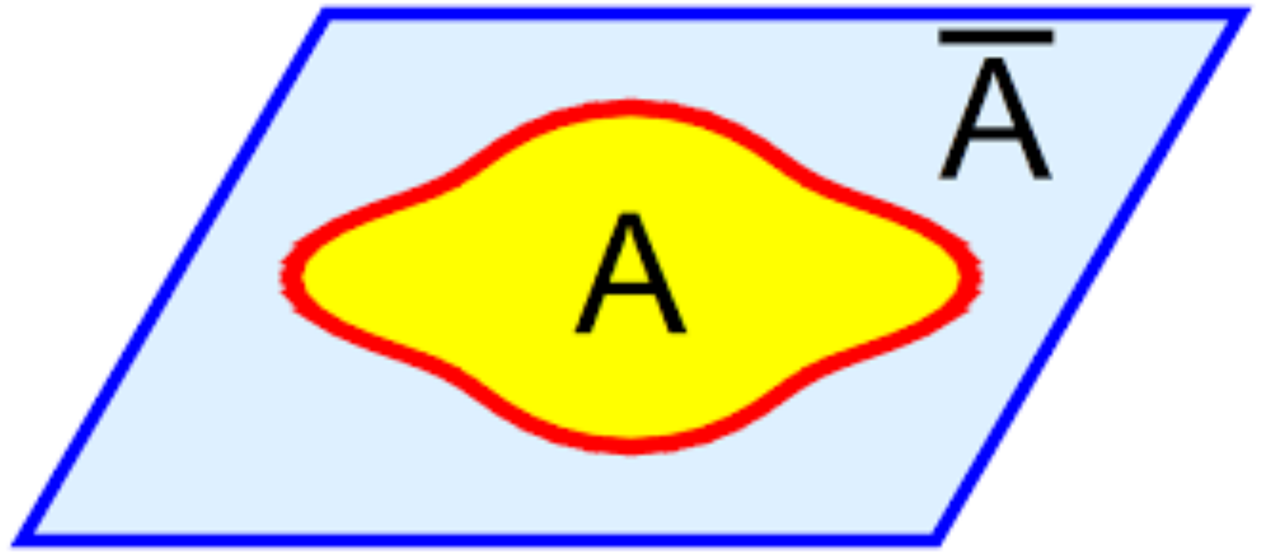
- The earliest papers where a measure of entanglement connected to symmetries is introduced are [A. Belin et al.'13; Caputa et al.'16]
- More intense interest starts with the work [Xavier, Alcaráz & Sierra'18] and specially [Goldstein & Sela'18] where the terminology, symmetry resolved entanglement/ entropies is used for the first time.
- A good introduction to the main ideas is provided by the abstract of Xavier's paper:

"The entanglement in a quantum system that possess an internal symmetry, characterized by the Sz-magnetization or U(1)-charge, is distributed among different sectors. [...] We find surprisingly that the entanglement entropy is equally distributed among the different magnetization sectors. Its value is given by the standard area law violating logarithmic term, that depends on the central charge c, minus a double logarithmic correction related to the zero temperature susceptibility. This result provides a new method to estimate simultaneously the central charge c and the critical exponents of U(1)-symmetric quantum chains."

- Interest in the SRE also comes from recent experimental work [Neven et al.'21; Vitale et al.'21]

Basic Definitions

- As usual, the basic building block is a reduced density matrix in a bipartite system:



$$\rho_A = \mathrm{Tr}_{\bar{A}}(|\Psi\rangle\langle\Psi|)$$

in a pure state $|\Psi\rangle$

- Then, the standard measures can be defined:

$$S_n = \frac{\mathrm{Tr}_A(\rho_A^n)}{1-n} \xrightarrow{n \to 1} S = -\mathrm{Tr}_A(\rho_A \log \rho_A)$$

