Many new conjectures on Fully-Packed Loop configurations

Andrea Sportiello
work in collaboration with L. Cantini
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\[ \# \{ \bigcirc \} + \# \{ \bigcirc \} = 2 \]
Part I

A short reminder of the Razumov–Stroganov conjecture(s)
The many Razumov–Stroganov conjectures

There exists a whole class of Razumov–Stroganov conjectures


Formulated in the early 2000’s, they relate the probabilities of some connectivity patterns in two different integrable models: the O(1) Dense Loop Model and the Fully-Packed Loop Model

A nice fact is that they can be formulated in purely combinatorial way, despite the fact that they are related to the physics of the XXZ Quantum Spin Chain and of the 6-Vertex Model.
A link pattern $\pi \in LP(2n)$ is a pairing of $\{1, 2, \ldots, 2n\}$ having no pairs $(a, c), (b, d)$ such that $a < b < c < d$ (i.e., the drawing consists of $n$ non-crossing arcs).

They are $C_n = \frac{1}{n+1} \binom{2n}{n}$ (the $n$-th Catalan number), and are in easy bijection with Dyck Paths of length $2n$ that is, integer partitions $\lambda \preceq \delta_n := (n-1, n-2, \ldots, 1)$

$$\pi = ((1, 6), (2, 3), (4, 5), (7, 10), (8, 9)) \quad \lambda(\pi) = (3, 3, 1) \preceq \delta_5$$
Consider dense loop configurations on a semi-infinite cylinder, i.e. tilings of \( \{1, \ldots, 2n\} \times \mathbb{N} \) with the two tiles \( , \, \) (with the uniform measure).

Link patterns are naturally associated to these configurations (despite the fact that they are infinite!)
Consider dense loop configurations on a semi-infinite cylinder i.e. tilings of \( \{1, \ldots, 2n\} \times \mathbb{N} \) with the two tiles \( \text{ } \) (with the uniform measure)

**Link patterns** are naturally associated to these configurations (despite the fact that they are infinite!)
Consider dense loop configurations on a semi-infinite cylinder i.e. tilings of \( \{1, \ldots, 2n\} \times \mathbb{N} \) with the two tiles (with the uniform measure)

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Many new conjectures on FPL's
Consider dense loop configurations on a semi-infinite cylinder i.e. tilings of \( \{1, \ldots, 2n\} \times \mathbb{N} \) with the two tiles and (with the uniform measure).

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Many new conjectures on FPL's
Fully-Packed Loops

Fully-Packed Loop configurations are tilings of the $n \times n$ square using the six tiles and with black/white alternating boundary conditions.

Again, a link pattern $\pi$ is naturally associated, according to the connectivities among the black terminations on the boundary.

Note that, by now, we ignore the link pattern associated to white, and the potential presence of loops.
Fully-Packed Loops configurations are tilings of the $n \times n$ square using the six tiles and with black/white alternating boundary conditions.

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The dihedral Razumov–Stroganov correspondence

\[ \tilde{\Psi}_n(\pi) : \text{probability of } \pi \]
\[ \text{in the } O(1) \text{ Dense Loop Model} \]
\[ \text{in the } \{1, ..., 2n\} \times \mathbb{N} \text{ cylinder} \]

\[ \Psi_n(\pi) : \text{probability of } \pi \]
\[ \text{for FPL with uniform measure} \]
\[ \text{in the } n \times n \text{ square} \]
The dihedral Razumov–Stroganov correspondence

\[ \tilde{\Psi}_n(\pi) : \text{probability of } \pi \text{ in the } O(1) \text{ Dense Loop Model in the } \{1, \ldots, 2n\} \times \mathbb{N} \text{ cylinder} \]

\[ \Psi_n(\pi) : \text{probability of } \pi \text{ for FPL with uniform measure in the } n \times n \text{ square} \]

Razumov–Stroganov correspondence
(conjecture: Razumov and Stroganov, 2001a for the \( n \times n \) square; proof: AS and Cantini, 2010, for all the ‘dihedral domains’)

\[ \tilde{\Psi}_n(\pi) = \Psi_n(\pi) \]
Dihedral symmetry of FPL

A corollary of the Razumov–Stroganov correspondence... (...that was known before the Razumov–Stroganov conjecture)

call $R$ the operator that rotates a link pattern by one position.

Dihedral symmetry of FPL (proof: Wieland, 2000)

$$\Psi_n(\pi) = \Psi_n(R\pi)$$
Domains with dihedral Razumov–Stroganov correspondence

In the case of the dihedral Razumov–Stroganov correspondence, Wieland gyration (and its generalisations) has been a crucial ingredient.

