Many new conjectures on Fully-Packed Loop configurations

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work in collaboration with L. Cantini

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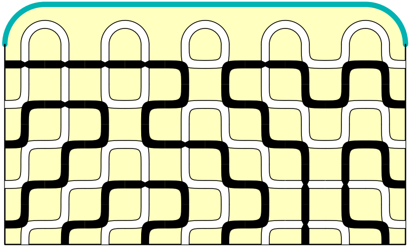
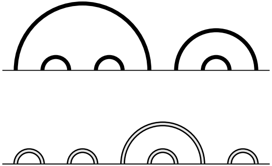
(this talk: May 3^{rd})





$$\#\{\bigcirc\} + \#\{\circledcirc\} = 2$$

Part I

A short reminder of the Razumov–Stroganov conjecture(s)

The many Razumov–Stroganov conjectures

There exists a whole class of Razumov–Stroganov conjectures

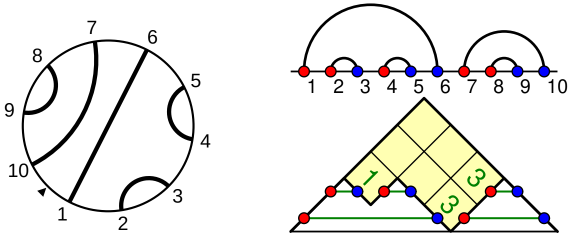
A.V. Razumov and Yu.G. Stroganov, *Combinatorial nature of ground state vector of $O(1)$ loop model*, Theor. Math. Phys. **138** (2004); —, *$O(1)$ loop model with different boundary conditions and symmetry classes of alternating-sign matrices*, Theor. Math. Phys. **142** (2005); J. de Gier, *Loops, matchings and alternating-sign matrices*, Discr. Math. **298** (2005); S. Mitra, B. Nienhuis, J. de Gier and M.T. Batchelor, *Exact expressions for correlations in the ground state of the dense $O(1)$ loop model*, JSTAT(2004); J. de Gier and V. Rittenberg, *Refined Razumov–Stroganov conjectures for open boundaries*, JSTAT(2004); Ph. Duchon, *On the link pattern distribution of quarter-turn symmetric FPL configurations*, FPSAC 2008

Formulated in the early 2000's, they relate the probabilities of some connectivity patterns in two different integrable models: the $O(1)$ Dense Loop Model and the Fully-Packed Loop Model

A nice fact is that they can be formulated in purely combinatorial way, despite the fact that they are related to the physics of the XXZ Quantum Spin Chain and of the 6-Vertex Model

Link patterns

A link pattern $\pi \in LP(2n)$ is a pairing of $\{1, 2, \dots, 2n\}$ having no pairs (a, c), (b, d) such that $a < b < c < d$ (i.e., the drawing consists of n non-crossing arcs).



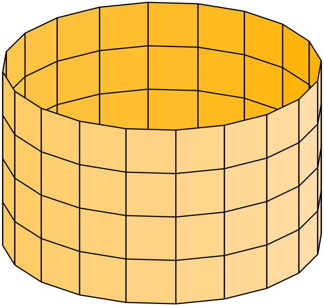
They are $C_n = \frac{1}{n+1}\binom{2n}{n}$ (the n-th *Catalan number*), and are in easy bijection with Dyck Paths of length $2n$ that is, integer partitions $\lambda \preceq \delta_n := (n-1, n-2, \dots, 1)$

$\pi = ((1,6),(2,3),(4,5),(7,10),(8,9))$ $\lambda(\pi) = (3,3,1) \preceq \delta_5$

$O(1)$ Dense Loop Model / XXZ $\Delta = -\frac{1}{2}$ spin chain

Consider dense loop configurations on a semi-infinite cylinder
i.e. tilings of $\{1, \dots, 2n\} \times \mathbb{N}$ with the two tiles  , 
(with the uniform measure)

Link patterns are naturally associated to these configurations
(despite the fact that they are infinite!)



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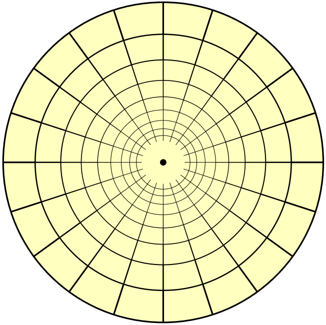

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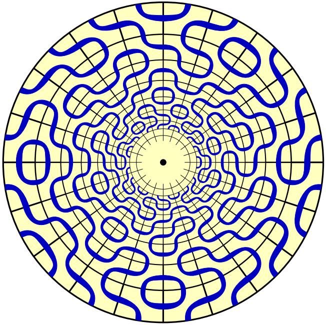

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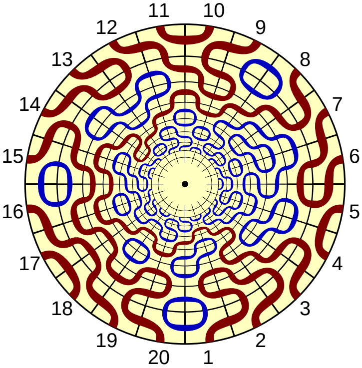
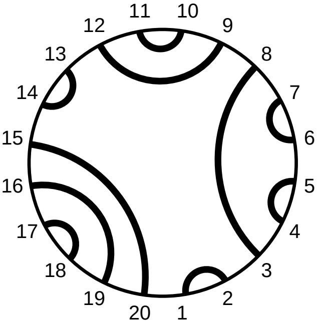
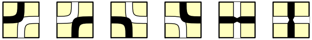
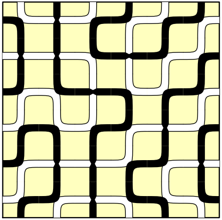




Fully-Packed Loops

Fully-Packed Loop configurations are tilings of the $n \times n$ square using the six tiles  and with black/white alternating boundary conditions

Again, a link pattern π is naturally associated, according to the connectivities among the black terminations on the boundary



Note that, by now, we ignore the link pattern associated to white, and the potential presence of loops

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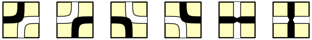
Fully-Packed Loop configurations are tilings of the $n \times n$ square using the six tiles  and with black/white alternating boundary conditions

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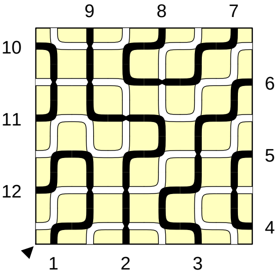

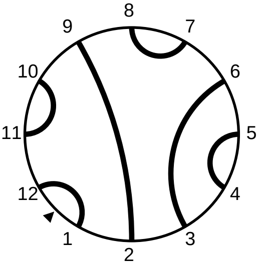

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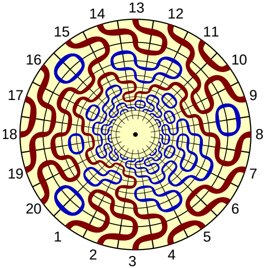
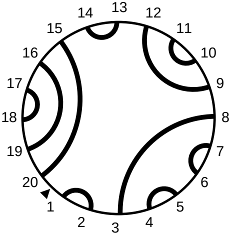
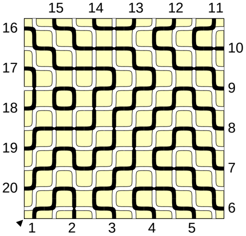
The dihedral Razumov–Stroganov correspondence







$\tilde{\Psi}_n(\pi)$: probability of π
in the $O(1)$ Dense Loop Model
in the $\{1, ..., 2n\} \times \mathbb{N}$ cylinder

$\Psi_n(\pi)$: probability of π
for FPL with uniform measure
in the $n \times n$ square

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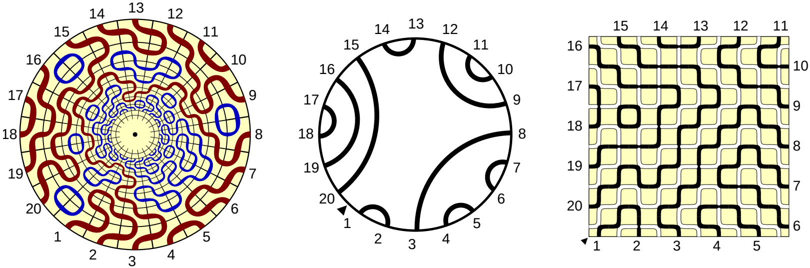

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Razumov–Stroganov correspondence

(conjecture: Razumov and Stroganov, 2001a for the $n \times n$ square;
proof: AS and Cantini, 2010, for all the 'dihedral domains')

$$\tilde{\Psi}_n(\pi) = \Psi_n(\pi)$$

Dihedral symmetry of FPL

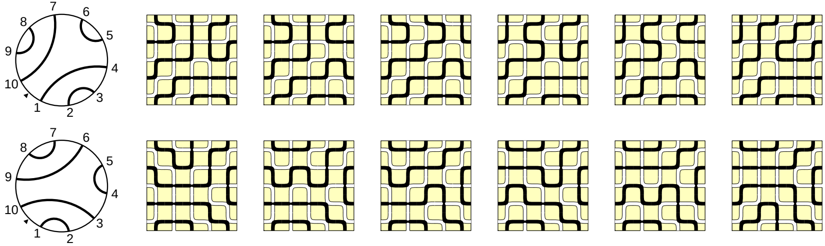
A corollary of the Razumov–Stroganov correspondence. . .