Rényi Entropies

Von Neumann Entropy

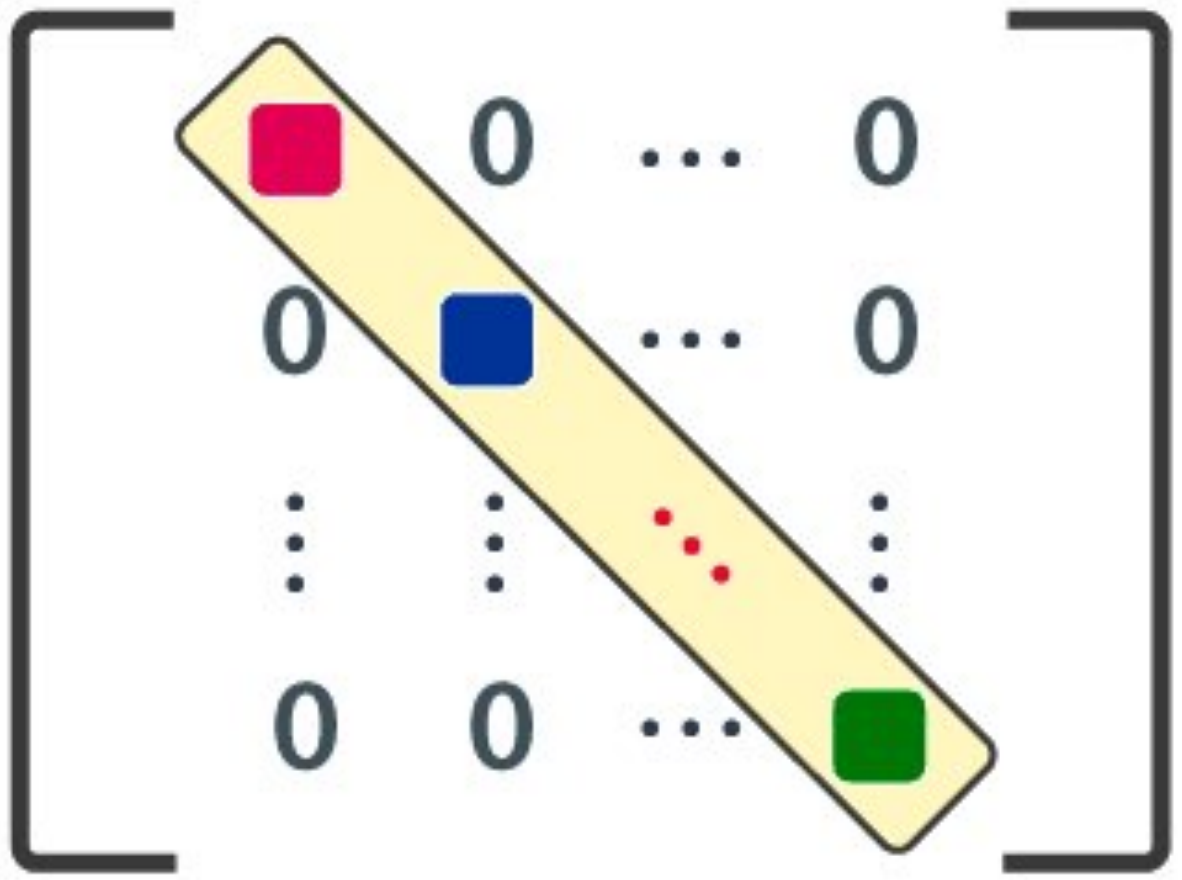
Interpretations, Fields and Geometry

- It is well-known that the trace $\mathcal{Z}_n/\mathcal{Z}_1^n = \mathrm{Tr}_A \rho_A^n$ admits a geometric interpretation as a partition function on an n-sheeted Riemann surface, where sheets are cyclicly connected along a branch cut of length $\ell(A)$ (in 1+1D)
- This Riemann surface in turn represents the space-time manifold associated to a replica theory, consisting on n copies of the original model
- In such a theory, the partition function above can be identified with a two-point function of symmetry fields $\mathcal{T}_n, \tilde{\mathcal{T}}_n = \mathcal{T}_n^\dagger$ [Calabrese & Cardy'04; Cardy, O. C.-A. & Doyon'08]

$$\boxed{\frac{\mathcal{Z}_n}{\mathcal{Z}_1^n} \propto {}_n\langle\Psi|\mathcal{T}_n(0)\tilde{\mathcal{T}}_n(\ell)|\Psi\rangle_n}$$

Symmetries

- If we have an internal symmetry and a local symmetry charge Q, such that $[Q, \rho] = 0$, then we have that $[\rho_A, Q_A] = 0$, where $Q = Q_A + Q_{\bar{A}}$ then
- ρ_A, ρ_A^n are block-diagonal matrices
- $\mathcal{Z}_n(q) = \mathrm{Tr}_A(\mathbb{P}_q \rho_A^n)$ is the symmetry resolved partition function



$$S_n(q) = \frac{1}{1-n} \log \frac{\mathcal{Z}_n(q)}{\mathcal{Z}_1(q)^n} \qquad S(q) = \lim_{n \to 1} S_n(q)$$

Symmetry Resolved Entanglement Entropy

$$S = \sum_q p(q) S(q) - \sum_q p(q) \log p(q)$$

How do we compute SREEs?

- A systematic approach based on composite twist fields was proposed by [Goldstein and Sela'18]
- The idea is that the quantities $Z_n(\alpha)$, known as charged moments can be obtained from correlation functions of (composite) twist fields.
- Then, the SREE follows from the Fourier transform, either continuous or discrete (depending on the type of symmetry)

$$\mathcal{Z}_n(q) = \int_{-\frac{1}{2}}^{\frac{1}{2}} Z_n(\alpha) e^{-2\pi i \alpha q} d\alpha$$