(. . . that was known *before* the Razumov–Stroganov conjecture)

call R the operator that rotates a link pattern by one position

Dihedral symmetry of FPL (proof: Wieland, 2000)

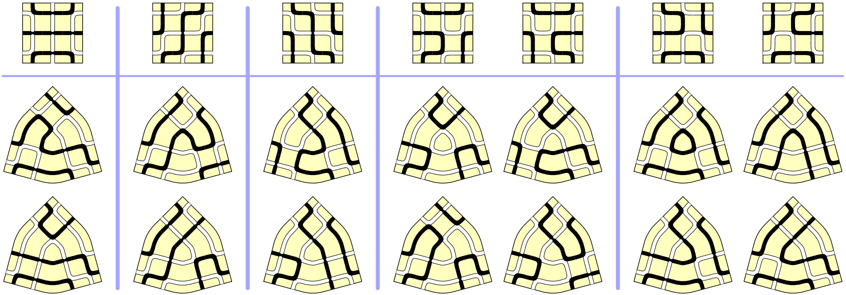
$$\Psi_n(\pi) = \Psi_n(R\pi)$$



Domains with dihedral Razumov–Stroganov correspondence

In the case of the dihedral Razumov–Stroganov correspondence, Wieland gyration (and its generalisations) has been a crucial ingredient.

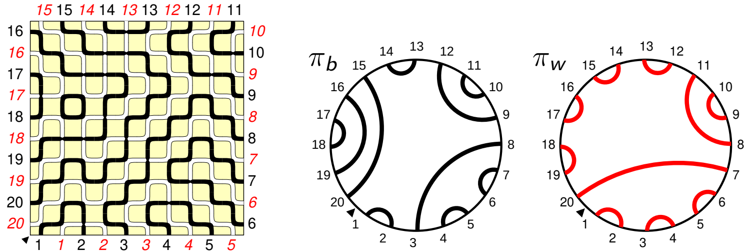
Not surprisingly, understanding the most general family of domains for which the correspondence holds has been inspiring



No black+white Razumov–Stroganov conjecture

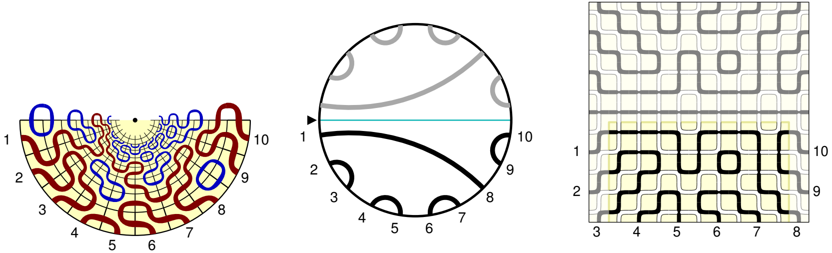
Remark: What is natural to consider in Wieland gyration lemma is the triple (π_b, π_w, ℓ) for the black and white link patterns, and the total number of loops (black+white)

However, we have no candidate replacing the $O(1)$ Dense Loop Model in a black+white version of the Razumov–Stroganov conjecture! (…no, the Rotor Model doesn't seem to work…)



$\ell = 1$

A Vertical Razumov–Stroganov Conjecture



$\tilde{\Psi}_n^V(\pi)$: probability of π
in the $O(1)$ Dense Loop Model
in the $\{1, \ldots, 2n\} \times \mathbb{N}$ strip

$\Psi_n^V(\pi)$: probability of π
for vertically-symmetric FPL
with uniform measure in the
$(2n+1) \times (2n+1)$ square

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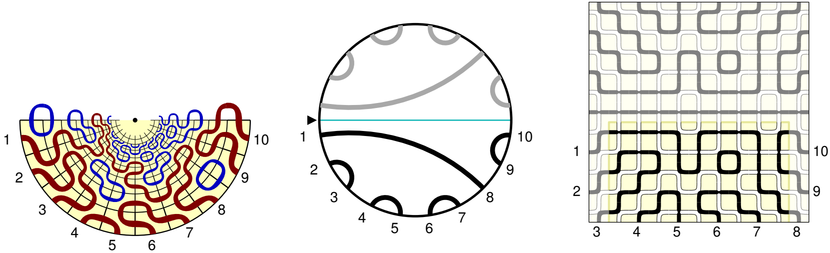

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Vertical Razumov–Stroganov conjecture
(Razumov and Stroganov, 2001b for the square of side $2n+1$)

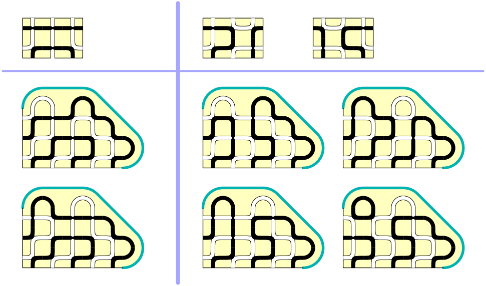
$$\tilde{\Psi}^V_n(\pi) = \Psi^V_n(\pi)$$

Domains with Vertical Razumov–Stroganov correspondence

The Vertical Razumov–Stroganov conjectures are a whole second family
They involve FPL with some version of reflecting wall and the $O(1)$ Dense Loop Model on a strip with a boundary

Our proof methods do not seem to work for any of the Vertical Razumov–Stroganov conjectures, which are all open at present

But at least we think we know the precise list of domains with Vertical RS



$3 + x + 7y + 2xy + 4y^2 + xy^2$ $\qquad$ $6 + 2x + 14y + 4xy + 8y^2 + 2xy^2$

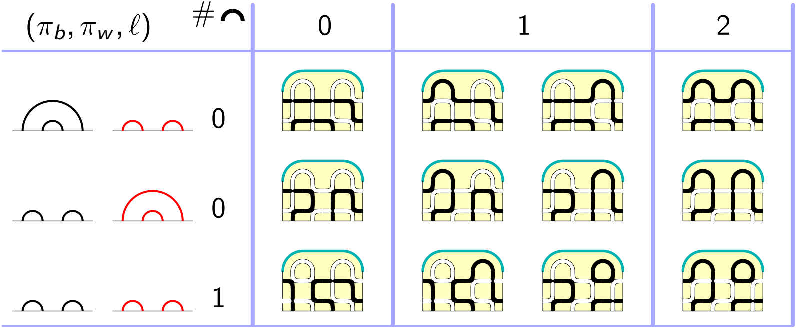
Part II

The many new conjectures...

Looking at UASM more closely

We shall "smash together the two failures" above: ❶ we haven't proven any flavour of the Vertical Razumov–Stroganov conjectures; ❷ we never devised any flavour of Razumov–Stroganov conjectures, not even dihedral, involving the triple enumeration $\Psi_n(\pi_b, \pi_w, \ell)$

We will look more closely at the full list of FPL's in one of the simplest instances of Vertical RS, that is U-turn ASM's (UASM).



The many conjectures on the enumerations $\Psi_{\pi_b,\pi_w}(\tau)$

Let us call $\Psi_n^V(\pi_b, \pi_w, \tau, y)$ the generating function of UASM's at size n, with black/white link patterns π_b and π_w, and weight $\tau^\ell y^{\#\cap}$

Known: $Z_n^V(y) = \sum_{\pi_b,\pi_w} \Psi_n^V(\pi_b, \pi_w, 1, y)$ has an overall factor $(1+y)^n$

G. Kuperberg, *Symmetry classes of alternating-sign matrices under one roof*, Ann. of Math. **156** (2002)

Luigi Cantini and myself conjectured, also long ago (and never published) that this factorisation holds for the RS components

$$\Psi_n^V(\pi_b, y) = \sum_{\pi_w} \Psi_n^V(\pi_b, \pi_w, 1, y) = (1+y)^n \, \tilde{\Psi}_n^V(\pi_b)$$

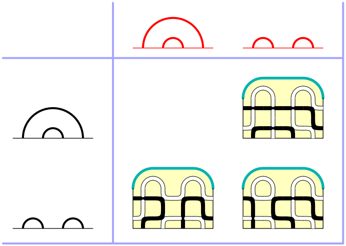
The new numerical investigation leads to the first of our "new conjectures":

Conjecture 1

$$\Psi_n^V(\pi_b, \pi_w, \tau, y) = (1+y)^n \, \Psi_{\pi_b,\pi_w}(\tau) \qquad \forall\ n, \tau, \pi_b, \pi_w$$

(only proven: $(1+y)^2$ divides $\Psi_n^V(\pi_b, \pi_w, \tau, y)$ for $n \geq 2$)

The many conjectures on the enumerations $\Psi_{\pi_b,\pi_w}(\tau)$

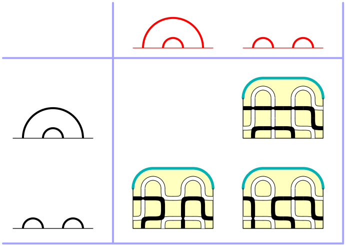


These sets of polynomials are better visualised as tables π_b vs. π_w...

$\longrightarrow$

	0	1
	1	τ

	0	0	0	0	1
	0	0	1	1	2τ
	0	1	τ	τ	τ^2+1
	0	1	τ	τ	τ^2+1
	1	2τ	τ^2+1	τ^2+1	$\tau^3+3\tau$



…or, equivalently, as tables λ vs. ρ with $\lambda = \lambda(\pi_b)$ and $\rho = \lambda(\pi_w)$

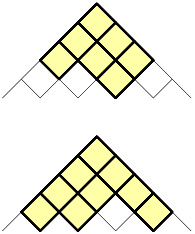
$\longrightarrow$

	0	1
0	0	1
1	1	τ

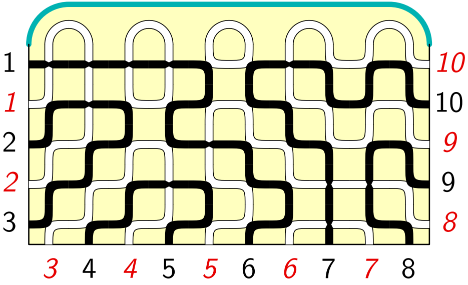
	0	1	2	2	3
0	0	0	0	0	1
1	0	0	1	1	2τ
2	0	1	τ	τ	τ^2+1
2	0	1	τ	τ	τ^2+1
3	1	2τ	τ^2+1	τ^2+1	$\tau^3+3\tau$

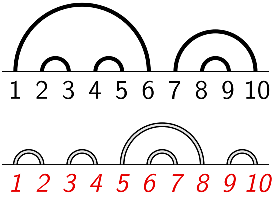
	0	1	2	2	3	3	3	4	4	4	5	5	5	6
0	0	0	0	0	0	0	0	0	0	0	0	0	0	1
1	0	0	0	0	0	0	0	0	0	0	1	1	1	3τ
2	0	0	0	0	0	0	1	1	1	2τ	2τ	2τ	2τ	$2+3\tau^2$
2	0	0	0	0	0	0	0	1	1	1	2τ	2τ	2τ	$2+3\tau^2$
3	0	0	0	0	0	0	1	τ	τ	τ	$1+\tau^2$	$1+\tau^2$	$1+\tau^2$	$\tau(3+\tau^2)$
3	0	0	0	0	0	0	1	τ	τ	τ	$1+\tau^2$	$1+\tau^2$	$1+\tau^2$	$\tau(3+\tau^2)$
3	0	0	0	0	1	1	2	4τ	3τ	4τ	$3+4\tau^2$	$2+4\tau^2$	$3+4\tau^2$	$\tau(10+5\tau^2)$
4	0	0	1	1	τ	τ	4τ	$2+3\tau^2$	$1+2\tau^2$	$2+3\tau^2$	$\tau(5+2\tau^2)$	$\tau(4+2\tau^2)$	$\tau(5+2\tau^2)$	$4+9\tau^2+2\tau^4$
4	0	0	1	1	τ	τ	3τ	$1+2\tau^2$	τ^2	$1+2\tau^2$	$\tau(2+\tau^2)$	$\tau(2+\tau^2)$	$\tau(2+\tau^2)$	$2+4\tau^2+\tau^4$
4	0	0	1	1	τ	τ	4τ	$2+3\tau^2$	$1+2\tau^2$	$2+3\tau^2$	$\tau(5+2\tau^2)$	$\tau(4+2\tau^2)$	$\tau(5+2\tau^2)$	$4+9\tau^2+2\tau^4$
5	0	1	2τ	2τ	$1+\tau^2$	$1+\tau^2$	$3+4\tau^2$	$\tau(5+2\tau^2)$	$\tau(2+\tau^2)$	$\tau(5+2\tau^2)$	$2+5\tau^2+\tau^4$	$2+5\tau^2+\tau^4$	$2+5\tau^2+\tau^4$	$\tau(10+7\tau^2+\tau^4)$
5	0	1	2τ	2τ	$1+\tau^2$	$1+\tau^2$	$2+4\tau^2$	$\tau(4+2\tau^2)$	$\tau(2+\tau^2)$	$\tau(4+2\tau^2)$	$2+5\tau^2+\tau^4$	$2+4\tau^2+\tau^4$	$2+5\tau^2+\tau^4$	$\tau(10+7\tau^2+\tau^4)$
5	0	1	2τ	2τ	$1+\tau^2$	$1+\tau^2$	$3+4\tau^2$	$\tau(5+2\tau^2)$	$\tau(2+\tau^2)$	$\tau(5+2\tau^2)$	$2+5\tau^2+\tau^4$	$2+5\tau^2+\tau^4$	$2+5\tau^2+\tau^4$	$\tau(10+7\tau^2+\tau^4)$
6	1	3τ	$2+3\tau^2$	$2+3\tau^2$	$\tau(3+\tau^2)$	$\tau(3+\tau^2)$	$\tau(10+5\tau^2)$	$4+9\tau^2+2\tau^4$	$2+4\tau^2+\tau^4$	$4+9\tau^2+2\tau^4$	$\tau(10+7\tau^2+\tau^4)$	$\tau(10+7\tau^2+\tau^4)$	$\tau(10+7\tau^2+\tau^4)$	$8+24\tau^2+9\tau^4+\tau^6$

A large example:



$$\Psi_{(3,3,1,0),(4,2,2,1)}(\tau) = \cdots + \tau^2 + \cdots$$





$\#\{\bigcirc\} + \#\{\circledcirc\} = 2$

The many conjectures on the enumerations $\Psi_{\pi_b,\pi_w}(\tau)$

In the following, with abuse of notation, $\Psi_{\lambda\,\rho}(\tau) \equiv \Psi_{\pi_b,\pi_w}(\tau)$

Conjecture 2

$$\deg\left(\Psi_{\lambda\,\rho}(\tau)\right) = |\lambda| + |\rho| - |\delta_n|$$

In particular, $\Psi_{\lambda\,\rho}(\tau) = 0$ if $|\lambda| + |\rho| < \binom{n}{2}$.

Conjecture 3

The $\Psi_{\lambda\,\rho}(\tau)$'s are polynomials of defined parity.

Conjecture 4

The table has three involutions: ❶ $\Psi_{\lambda\,\rho}(\tau) = \Psi_{\rho\,\lambda}(\tau)$;
❷ $\Psi_{\lambda\,\rho}(\tau) = \Psi_{\rho'\,\lambda'}(\tau)$; ❸ $\Psi_{\lambda\,\rho}(\tau) = \Psi_{\lambda\,\rho'}(\tau)$.

❶: easily proven (Wieland + swap b/w);
❷: easily corollary of Conjecture 1 (vertical reflection + swap b/w);
❸: rather mysterious.

Conjecture 5

The entries s.t. $|\lambda| + |\rho| = |\delta_n|$ are the Littlewood–Richardson coefficients $\Psi_{\lambda\,\rho}(\tau) = c^{\delta_n}_{\lambda\rho}$.

	0	1
	1	τ

	0	0	0	0	1
	0	0	1	1	2τ
	0	1	τ	τ	τ^2+1
	0	1	τ	τ	τ^2+1
	1	2τ	τ^2+1	τ^2+1	$\tau^3+3\tau$

	0	1	2	2	3	3	3	4	4	4	5	5	5	6
0	0	0	0	0	0	0	0	0	0	0	0	0	0	1
1	0	0	0	0	0	0	0	0	0	0	1	1	1	3τ
2	0	0	0	0	0	0	1	1	1	2τ	2τ	2τ	$2+3\tau^2$
2	0	0	0	0	0	0	0	1	1	1	2τ	2τ	2τ	$2+3\tau^2$
3	0	0	0	0	0	0	1	τ	τ	τ	$1+\tau^2$	$1+\tau^2$	$1+\tau^2$	$\tau(3+\tau^2)$
3	0	0	0	0	0	0	1	τ	τ	τ	$1+\tau^2$	$1+\tau^2$	$1+\tau^2$	$\tau(3+\tau^2)$
3	0	0	0	0	1	1	2	4τ	3τ	4τ	$3+4\tau^2$	$2+4\tau^2$	$3+4\tau^2$	$\tau(10+5\tau^2)$
4	0	0	1	1	τ	τ	4τ	$2+3\tau^2$	$1+2\tau^2$	$2+3\tau^2$	$\tau(5+2\tau^2)$	$\tau(4+2\tau^2)$	$\tau(5+2\tau^2)$	$4+9\tau^2+2\tau^4$
4	0	0	1	1	τ	τ	3τ	$1+2\tau^2$	τ^2	$1+2\tau^2$	$\tau(2+\tau^2)$	$\tau(2+\tau^2)$	$\tau(2+\tau^2)$	$2+4\tau^2+\tau^4$
4	0	0	1	1	τ	τ	4τ	$2+3\tau^2$	$1+2\tau^2$	$2+3\tau^2$	$\tau(5+2\tau^2)$	$\tau(4+2\tau^2)$	$\tau(5+2\tau^2)$	$4+9\tau^2+2\tau^4$
5	0	1	2τ	2τ	$1+\tau^2$	$1+\tau^2$	$3+4\tau^2$	$\tau(5+2\tau^2)$	$\tau(2+\tau^2)$	$\tau(5+2\tau^2)$	$2+5\tau^2+\tau^4$	$2+5\tau^2+\tau^4$	$2+5\tau^2+\tau^4$	$\tau(10+7\tau^2+\tau^4)$
5	0	1	2τ	2τ	$1+\tau^2$	$1+\tau^2$	$2+4\tau^2$	$\tau(4+2\tau^2)$	$\tau(2+\tau^2)$	$\tau(4+2\tau^2)$	$2+5\tau^2+\tau^4$	$2+4\tau^2+\tau^4$	$2+5\tau^2+\tau^4$	$\tau(10+7\tau^2+\tau^4)$
5	0	1	2τ	2τ	$1+\tau^2$	$1+\tau^2$	$3+4\tau^2$	$\tau(5+2\tau^2)$	$\tau(2+\tau^2)$	$\tau(5+2\tau^2)$	$2+5\tau^2+\tau^4$	$2+5\tau^2+\tau^4$	$2+5\tau^2+\tau^4$	$\tau(10+7\tau^2+\tau^4)$
6	1	3τ	$2+3\tau^2$	$2+3\tau^2$	$\tau(3+\tau^2)$	$\tau(3+\tau^2)$	$\tau(10+5\tau^2)$	$4+9\tau^2+2\tau^4$	$2+4\tau^2+\tau^4$	$4+9\tau^2+2\tau^4$	$\tau(10+7\tau^2+\tau^4)$	$\tau(10+7\tau^2+\tau^4)$	$\tau(10+7\tau^2+\tau^4)$	$8+24\tau^2+9\tau^4+\tau^6$

Part III

Schur functions, Littlewood–Richardson coefficients and all that

Semi-Standard Young Tableaux $SSYT(\lambda, n)$:

Fillings of λ with the integers $\{1, 2, \ldots, n\}$, repetitions allowed, satisfying $\begin{matrix}\bullet & \leq & \bullet \\ \wedge & & \\ \bullet & & \end{matrix}$