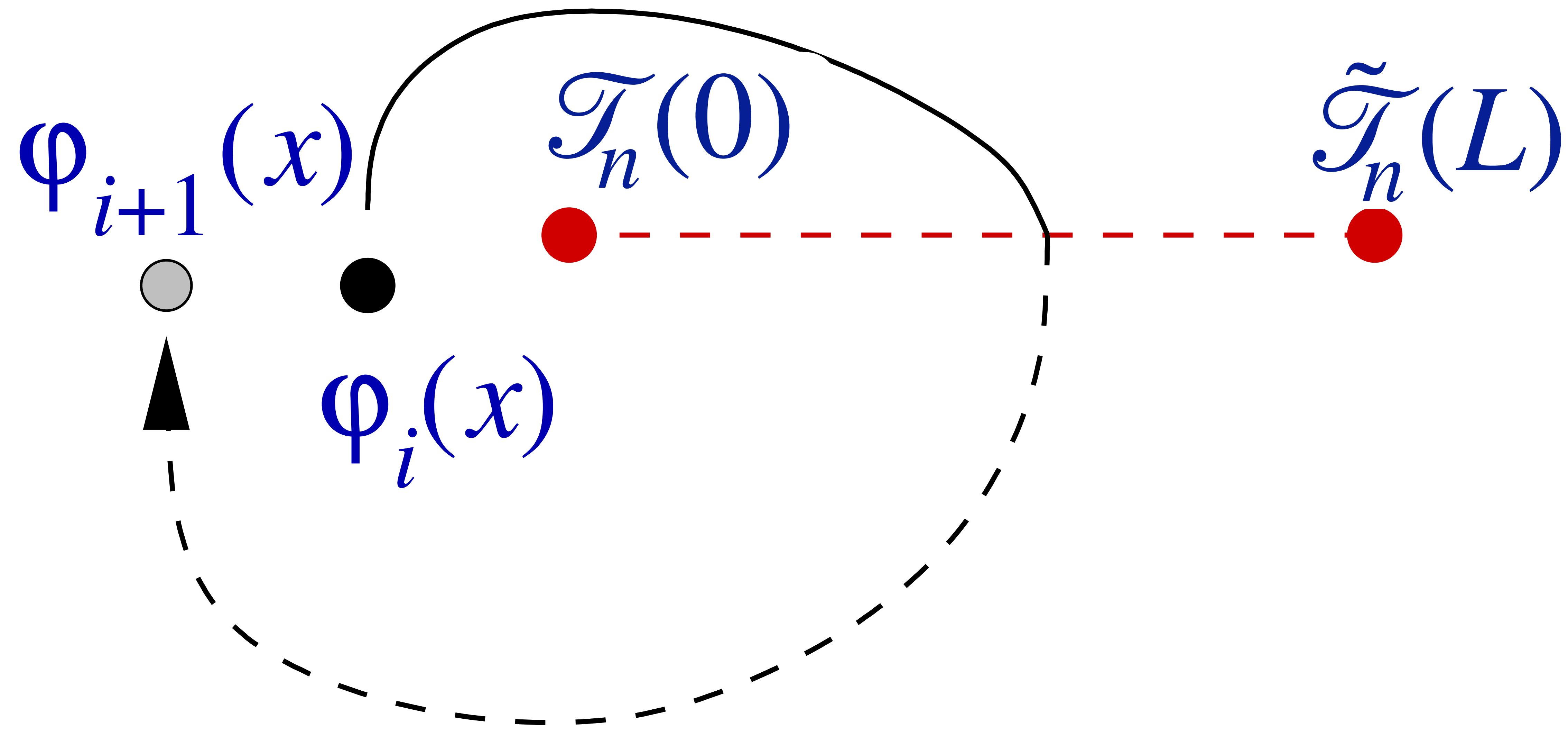
Partition Function vs
Charged Moments (for U(1) Symmetry)

(Equal-Time) Exchange Relations for Standard Branch Point Twist Fields on Replica Theories

$$\varphi_i(x)\mathcal{T}_n(y) = \mathcal{T}_n(y)\varphi_{i+1}(x) \quad x > y$$

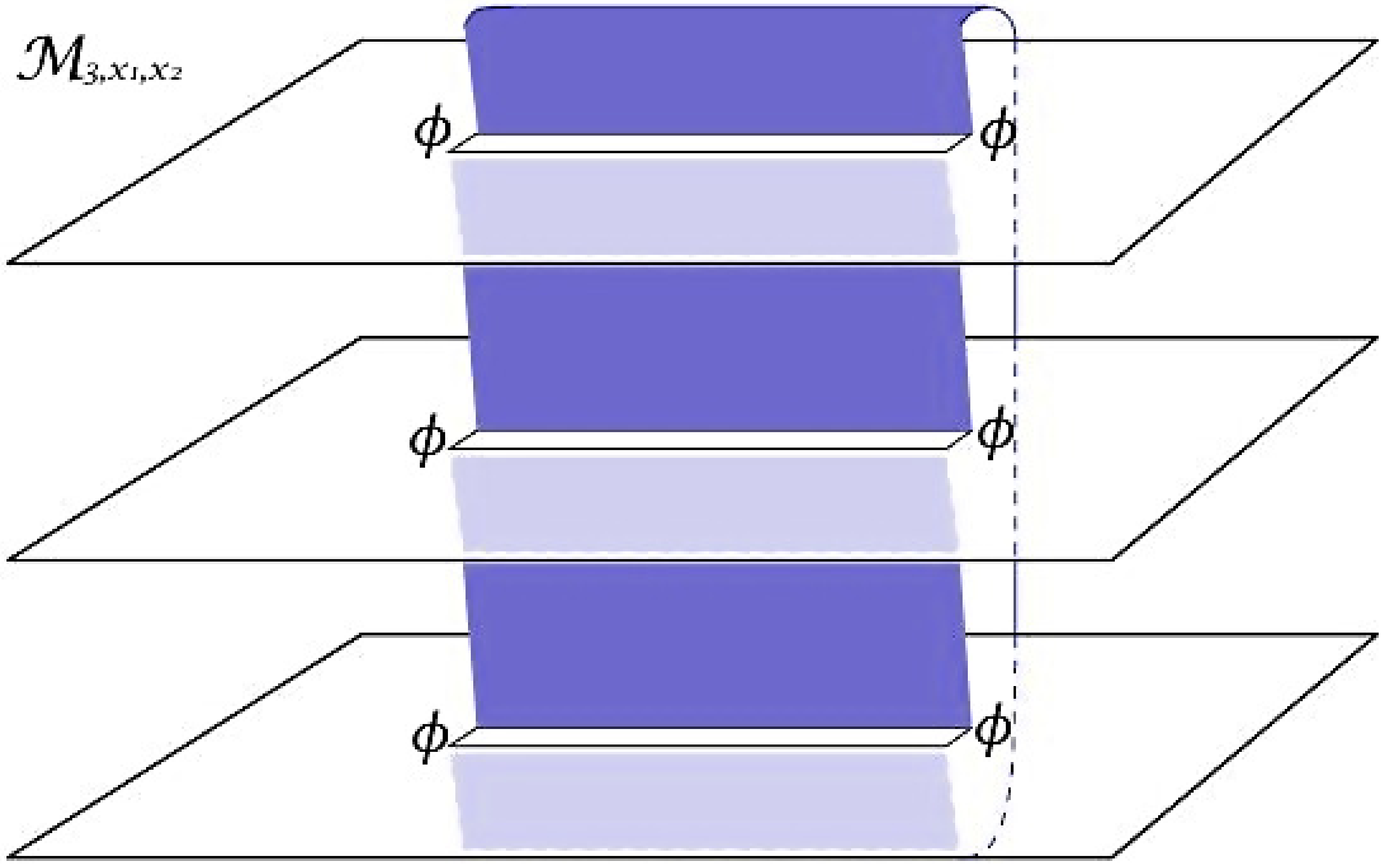
$$\varphi_i(x)\mathcal{T}_n(y) = \mathcal{T}_n(y)\varphi_i(x) \quad x < y$$