Play a crucial role in the representation theory of the general linear group $GL(n)$

1	1	3	4	4
2	3			
5	6			
6				

Remark: $SSYT(\lambda, n) = \varnothing$ if $n < \ell(\lambda)$

Schur polynomials are the 'generating functions' of SSYT's:

$$s_\lambda(x_1, \ldots, x_n) = \sum_{T \in SSYT(\lambda,n)} \prod_{i=1}^{n} x_i^{\#\{i \in T\}}$$

s  $(x_1, \ldots, x_6) = \cdots + x_1^2 x_2 x_3^2 x_4^2 x_5 x_6^2 + \cdots$

Many beautiful facts about Schur Functions

❶ Schur polynomials are homogeneous of degree $|\lambda|$ and symmetric (seen via the Bender–Knuth involution). They form a basis of the algebras of symmetric polynomials

$$\Lambda_{n,\mathbb{K}}(\vec{x}) = \left[\begin{matrix} \text{algebra of symm.} \\ \text{polyn. in } x_1, \dots, x_n \end{matrix} \right] = span_{\mathbb{K}}\big(s_\lambda(x_1, \dots, x_n)\big)_{\lambda : \ell(\lambda) \leq n}$$

❷ The Weyl character formula tells that the Schur polynomials can be written as the ratio of two determinants

$$s_\lambda(x_1, \dots, x_n) = \frac{1}{\Delta(\vec{x})} \det \left(\left(x_i^{(\lambda+\delta_n)_j} \right)_{i,j=1,\dots,n} \right)$$

$$\Delta(\vec{x}) = \det \left(\left(x_i^{(\delta_n)_j} \right)_{i,j=1,\dots,n} \right) = \prod_{i<j} (x_i - x_j)$$

Many beautiful facts about Schur Functions

❸ Call $\begin{cases} e_k(\vec{x}) = \sum_{i_1 < i_2 < \cdots < i_k} x_{i_1} \cdots x_{i_k} \\ h_k(\vec{x}) = \sum_{i_1 \leq i_2 \leq \cdots \leq i_k} x_{i_1} \cdots x_{i_k} \end{cases}$

We can write $s_\lambda(x_1, \dots, x_n)$ as polynomials in the $e_k(x_1, \dots, x_n)$'s, or the $h_k(x_1, \dots, x_n)$'s. As soon as $n \geq \ell(\lambda)$, these expressions are given by the Jacobi–Trudi and dual Jacobi–Trudi formulas

$$\begin{aligned} s_\lambda &= \det\left(\left(h_{\lambda_i + j - i} \right)_{i,j=1,\dots,\ell(\lambda)} \right) \quad (JT) \\ &= \det\left(\left(e_{\lambda'_i + j - i} \right)_{i,j=1,\dots,\lambda_1} \right) \quad (dJT) \end{aligned}$$

In particular, they stabilise (i.e., become independent of n)

This allows to define Schur functions, defined also for infinite alphabets

Many beautiful facts about Schur Functions

One useful class of infinite alphabets is induced by the ('supersymmetry') ω-involution, that exchanges e_k's and h_k's. That is, we have Schur functions (in fact, polynomials) depending on a 'finite supersymmetric alphabet', $s_\lambda(x_1, \dots, x_n | y_1, \dots, y_m)$ (the name is 'legitimate' as these are super-characters of $GL(n|m)$)

These functions can be defined through the JT or dJT formulas, setting $h_k(x_1, \dots, x_n | y_1, \dots, y_m) = [z^k] \big(\prod_j (1 + z y_j) \big) / \big(\prod_i (1 - z x_i) \big)$ and $e_k(x_1, \dots, x_n | y_1, \dots, y_m) = [z^k] \big(\prod_i (1 + z x_i) \big) / \big(\prod_j (1 - z y_j) \big)$

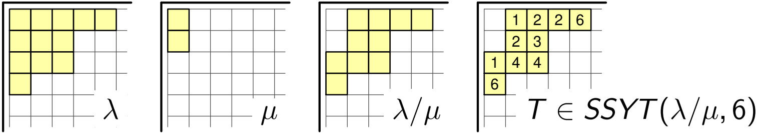
Exchanging the 'bosonic' and 'fermionic' parts of the alphabet accounts to take the transpose Young diagrams

$$s_{\lambda/\mu}(x_1, \dots, x_n | y_1, \dots, y_m) = s_{\lambda'/\mu'}(y_1, \dots, y_m | x_1, \dots, x_n)$$

Many beautiful facts about Schur Functions

❹ Define the skew Schur polynomials as

$$s_{\lambda/\mu}(x_1,\dots,x_n) = \sum_{T\in SSYT(\lambda/\mu,n)} \prod_{i=1}^{n} x_i^{\#\{i\in T\}}$$



We have

$$\begin{aligned} s_{\lambda/\mu} &= \det\left(\left(h_{\lambda_i-\mu_j+j-i}\right)_{i,j=1,\dots,\ell(\lambda)}\right) \quad (JT)\\ &= \det\left(\left(e_{\lambda'_i-\mu'_j+j-i}\right)_{i,j=1,\dots,\lambda_1}\right) \quad (dJT)\end{aligned}$$

In the scalar product $\langle \cdot | \cdot \rangle$ such that the Schur basis is self-dual

$$\langle s_\lambda | s_\mu \rangle = \delta_{\lambda\mu}$$

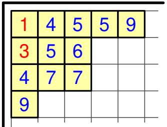
(this is called the Hall scalar product)

these polynomials have the property $\langle h | s_{\lambda/\mu} \rangle = \langle h\, s_\mu | s_\lambda \rangle \; \forall h$

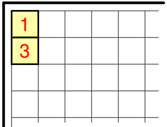
Many beautiful facts about Schur Functions

It follows that

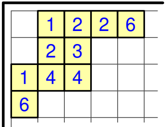
$$s_\lambda(x_1,\dots,x_n,x_{n+1},\dots,x_{n+m}) = \sum_\mu s_\mu(x_1,\dots,x_n)\,s_{\lambda/\mu}(x_{n+1},\dots,x_{n+m})$$



$T_1 \in SSYT(\lambda, n+m)$



$T_2 \in SSYT(\mu, n)$



$T_3 \in SSYT(\lambda/\mu, m)$

(this is evident for finite alphabets, but the formula $s_\lambda(\vec{x}\cup\vec{y}) = \sum_\mu s_\mu(\vec{x})s_{\lambda/\mu}(\vec{y})$ holds also for infinite alphabets)

❺ The structure constants $c^\lambda_{\mu\nu}$ of the algebra $\Lambda = span_{\mathbb{K}}\big(s_\lambda(\vec{x})\big)_\lambda$ are non-negative integers known as Littlewood–Richardson coefficients

$$s_\mu(\vec{x})s_\nu(\vec{x}) = \sum_\lambda c^\lambda_{\mu\nu}\, s_\lambda(\vec{x}) \qquad c^\lambda_{\mu\nu} \in \mathbb{N}$$

Many beautiful facts about Schur Functions

What we said above implies that the three problems

$$\begin{cases} s_\mu(\vec{x})s_\nu(\vec{x}) &= \sum_\lambda c^\lambda_{\mu\nu}\, s_\lambda(\vec{x}) \\ s_{\lambda/\mu}(\vec{x}) &= \sum_\nu c^\lambda_{\mu\nu}\, s_\nu(\vec{x}) \\ s_\lambda(\vec{x}\cup\vec{y}) &= \sum_{\mu,\nu} c^\lambda_{\mu\nu}\, s_\mu(\vec{x})s_\nu(\vec{y}) \end{cases}$$

are all solved by the same Littlewood–Richardson coefficients

Many other interesting basis of symmetric functions (Hall–Littlewood, Grothendieck, ...) generalise the Schur case in some sense, but, if we insist on keeping the Hall ($\langle s_\lambda | s_\mu \rangle = \delta_{\lambda\mu}$) scalar product, self-duality is not present in general.

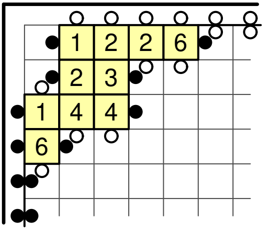
We have two basis of functions, $\{f_\lambda\}$ and $\{g^\lambda\}$, such that $\left\langle g^\lambda | f_\mu \right\rangle = \delta^\lambda_\mu$, and two different sets of structure constants

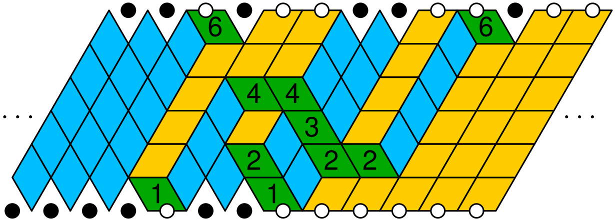
$$f_\lambda\, f_\mu = \sum_\nu c^\nu_{\lambda\mu}\, f_\nu \qquad g^\lambda\, g^\mu = \sum_\nu d^{\lambda\mu}_\nu\, g^\nu$$

Representation of Schur polynomials as Vertex Models

(Skew-)Schur polynomials can be represented as partition functions of tiling models, namely as free-fermionic $\mathcal{U}_q(\widehat{sl}_2)$ Yang–Baxter integrable Vertex Models with homogeneous vertical spectral parameters, the horizontal ones determine the alphabet

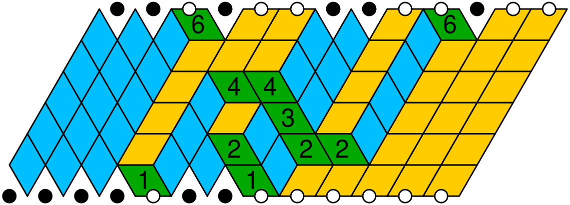
$s_{\lambda/\mu}(x_1, \dots, x_n)$ is described by an infinite horizontal strip, of height n, where all non-trivial tiles occur within a width $\lambda_1 + \ell(\lambda)$
The partitions λ and μ fix the top and bottom boundary conditions





Representation of Schur polynomials as Vertex Models

Lozenge tilings are nice, but, in order to describe in a symmetric way the 'supersymmetric' (skew-)Schur functions, we shall rather shear the triangular lattice into the square lattice



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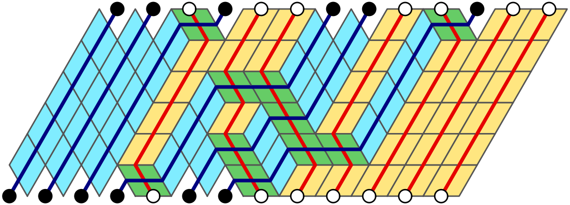
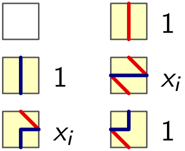
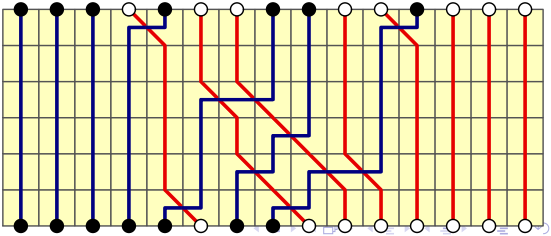


$T(x_i)$:





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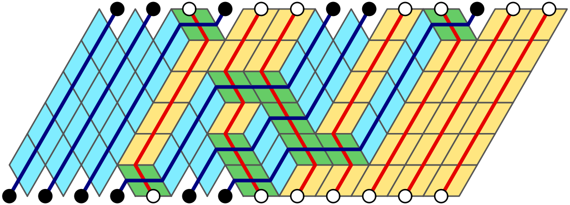
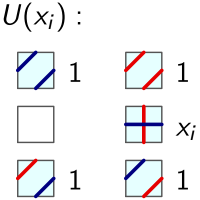
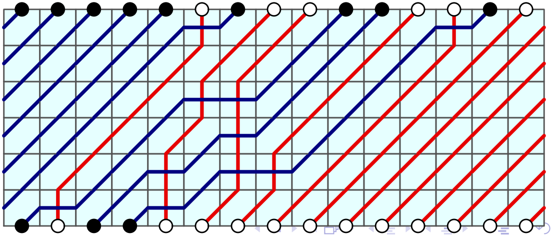


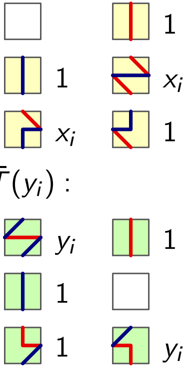




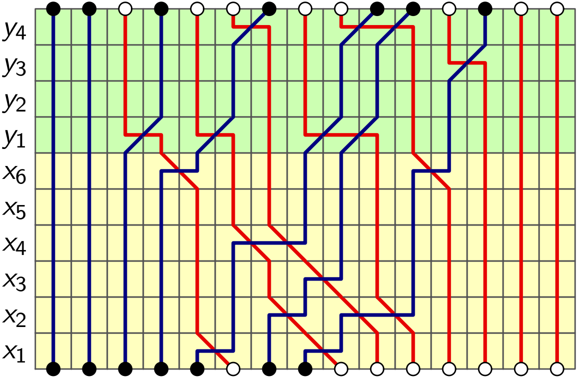
Representation of Schur polynomials as Vertex Models

$T(x_i)$:

$\bar{T}(y_i)$:

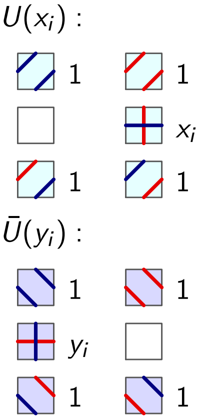


A supersymmetric skew Schur polynomial:



s  $(x_1, \ldots, x_6 | y_1, \ldots, y_4) = \cdots + x_1^2 x_2^3 x_3 x_4^2 x_6^2 \; y_1^4 y_3 y_4^3 + \cdots$

Representation of Schur polynomials as Vertex Models



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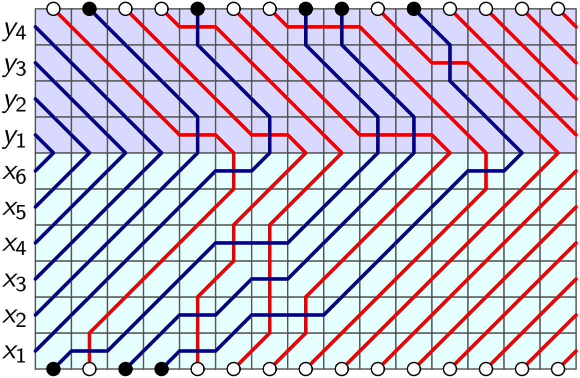


s  $(x_1, \ldots, x_6 | y_1, \ldots, y_4) = \cdots + x_1^2 x_2^3 x_3 x_4^2 x_6^2 \; y_1^4 y_3 y_4^3 + \cdots$

Representation of Schur polynomials as Vertex Models

The operators $T(x)$ and $\bar{T}(y)$ are 'transfer matrices'.
They act on the Hilbert space indexed by integer partitions, as

$$\langle \mu | T(x) | \lambda \rangle = \begin{cases} x^{|\lambda/\mu|} & \mu \preceq \lambda \,;\, \lambda/\mu \text{ is a 'horizontal strip' (no } \begin{array}{|c|}\hline \\ \hline \\ \hline \end{array} \text{)} \\ 0 & \text{otherwise} \end{cases}$$

$$\langle \mu | \bar{T}(y) | \lambda \rangle = \begin{cases} y^{|\lambda/\mu|} & \mu \preceq \lambda \,;\, \lambda/\mu \text{ is a 'vertical strip' (no } \begin{array}{|c|c|}\hline & \\ \hline \end{array} \text{)} \\ 0 & \text{otherwise} \end{cases}$$

$$s_{\lambda/\mu}(x_1, \dots, x_n | y_1, \dots, y_m) = \langle \mu | T(x_1) \cdots T(x_n) \bar{T}(y_1) \cdots \bar{T}(y_m) | \lambda \rangle$$

In particular $\langle \mu | T(x) | \lambda \rangle = \langle \mu' | \bar{T}(x) | \lambda' \rangle$

Of course, by definition of transpose operator,
$\langle \mu | T^{+}(x) | \lambda \rangle = \langle \lambda | T(x) | \mu \rangle$ and $\langle \mu | \bar{T}^{+}(x) | \lambda \rangle = \langle \lambda | \bar{T}(x) | \mu \rangle$

Schur processes

Operators $T(x)$, $\bar{T}(y)$ and their transpose form an interesting algebra

$$T(x)|\varnothing\rangle = \bar{T}(x)|\varnothing\rangle = |\varnothing\rangle \qquad \langle\varnothing|T^{+}(x) = \langle\varnothing|\bar{T}^{+}(x) = \langle\varnothing|$$

$$\left[T(x), T(y)\right] = \left[\bar{T}(x), \bar{T}(y)\right] = \left[T(x), \bar{T}(y)\right] = 0$$

$$T(x)\,T^{+}(y) = \frac{1}{1-xy}\,T^{+}(y)\,T(x) \qquad \bar{T}(x)\,\bar{T}^{+}(y) = \frac{1}{1-xy}\,\bar{T}^{+}(y)\,\bar{T}(x)$$

$$T(x)\,\bar{T}^{+}(y) = (1+xy)\,\bar{T}^{+}(y)\,T(x) \qquad \bar{T}(x)\,T^{+}(y) = (1+xy)\,T^{+}(y)\,\bar{T}(x)$$

This is proven through the Yang–Baxter equation for the corresponding 'free-fermionic 5-Vertex Model with electric fields'.

Partition functions and correlation functions of several dimer models (lozenges, domino tilings,...) can be calculated in this way

✍ A. Okounkov and N. Reshetikhin, *Correlation function of Schur process with application to local geometry of a random 3-dimensional Young diagram*, J. Amer. Math. Soc. **16** (2003)

Littlewood–Richardson coefficients as a Vertex Model

Remarkably, also the Littlewood–Richardson coefficients are described by an integrable Vertex Model, this time of square-triangle tilings, with underlying $\mathcal{U}_q(\widehat{sl}_3)$ symmetry.