$$i = 1, \ldots, n \qquad i \equiv n + i$$



(Equal-Time) Exchange Relations for Composite Branch Point Twist Fields on Replica Theories

- If the internal symmetry is U(1) such as for complex free theories [Horváth, Capizzi & Calabrese'21] or for the sine-Gordon model [Horváth, Calabrese & O.C.-A.'21]



$$Z_n(\alpha) \propto {}_n\langle \Psi | \mathcal{T}_n^\alpha(0)\tilde{\mathcal{T}}_n^\alpha(L) | \Psi \rangle_n$$

$$\varphi_i(x)\mathcal{T}_n^\alpha(y) = e^{2\pi i\alpha}\mathcal{T}_n^\alpha(y)\varphi_{i+1}(x) \quad x > y$$

$$\varphi_i(x)\mathcal{T}_n^\alpha(y) = \mathcal{T}_n^\alpha(y)\varphi_i(x) \quad x < y$$

$$i = 1,\ldots,n \qquad i \equiv n+i$$

Composite Twist Fields

- Formally, these composite fields can be defined in CFT as

$$\mathcal{T}_n^{\alpha}(y) \propto \lim_{x \to y} |x-y|^{2\Delta_\alpha(1-\frac{1}{n})} \sum_{j=1}^{n} \mathcal{T}_n(y)\mathcal{V}_\alpha^j(x)\,,$$

- That is, the leading field in the OPE of the standard BPTF and the U(1) field $\mathcal{V}_\alpha^j$ acting on copy j.
- The conformal dimensions of such fields were first obtained in the context of entanglement [O.C.-A., Doyon & Levi'11; Levi'12;Bianchini et al'14]and later again in [Goldstein & Sela'18] for the case of two symmetry fields.
- Matrix elements can be computed via a generalised form factor program proposed in [Horváth & Calabrese'20]
- This has been done for a large number of examples...

$$\Delta_n^{\alpha} = \Delta_n + \frac{\Delta_\alpha}{n}$$

Main Ideas/Results

- Using IQFT techniques the form factors of fields $\mathcal{T}_n^\alpha, \tilde{\mathcal{T}}_n^\alpha$ can be computed and the two-point function obtain through a form factor expansion

$$\langle 0|\mathcal{T}_n^\alpha(0)\tilde{\mathcal{T}}_n^\alpha(\ell)|0\rangle \sim \sum_{p=0}^{\infty}\sum_{\mu_p=1}^{N}\int d\theta_1 \ldots d\theta_p |F_p^\alpha(\theta_{\mu_1},\ldots,\theta_{\mu_p};n)|^2 e^{-\sum_{j=1}^p m_{\mu_j}\ell\cosh\theta_{\mu_j}}$$

- This is complicated (as usual) but even more so for the SREEs because they are related to the Fourier Transform of this function.
- In many examples, the only term that can be analysed in detail is the zeroth order of this expansion, that is the disconnected part of the two-point function: $\langle \mathcal{T}_n^\alpha\rangle^2$.
- In massive 1+1D QFT we know that $\langle \mathcal{T}_n^\alpha\rangle = v(\alpha;n)m^{2\Delta_n^\alpha}$ and one can often argue that the leading contribution to the Fourier Transform comes from $\alpha \sim 0$.

Leading Order

- Then a Saddle-Point analysis gives the sort of result Xavier et al. anticipated. For instance, in the sine-Gordon model (which has continuous U(1) symmetry):

$$\boxed{S_n(q,\ell) = -\frac{n+1}{6n}\ln(m\varepsilon) + \frac{\ln\sqrt{n}D_n^0}{1-n} - \frac{1}{2}\ln\frac{\Delta|\ln(m\varepsilon)|}{\pi} + \mathcal{O}(|\ln(m\varepsilon)|^{-\frac{1}{2}}, e^{-m_1\ell})}$$

- Whereas, for a theory with a discrete symmetry (like the $\mathbb{Z}_3$ Potts model) we have:

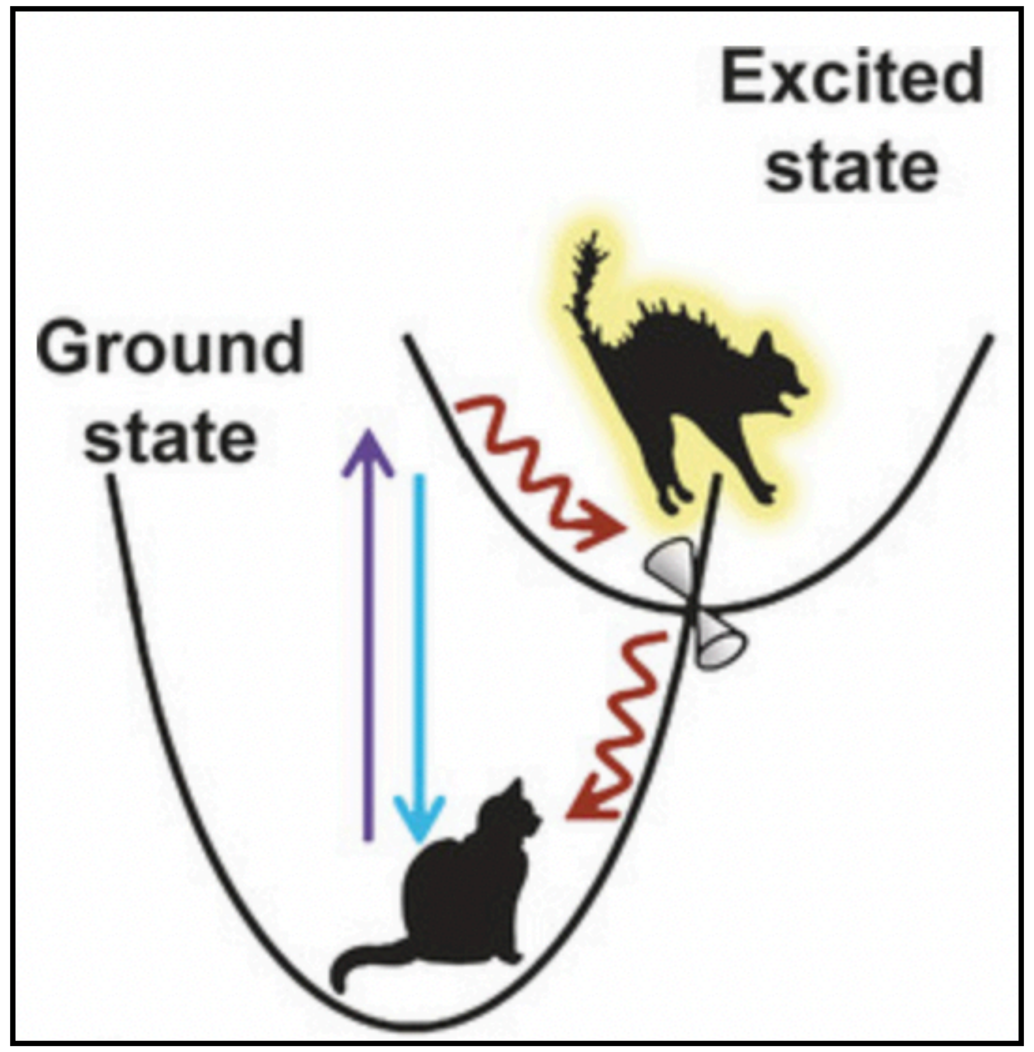
$$\boxed{S_n(q) = S_n - \log 3 + \frac{1}{1-n} 2\cos\frac{2\pi q}{3}\epsilon^{\frac{4}{n}\Delta_+}\frac{\left\langle \mathcal{T}_n^+(0)\tilde{\mathcal{T}}_n^+(\ell)\right\rangle}{\left\langle \mathcal{T}_n(0)\tilde{\mathcal{T}}_n(\ell)\right\rangle} + o(\epsilon^{\frac{4}{n}\Delta_+})}$$

- Here, the Fourier transform is discrete so the sum can actually be carried out exactly but one still approximates the two-point functions involved….

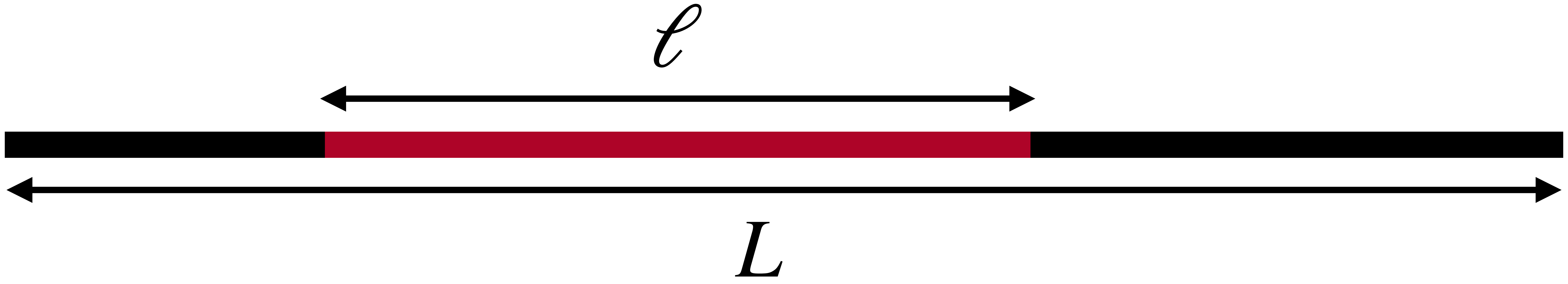
Excited States

- I will now discuss some recent work where we computed the SREE for a certain type of excited states.
- We want to look at states of massive QFT where there is a finite number of excitations above a (generally non-trivial) ground state.

$$|\Psi\rangle_n = |\theta_1 \ldots \theta_k\rangle \otimes \ldots \otimes |\theta_1 \ldots \theta_k\rangle$$