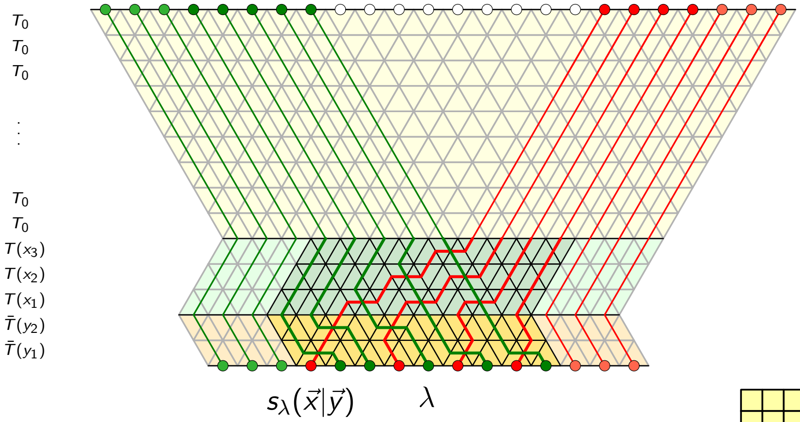
A. Knutson and T. Tao, *Puzzles and (equivariant) cohomology of Grassmannians*, Duke Math. J. **119** (2003); P. Zinn-Justin, *Littlewood–Richardson Coefficients and Integrable Tilings*, EJC **16** (2009)

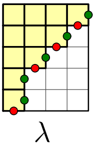
The key idea is to express the two sides of the coproduct identity $s_\lambda(\vec{x}|\vec{y}) = \sum_{\mu,\nu} c^\lambda_{\mu\nu}\, s_\mu(\vec{x}) s_{\nu'}(\vec{y})$ as partition functions in a rank-2 model (i.e., with particles of three colours)

The three Schur terms, $s_\lambda(\vec{x}|\vec{y})$, $s_\mu(\vec{x})$ and $s_{\nu'}(\vec{y})$, are realised within the three possible embeddings of $\widehat{sl}_2$ in $\widehat{sl}_3$ that is, the three choices of two colours among three

The identity is a consequence of commutation of transfer matrices, which in turns comes from the Yang–Baxter equation of the rank-2 model

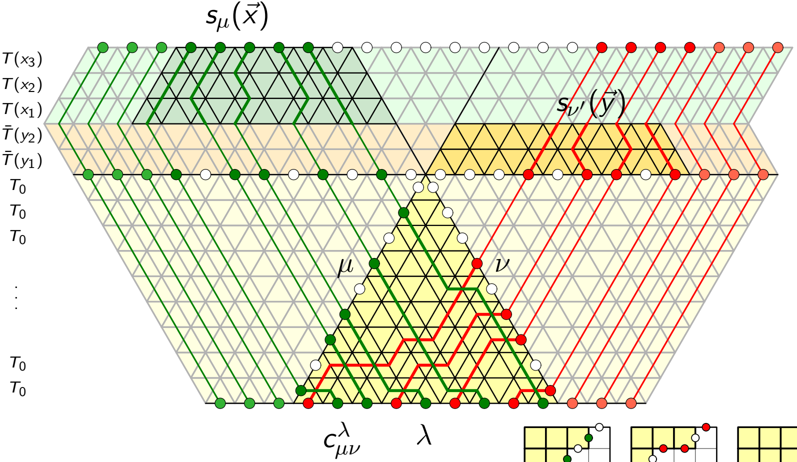
Littlewood–Richardson coefficients as a Vertex Model



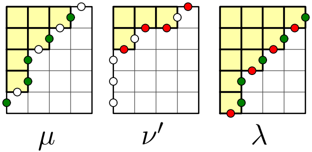


$$s_\lambda(\vec{x}|\vec{y}) = \sum_{\mu,\nu} c^{\lambda}_{\mu\nu}\, s_\mu(\vec{x}) s_{\nu'}(\vec{y})$$

Littlewood–Richardson coefficients as a Vertex Model



$$s_\lambda(\vec{x}|\vec{y}) = \sum_{\mu,\nu} c^\lambda_{\mu\nu}\, s_\mu(\vec{x}) s_{\nu'}(\vec{y})$$



A property of the Littlewood–Richardson coefficients

Let us come back to our "many new conjectures"...

Conjecture 4

❶ $\Psi_{\lambda\,\rho} = \Psi_{\rho\,\lambda}$; ❷ $\Psi_{\lambda\,\rho} = \Psi_{\rho'\,\lambda'}$; ❸ $\Psi_{\lambda\,\rho} = \Psi_{\lambda\,\rho'}$.

Conjecture 5

When $|\lambda| + |\rho| = |\delta_n|$ we have $\Psi_{\lambda\,\rho} = c^{\delta_n}_{\lambda\rho}$ (Littlewood–Richardson)

Are these two conjectures even compatible?
Indeed, ❶ and ❷ are simple symmetries of LR coeffs
(with ❷ using the fact $\delta_n = (\delta_n)'$),
but why on Earth should we have $c^{\lambda}_{\mu\nu} = c^{\lambda}_{\mu\nu'}$?

Call $\mathcal{T} = \{\delta_n\}_{n\geq 1}$ and $\mathcal{M} = \{\lambda \,|\, c^{\lambda}_{\mu\nu} = c^{\lambda}_{\mu\nu'}\ \forall \mu, \nu\}$

Lemma

$$\mathcal{T} = \mathcal{M}$$

A property of the Littlewood–Richardson coefficients

Lemma

$\mathcal{T} = \{\delta_n\}_{n\geq 1}$ and $\mathcal{M} = \{\lambda \,|\, c^{\lambda}_{\mu\nu} = c^{\lambda}_{\mu\nu'} \ \forall \mu, \nu\}$ coincide.

Proof. The implication $\lambda \notin \mathcal{T} \Rightarrow \lambda \notin \mathcal{M}$ is easy (recognise that $\lambda \notin \mathcal{T} \Leftrightarrow \lambda = [\alpha \circ \circ \bullet \beta]$ or $\lambda = [\alpha \circ \bullet \bullet \beta]$, call $\mu = [\alpha \bullet \circ \circ \beta]$ or $\mu = [\alpha \bullet \bullet \circ \beta]$, and evaluate $c^{\lambda}_{\mu\,(2)}$, $c^{\lambda}_{\mu\,(1,1)}$)

The implication $\lambda \in \mathcal{T} \Rightarrow \lambda \in \mathcal{M}$ is interesting.
The crucial observation is that $T(x)|\delta_n\rangle = \bar{T}(x)|\delta_n\rangle$

that, using the commutation of T's and $\bar{T}$'s, implies on supersymmetric skew Schur functions $s_{\delta_n/\mu}(\vec{x}|\vec{y}) = s_{\delta_n/\mu}(\vec{y}|\vec{x})$

by the coproduct definition of LR's:
$\sum_\nu c^{\delta_n}_{\mu\nu}\, s_\nu(\vec{x}|\vec{y}) = s_{\delta_n/\mu}(\vec{x}|\vec{y}) = s_{\delta_n/\mu}(\vec{y}|\vec{x}) = \sum_\nu c^{\delta_n}_{\mu\nu}\, s_\nu(\vec{y}|\vec{x}) = \sum_\nu c^{\delta_n}_{\mu\nu}\, s_{\nu'}(\vec{x}|\vec{y}) = \sum_\nu c^{\delta_n}_{\mu\nu'}\, s_\nu(\vec{x}|\vec{y})$. By the linear independence of Schur functions $c^{\delta_n}_{\mu\nu} = c^{\delta_n}_{\mu\nu'}$ □

A mystery plot

We have mentioned that there exists several deformations of Schur functions (Grothendiek, Hall–Littlewood, ...), many of them allow for a representation as an integrable Vertex Model, and even some representation *à la* Zinn-Justin of the corresponding structure constants (i.e., with the trick "sl_2 embeds into sl_3 in three ways").

✍ M. Wheeler and P. Zinn-Justin, *Littlewood–Richardson coefficients for Grothendieck polynomials from integrability*, J. für die Reine und Angewandte Math. **757** (2017); — *Hall polynomials, inverse Kostka polynomials and puzzles*, JCT-A **159** (2018).

Maybe there exists a basis/dual-basis of symmetric functions $\{f_\lambda\}$, $\{g^\lambda\}$, which are a τ-deformation of Schur fns., such that $\Psi_{\lambda\,\rho}(\tau) = c^{\delta_n}_{\lambda\rho}$ or $\Psi_{\lambda\,\rho}(\tau) = d^{\lambda\rho}_{\delta_n}$, for all pairs $\lambda, \rho \preceq \delta_n$?

Maybe we will have a result of the form $\Psi_{\lambda\,\rho}(\tau) = \sum_{P \in \mathcal{P}_{\lambda,\rho,\delta_n}} \tau^{x(P)}$ with $\mathcal{P}_{\lambda,\rho,\delta_n}$ some variant of Knutson–Tao puzzles, and $x(P)$ the number of tiles of some kind?

A mystery plot: collecting the hints

We shall suppose that these new functions exist, are still described by an integrable Vertex Model, and are given by a 'minimal' deformation of $T(x)$ and $\bar{T}(y)$ operators.

Which properties shall we reproduce?

1. The degree condition (and its corollary on which $\Psi_{\lambda\,\rho}$ do vanish)
2. Polynomials of defined parity
3. The mysterious extra symmetry $\Psi_{\lambda\,\rho} = \Psi_{\lambda\,\rho'}$
4. The new T and $\bar{T}$ must still constitute a commuting family
5. $\langle \mu | T(x) | \lambda \rangle$ well-defined on infinite strings $\cdots \bullet \bullet \bullet [\cdots] \circ \circ \circ \cdots$

Which generalisations we do not want?

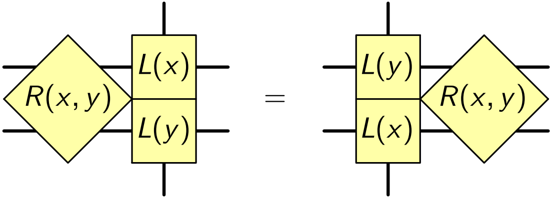
1. We do not "change δ_n" (e.g., try $\Psi_{\lambda\,\rho}(\tau) = \sum_{\theta \succeq \delta_n} c^{\theta}_{\lambda\rho} \tau^{|\theta/\delta_n|}$)
2. We only investigate Vertex Models with "spin $\frac{1}{2}$" horizontal and vertical spaces

The reason is that

we want our proof of $c^{\delta_n}_{\lambda\rho} = c^{\delta_n}_{\lambda\rho'}$ to extend to $\Psi_{\lambda\,\rho}(\tau)$ almost verbatim

5VM and 6VM $RLL = LLR$ relations

The standard technique from Integrable Systems is to construct a $RLL = LLR$ relation (a version of Yang–Baxter when the spaces are not all equal), that is, for L the tile-weights appearing in the transfer matrices T and $\bar{T}$, devise a matrix R such that