- We will compute the charged moments and then the SREEs for such states in free complex theories (which have U(1) symmetry).
- We will consider a 1+1D system in the following scaling limit:



$$\ell, L \to \infty, \quad \frac{\ell}{L} = r \in [0,1]$$

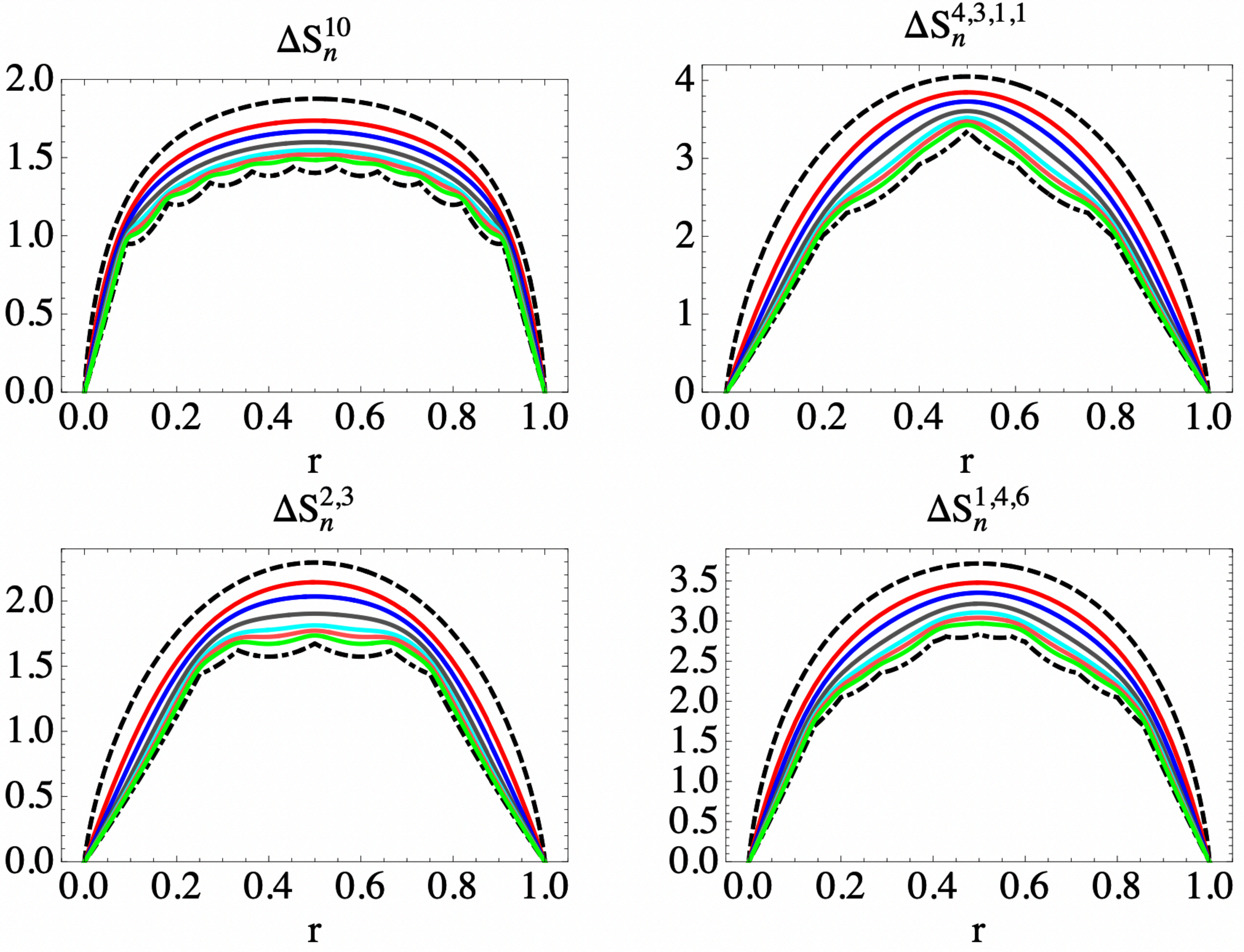
Entanglement Entropy of Excited States

- Recently we did a lot of work trying to understand the entanglement entropy of these states in free QFTs, magnon and certain qubit-based states [O. C.-A., De Fazio, Doyon & Szécsényi'18-19].
- There is also a lot of work by [Zhang & Rajabpour'20-22], especially for spin chains.
- These works showed that, **once the ground state contribution is subtracted**, the entropy that remains is a simple function of r, the number of excitations and their statistics. For example:

$$\Delta S_n^1 = \frac{\log(r^n + (1-r)^n)}{1-n} \qquad \Delta S_1^1 = -r\log r - (1-r)\log(1-r)$$

For a state of one excitation

- And similarly for states of more excitations...


ΔS_n^{10}
$\Delta S_n^{4,3,1,1}$
$\Delta S_n^{2,3}$
$\Delta S_n^{1,4,6}$
r
r
r
r

Computation Techniques/Probabilistic Arguments

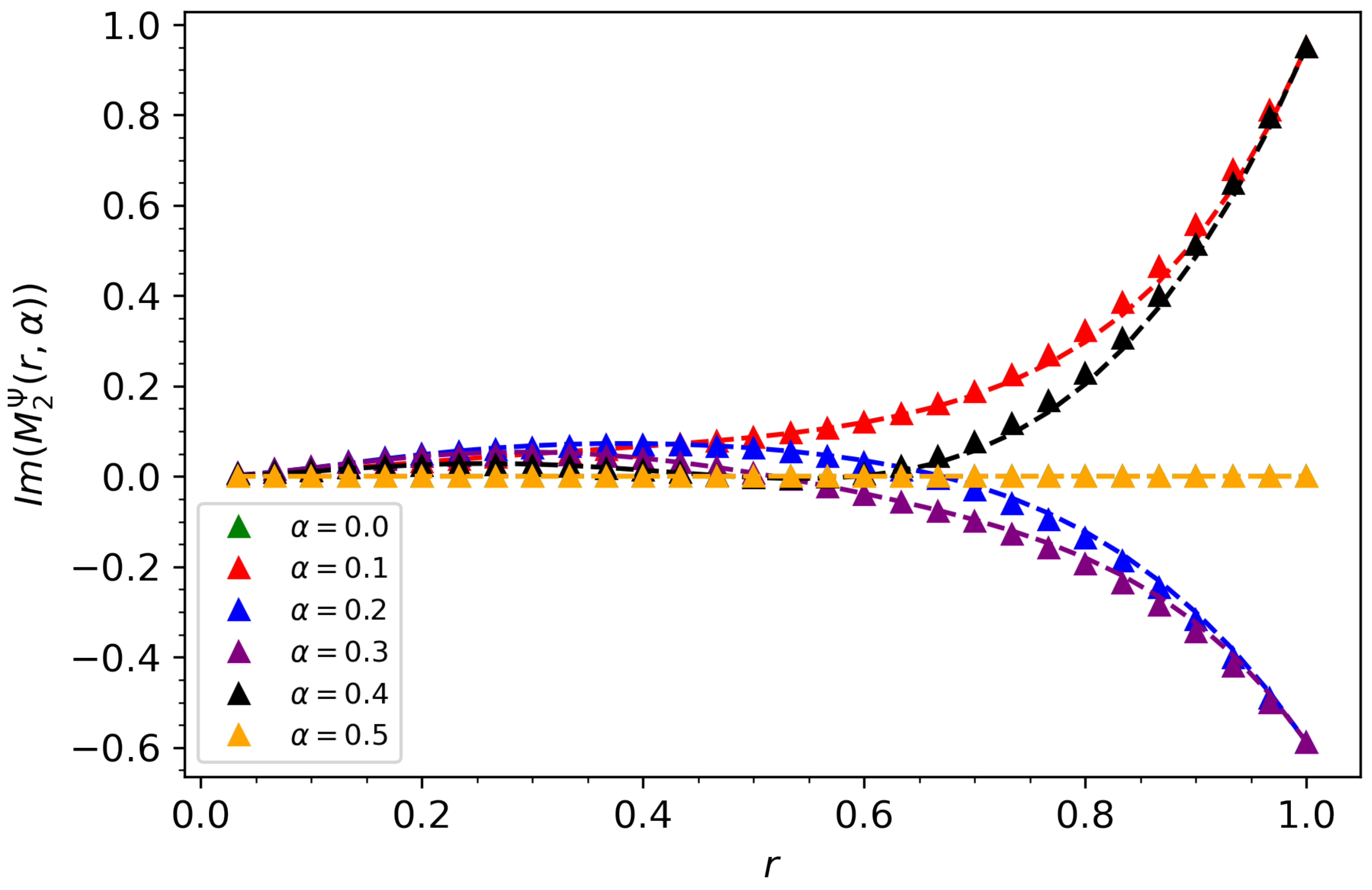
- There are many different ways of obtaining these formulae:

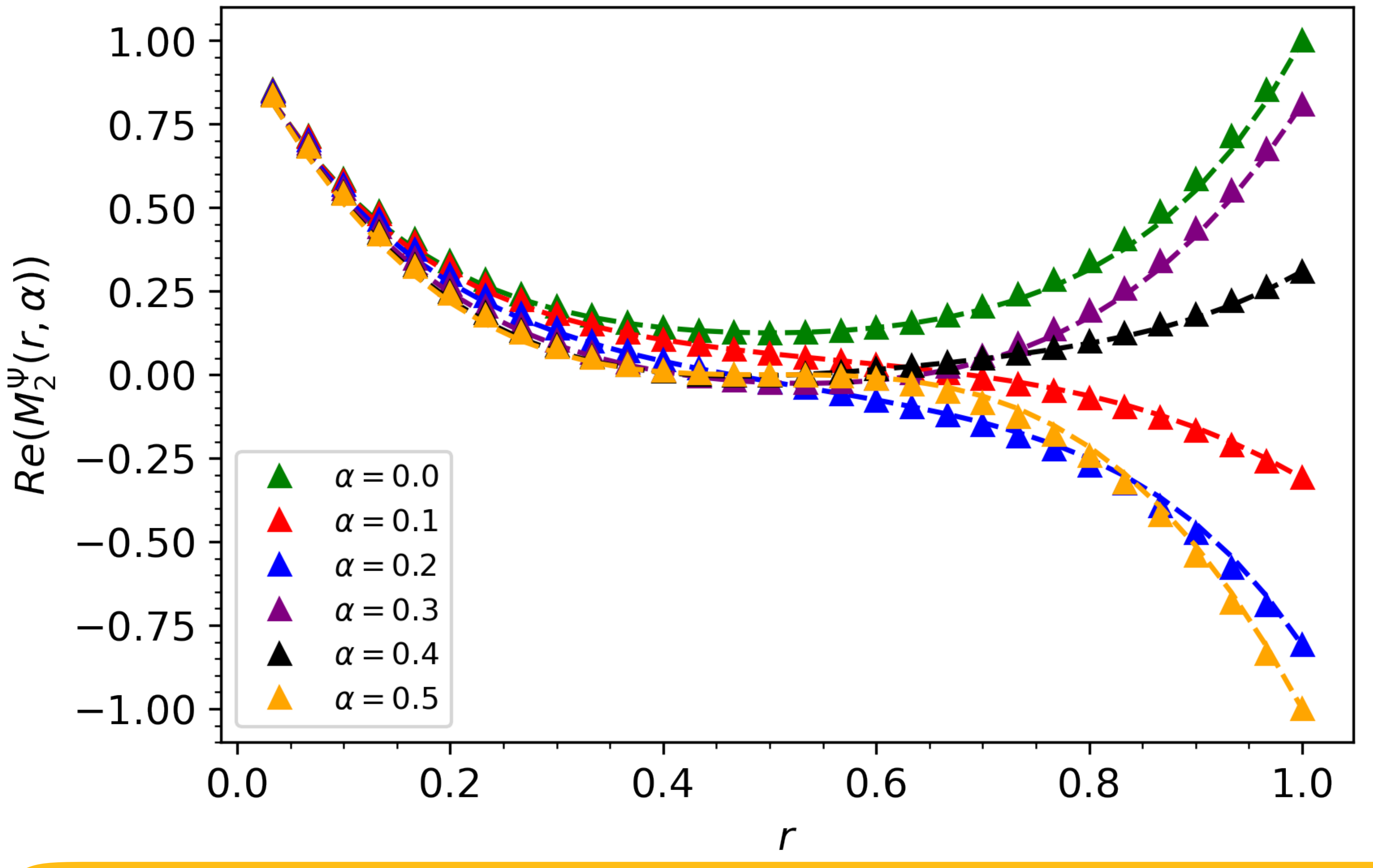
    1. semiclassical/probabilistic arguments,
    2. a QFT computation based on form factors,
    3. computing the entanglement of magnon states in interacting theories,
    4. from simple qubit states ($|\Psi\rangle = \sqrt{r}\,|01\rangle + \sqrt{1-r}\,|10\rangle$)
    5. in free theories (in any dimension, with r appropriately redefined),
    6. in highly excited states of CFT [Capizzi, Ruggiero & Calabrese'20]