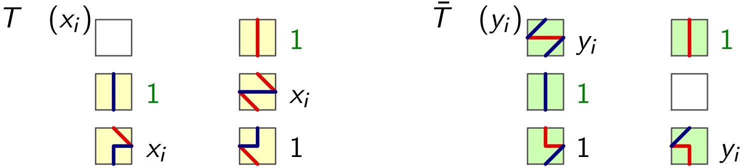
5VM and 6VM $RLL = LLR$ relations

For the (non–free-fermionic) 5-Vertex Model, this can be done unambiguously, once we take into account:

❶ weight well-defined on infinite strings;

❷ gauge invariance;

❸ covariance under reparametrisation;



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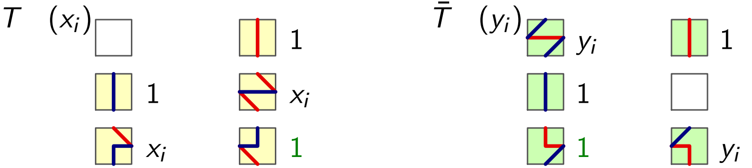

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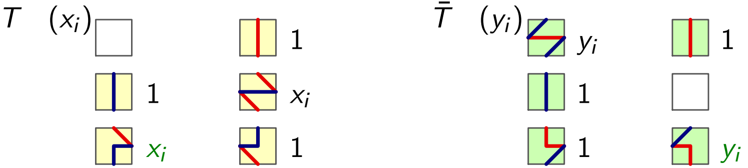

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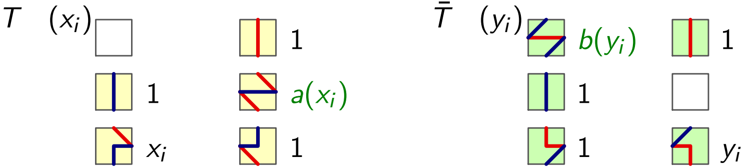

$$a(x_1) - x_1 = a(x_2) - x_2 = b(y_1) - y_1 = b(y_2) - y_2$$

5VM and 6VM $RLL = LLR$ relations

For the (non–free-fermionic) 5-Vertex Model, this can be done unambiguously, once we take into account:

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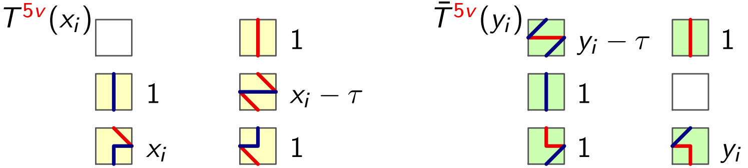


Non-FF 5VM and dual Canonical Grothendieck polynomials

The FF 5VM operators T and $\bar{T}$ act on integer partitions as

$$\langle\mu|T(x)|\lambda\rangle = \begin{cases} x^{|\lambda/\mu|} & \mu \preceq \lambda\,;\,\lambda/\mu \text{ hor. strip} \\ 0 & \text{otherwise} \end{cases}$$

$$\langle\mu|\bar{T}(y)|\lambda\rangle = \begin{cases} y^{|\lambda/\mu|} & \mu \preceq \lambda\,;\,\lambda/\mu \text{ vert. strip} \\ 0 & \text{otherwise} \end{cases}$$

$$s_{\lambda/\mu}(x_1,\dots,x_n|y_1,\dots,y_m) = \langle\mu|T(x_1)\cdots T(x_n)\bar{T}(y_1)\cdots\bar{T}(y_m)|\lambda\rangle$$

1	1	3	4	4	4
2	3				
4	6				
6					

$x_1^2\, x_2\, x_3^2\, x_4^4\, x_6^2$

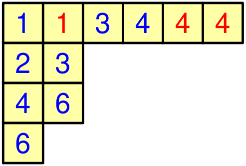
Non-FF 5VM and dual Canonical Grothendieck polynomials

The non-FF 5VM operators T and $\bar{T}$ act on integer partitions as

$$\langle \mu | T^{5v}(x) | \lambda \rangle = \begin{cases} x^{K(\lambda/\mu)} (x-\tau)^{|\lambda/\mu| - K(\lambda/\mu)} & \mu \preceq \lambda \, ; \lambda/\mu \text{ hor. strip} \\ 0 & \text{otherwise} \end{cases}$$

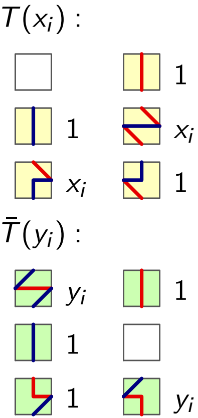
$$\langle \mu | \bar{T}^{5v}(y) | \lambda \rangle = \begin{cases} y^{K(\lambda/\mu)} (y-\tau)^{|\lambda/\mu| - K(\lambda/\mu)} & \mu \preceq \lambda \, ; \lambda/\mu \text{ vert. strip} \\ 0 & \text{otherwise} \end{cases}$$

$$f_{\lambda/\mu}(x_1, \dots, x_n | y_1, \dots, y_m) = \langle \mu | T^{5v}(x_1) \cdots T^{5v}(x_n) \bar{T}^{5v}(y_1) \cdots \bar{T}^{5v}(y_m) | \lambda \rangle$$

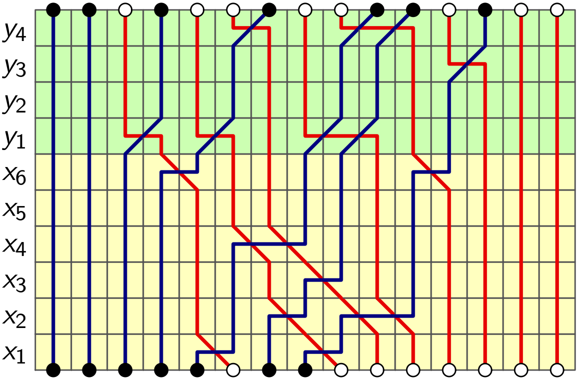


$$x_1 (x_1 - \tau) x_2 x_3^2 x_4^2 (x_4 - \tau)^2 x_6^2$$

Schur vs. f_λ: an example



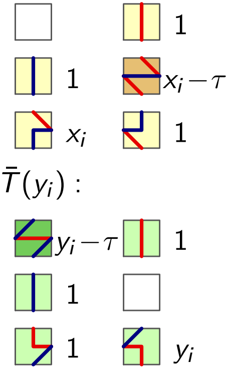
A supersymmetric skew Schur function:



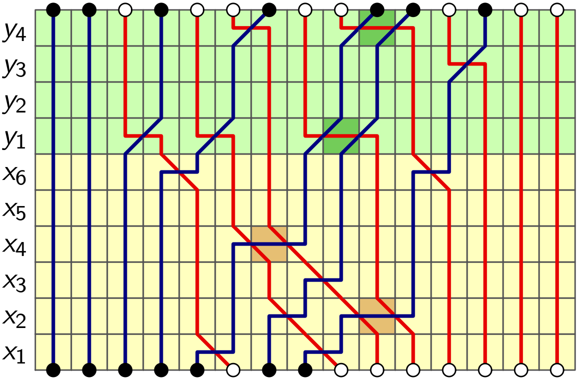
s  $(x_1, \ldots, x_6 | y_1, \ldots, y_4) = \cdots + x_1^2 x_2^3 x_3 x_4^2 x_6^2 \, y_1^4 y_3 y_4^3 + \cdots$

Schur vs. f_λ: an example

$T(x_i)$:



A supersymmetric skew f_λ function:



s  $(x_1, \dots, x_6 | y_1, \dots, y_4) = \cdots + x_1^2 x_2^2 x_3 x_4 x_6^2 \, y_1^3 y_3 y_4^2$

$\cdot (x_2 - \tau)(x_4 - \tau)(y_1 - \tau)(y_4 - \tau) + \cdots$

How f_λ's could possibly relate to our $\Psi_{\pi_b,\pi_w}(\tau)$'s

Remark: $f_{\lambda/\mu}(\vec{x}|\vec{y})$ are homogeneous of degree $|\lambda/\mu|$ in x_i's, y_j's and τ (so that in fact only the cases $\tau = 0$ (Schur) and $\tau = 1$ do matter)

As a result, we cannot hope that the structure constants $c^{\nu}_{\lambda\mu}$ or $d^{\lambda\mu}_{\nu}$ of the f_λ's are *tout court* our $\Psi_{\lambda\,\rho}(\tau)$'s. Our best hope is that they reproduce the leading coefficient of the polynomials, i.e. the coefficient of degree $|\lambda| + |\rho| - \binom{n}{2}$ in τ.

Indeed, we have some preliminary results, that go beyond this limitation, and involve some 6-Vertex Model generalisation of our transfer matrices

(but this is not really working so far, so I do not show you this...)