- The only important assumption is that the excitations are localised. This can be achieved in different ways: $\xi \ll \ell, L$ (gapped systems) or $\dfrac{2\pi}{P} \ll \ell, L$ (CFT).
- The semiclassical viewpoint was known well before our work and has recently been extended [Mussardo & Viti'21].

Charged Moments for Excited States

- These techniques extend seamlessly to the SREE with the caveat that the natural generalisation of the FF formulae now gives the ratio of the charged moments.





$$M^{\Psi}(r;\alpha) = \frac{Z^{\Psi}(r;\alpha)}{Z^{0}(r;\alpha)}$$

In general

$$M^{1^+1^+}(r;\alpha) = (e^{2\pi i\alpha} r^n + (1-r)^n)^2$$

For a state of two excitations of the same charge but different momenta

Symmetry Resolved Entanglement Entropies

- A particularly nice feature of these states is that (unlike most other cases) one can actually write exact formulae for the SREEs.
- It is easy to argue why this is the case for massive QFT but it is a rather general result.
- To compute the SREEs of the excited state, we need to isolate the charged moments of the excited state only. So we need the product $Z_n^0(r;\alpha)M^\Psi(r;\alpha)$.
- This seems simple, but because we are taking a scaling limit $\ell, L \to \infty$ we need to take care that this limit is finite (separately) for $Z_n^0(r;\alpha)$ and $Z_n^\Psi(r;\alpha)$.
- In massive QFT it is easy to show that the infinite volume limit in the ground state isthe VEV of the composite twist field, and this is a function of α but independent of r.
- The Fourier Transform can be computed exactly and the SREE can be expressed in terms of entropies and partition functions in the ground state.

Example

- The SR partition function for a state of one excitation of charge ϵ is:

$$\mathcal{Z}_n^{1^\epsilon}(r;q) = \int_{-\frac{1}{2}}^{\frac{1}{2}} Z_n^0(\alpha)(e^{2\pi i\epsilon\alpha} r^n + (1-r)^n) e^{-2\pi i\alpha q} d\alpha$$
$$= \mathcal{Z}_n^0(q-\epsilon) r^n + \mathcal{Z}_n^0(q)(1-r)^n$$

- So the SREE is simply:

$$S_n^{1^\epsilon}(r;q) = \frac{1}{1-n} \log \frac{\mathcal{Z}_n^{1^\epsilon}(r;q)}{\mathcal{Z}_1^{1^\epsilon}(r;q)^n}$$
$$= \frac{1}{1-n} \log \frac{\mathcal{Z}_n^0(q-\epsilon) r^n + \mathcal{Z}_n^0(q)(1-r)^n}{[\mathcal{Z}_1^0(q-\epsilon) r + \mathcal{Z}_1^0(q)(1-r)]^n}$$

- The partition functions can be related back to the SREE of the ground state.
- Other states are more complicated but in essence, the same idea holds.
- For qubit and magnon states the ground state is trivial and all these formulae become explicit functions of polynomials in r, which admit interesting probabilistic interpretations

Conclusion

- In this talk I reviewed the definition and some properties of the symmetry resolved entanglement entropies.
- I have tried to give an idea of how these may be computed for IQFTs and how an approach based on quantum fields can be useful in this context.
- In general, it is technically quite difficult to obtain analytical results for this quantity (sG, Potts), but the excited states that we have studied recently provide a very nice example where a lot can be done analytically.
- For these special states the SREEs can be fully expressed in terms of the entropies of the ground state, so the more complex the entanglement structure of GS is, the more complex the contribution from excited states will be.
- An immediate follow up problem are the SR versions of other entanglement measures.
- More generally, we would like to understand the structure of higher order corrections in integrable QFT.


Thank You!