Towards an expansion of f_λ's over Schur functions

It is easily seen that $f_\lambda = \sum_{\mu\,:\,|\mu|\leq|\lambda|} \tau^{|\lambda|-|\mu|}\, s_\mu\, B^\mu_{\ \lambda}$, with $B^\mu_{\ \lambda} \in \mathbb{Z}$.
Some more work shows that (call $\ell = \ell(\lambda)$)

1. ${B_\lambda}^\mu \neq 0$ only if $\ell(\lambda) = \ell(\mu)$ and $|\mu| \preceq |\lambda|$
   (where $\preceq$ is the inclusion order)
2. $\prod_{i=1}^{\ell} x_i$ divides $f_\lambda(x_1, \dots, x_\ell)$
3. If $\lambda_\ell \geq 2$, then $f_\lambda(x_1, \dots, x_\ell) = f_{\lambda_\diamond}(x_1, \dots, x_\ell) \prod_{i=1}^{\ell}(x_i - \tau)$,
   with $\lambda_\diamond = (\lambda_1 - 1, \dots, \lambda_\ell - 1)$
4. If $\lambda_\ell = 1$, then $f_\lambda(x_1, \dots, x_\ell) = x_\ell f_{\lambda_\circ}(x_1, \dots, x_{\ell-1}) + \mathcal{O}(x_\ell^2)$,
   with $\lambda_\circ = (\lambda_1, \dots, \lambda_{\ell-1})$

This leads to a heuristic formula for $B^\mu_{\ \lambda}$, valid for a generic minimal alphabet $(x_1, \dots, x_\ell)$, that you can then prove right by determining the inverse $(B^{-1})_\lambda^{\ \mu}$ and showing that (at $\tau = 1$)

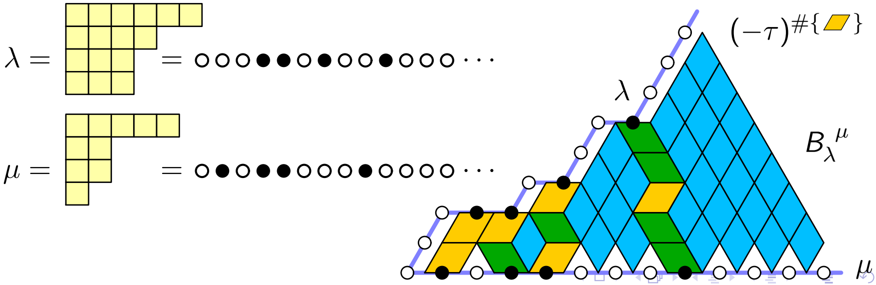
$$T^{5v}(x) = B^{-1}\, T(x) B$$

Expansion of f_λ's and g^λ's over Schur functions

$$f_\lambda = \sum_{\substack{\mu \preceq \lambda \\ \ell(\mu)=\ell(\lambda)}} \tau^{|\lambda/\mu|} s_\mu B^\mu_{\ \lambda} \qquad g^\nu = \sum_{\substack{\mu \succeq \nu \\ \ell(\mu)=\ell(\nu)}} \tau^{|\mu/\nu|} (B^{-1})^\nu_{\ \mu} s_\mu$$

$$B^\mu_{\ \lambda} = (-1)^{|\lambda/\mu|} \det \left[\binom{\lambda_j - 1}{\mu_i - i + j - 1} \right]_{i,j=1\dots,\ell}$$

$$(B^{-1})^\lambda_{\ \mu} = \det \left[\binom{\lambda_i - i + j - 1}{\mu_j - 1} \right]_{i,j=1\dots,\ell}$$



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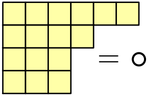
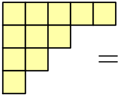
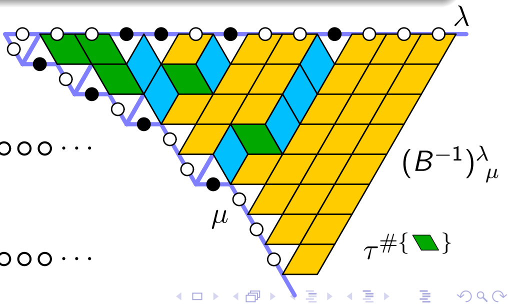
$\lambda =$  $=$ ○○○●●○●○○●○○○ $\cdots$

$\mu =$  $=$ ○●○●○●○○●○○○○ $\cdots$



Determinantal formulas for the f_λ's

Weyl-type determinantal formula for f_λ

$$f_\lambda(x_1, \dots, x_n) = \frac{1}{\Delta(\vec{x})} \det \left[(x_j - \tau)^{\lambda_i - 1} (x_j^{n-i+1} - \tau^{n-i+1} \delta_{\lambda_j, 0}) \right]_{i,j=1,\dots,n}$$

Jacobi–Trudi-type determinantal formula for $f_{\lambda/\mu}$

$$f_{\lambda/\mu}(\vec{x}) = \det \left(\left(h_{[\lambda_i - \mu_j - 1, j-i+1]} \right)_{i,j=1,\dots,\ell(\lambda)} \right)$$

$$h_{[a,b]} := \sum_{c=0}^{a} \binom{a}{c} (-\tau)^c h_{a+b-c} = [z^{a+b}](1 - \tau z)^a \prod \frac{1}{1 - z x_i}$$

The Jacobi–Trudi-type formula indeed generalises the one for Schur, recalling that $s_{\lambda/\mu} = \det \left(\left(h_{\lambda_i - \mu_j + j - i} \right)_{i,j=1,\dots,\ell(\lambda)} \right)$ and observing that $h_{[a,b]} = h_{a+b}$ when $\tau = 0$.

Also, it is stable, i.e. you can take matrices of dimension $d \geq \ell(\lambda)$

The f_λ are $(\alpha, \beta) = (-1, 0)$ Canonical Grothendieck poly's

All these results allow to identify the f_λ's with functions that have already arised in various places in the literature

✍ A. Borodin, *On a family of symmetric rational functions*, Adv. in Math. **306** (2014) [Sect. 8.4, identified by the Weyl-type formula]

✍ K. Motegi and T. Scrimshaw, *Refined Dual Grothendieck Polynomials, Integrability, and the Schur Measure*, SLC **85** (2021) [ex. 3.7, with $t_i \to \tau$, identified by the formula for $B^{\mu}{}_{\lambda}$]

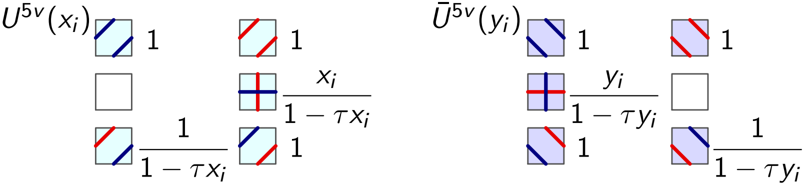
✍ A. Gunna and P. Zinn-Justin, *Vertex models for Canonical Grothendieck polynomials and their duals*, `arXiv:2009.13172` (Sept. 2020) [Sect. 3.4.3, identified from the branching rule] (see also ✍ D. Yeliussizov, *Symmetric Grothendieck polynomials, skew Cauchy identities, and dual filtered Young graphs*, JCT-A **161** (2019))

Note that in these papers the f_λ's arise from a bosonic Vertex Model!

What about the g^{λ}'s?

Now that we have our favourite f_{λ}'s, how can we determine the duals g^{λ}'s?
(1) you feel lucky, and search for a τ-deformation of $U(x)$ and $\bar{U}(y)$;
(2) you go the safe way, and evaluate the branching rule of the g^{λ}'s, that is

$$U^{5v}(x) = BT(x)B^{-1} \qquad (\tau = 1)$$



Remark: $g^{\lambda/\mu}(\vec{x}|\vec{y})$ are polynomials in x_i's, y_j's and τ,
and homogeneous of degree $|\lambda/\mu|$ in x_i's, y_j's and τ^{-1}

Determinantal formulas for the g^λ's

Weyl-type determinantal formula for g^λ

$$g^\lambda(x_1,\dots,x_n) = \frac{1}{\Delta(\vec{x})} \det\left[\left(\frac{x_j}{1-\tau x_j}\right)^{\lambda_i} x_j^{n-i}\right]_{i,j=1\dots,n}$$

Jacobi–Trudi-type determinantal formula for $g^{\lambda/\mu}$

$$g^{\lambda/\mu}(\vec{x}) = \det\left(\left(h_{\{\,\lambda_i-\mu_j-1\,,\,j-i+1\,\}}\right)_{i,j=1,\dots,\ell(\lambda)}\right)$$

$$h_{\{a,b\}} := \sum_{c\geq 0} \binom{a+c}{a} \tau^c h_{a+b+c} = [z^{a+b}](1-\tau/z)^{-a-1} \prod \frac{1}{1-zx_i}$$

Again, the Jacobi–Trudi-type formula generalises the one for Schur, because also $h_{\{a,b\}} = h_{a+b}$ when $\tau = 0$.

Our best conjecture so far...

So, we had hopes that the structure constants of our new basis $\{f_\lambda\}$ may be related to our UASM enumeration vectors, but, due to the homogeneity in $\deg(\vec{x}) + \deg(\tau)$, only for the leading coefficient of the enumeration polynomials, namely

Conjecture 6

$$f_\mu(\vec{x}) f_\nu(\vec{x}) = \sum_\lambda c^\lambda_{\mu\nu} f_\lambda(\vec{x}) \qquad [\tau^{|\lambda|+|\rho|-\binom{n}{2}}]\Psi_{\lambda\,\rho}(\tau) = c^{\delta_n}_{\lambda\rho}$$

This conjecture indeed holds up to $n = 5$

Recall that consistency with our conjectures requires

$$[\tau^{|\lambda|+|\rho|-\binom{n}{2}}]\big(\Psi_{\lambda\,\rho}(\tau) - \Psi_{\lambda\,\rho'}(\tau)\big) = c^{\delta_n}_{\lambda\rho} - c^{\delta_n}_{\lambda\rho'} = 0$$

Indeed our proof works out of the box for the coproduct coefficients, i.e., starting from $g^\lambda(\vec{x} \cup \vec{y}) := \sum_{\mu,\nu} c^\lambda_{\mu\nu}\, g^\mu(\vec{x}) g^\nu(\vec{y})$, and establishing $U(x)|\delta_n\rangle = \bar{U}(x)|\delta_n\rangle$, which implies a "triangular=magic" lemma also in this case.

A work in progress

This is clearly a work in progress, with many things going on...
I recall you our three main open questions:

- How can we prove our conjectures on the $\Psi_{\lambda\,\rho}(\tau)$ enumerations?
- There is any hope for a conjecture of the form $\Psi_{\lambda\,\rho}(\tau) = c^{\delta_n}_{\lambda\rho}$, for some family of functions?
- There is a puzzle description of the $c^{\lambda}_{\mu\nu}$ and $d^{\mu\nu}_{\lambda}$ structure constants for the canonical Grothendieck polynomials?
  [see also the work in progress of A. Gunna and P. Zinn-Justin]

Thank you for listening!