Not surprisingly, understanding the most general family of domains for which the correspondence holds has been inspiring.
**Remark:** What is natural to consider in Wieland gyration lemma is the triple $(\pi_b, \pi_w, \ell)$ for the black and white link patterns, and the total number of loops (black+white)

However, we have no candidate replacing the $O(1)$ Dense Loop Model in a black+white version of the Razumov–Stroganov conjecture! (...no, the Rotor Model doesn’t seem to work...)
A Vertical Razumov–Stroganov Conjecture

\[ \tilde{\Psi}^V_n(\pi) : \text{probability of } \pi \]

in the \( O(1) \) Dense Loop Model

in the \( \{1, \ldots, 2n\} \times \mathbb{N} \) strip

\[ \Psi^V_n(\pi) : \text{probability of } \pi \]

for vertically-symmetric FPL

with uniform measure in the

\( (2n + 1) \times (2n + 1) \) square

Many new conjectures on FPL's
A Vertical Razumov–Stroganov Conjecture

\[ \tilde{\Psi}_n^V(\pi) : \text{probability of } \pi \]
\[ \text{in the } O(1) \text{ Dense Loop Model} \]
\[ \text{in the } \{1, \ldots, 2n\} \times \mathbb{N} \text{ strip} \]

\[ \Psi_n^V(\pi) : \text{probability of } \pi \]
\[ \text{for vertically-symmetric FPL} \]
\[ \text{with uniform measure in the} \]
\[ (2n+1) \times (2n+1) \text{ square} \]

**Vertical Razumov–Stroganov conjecture**

(Razumov and Stroganov, 2001b for the square of side $2n+1$)

\[ \tilde{\Psi}_n^V(\pi) = \Psi_n^V(\pi) \]
The Vertical Razumov–Stroganov conjectures are a whole second family. They involve FPL with some version of reflecting wall and the $O(1)$ Dense Loop Model on a strip with a boundary.

Our proof methods do not seem to work for any of the Vertical Razumov–Stroganov conjectures, which are all open at present.

But at least we think we know the precise list of domains with Vertical RS:

$$3 + x + 7y + 2xy + 4y^2 + xy^2$$

$$6 + 2x + 14y + 4xy + 8y^2 + 2xy^2$$
Part II

The many new conjectures...
Looking at UASM more closely

We shall “smash together the two failures” above: ① we haven’t proven any flavour of the Vertical Razumov–Stroganov conjectures; ② we never devised any flavour of Razumov–Stroganov conjectures, not even dihedral, involving the triple enumeration $\Psi_n(\pi_b, \pi_w, \ell)$

We will look more closely at the full list of FPL’s in one of the simplest instances of Vertical RS, that is U-turn ASM’s (UASM).

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The many conjectures on the enumerations $\Psi_{\pi_b, \pi_w}(\tau)$

Let us call $\Psi_n^V(\pi_b, \pi_w, \tau, y)$ the generating function of UASM’s at size $n$, with black/white link patterns $\pi_b$ and $\pi_w$, and weight $\tau \ell y \#\cap$

Known: $Z_n^V(y) = \sum_{\pi_b, \pi_w} \Psi_n^V(\pi_b, \pi_w, 1, y)$ has an overall factor $(1 + y)^n$


Luigi Cantini and myself conjectured, also long ago (and never published) that this factorisation holds for the RS components

$$\Psi_n^V(\pi_b, y) = \sum_{\pi_w} \Psi_n^V(\pi_b, \pi_w, 1, y) = (1 + y)^n \tilde{\Psi}_n^V(\pi_b)$$

The new numerical investigation leads to the first of our “new conjectures”:

Conjecture 1

$$\Psi_n^V(\pi_b, \pi_w, \tau, y) = (1 + y)^n \Psi_{\pi_b, \pi_w}(\tau) \quad \forall \ n, \tau, \pi_b, \pi_w$$

(only proven: $(1 + y)^2$ divides $\Psi_n^V(\pi_b, \pi_w, \tau, y)$ for $n \geq 2$)
The many conjectures on the enumerations $\Psi_{\pi_b,\pi_w}(\tau)$

These sets of polynomials are better visualised as tables $\pi_b$ vs. $\pi_w$...

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L. Cantini and A. Sportiello

Many new conjectures on FPL's
The many conjectures on the enumerations $\Psi_{\pi_b,\pi_w}(\tau)$

...or, equivalently, as tables $\lambda$ vs. $\rho$ with $\lambda = \lambda(\pi_b)$ and $\rho = \lambda(\pi_w)$

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\begin{array}{c|ccc}
\lambda & 0 & 1 \\
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1 & 0 & 0 & 1 & 1 & 2\tau \\
2 & 0 & 1 & \tau & \tau & \tau^2 + 1 \\
3 & 0 & 1 & \tau & \tau & \tau^2 + 1 \\
\end{array}
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\begin{array}{c|cccc}
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Many new conjectures on FPL's
A large example:

$$\Psi_{3,3,1,0),(4,2,2,1)}(\tau) = \cdots + \tau^2 + \cdots$$

$$\# \{\bigcirc\} + \# \{\bigcirc\} = 2$$
The many conjectures on the enumerations $\Psi_{\pi_b,\pi_w}(\tau)$

In the following, with abuse of notation, $\Psi_{\lambda \rho}(\tau) \equiv \Psi_{\pi_b,\pi_w}(\tau)$

**Conjecture 2**

$$\deg (\Psi_{\lambda \rho}(\tau)) = |\lambda| + |\rho| - |\delta_n|$$

In particular, $\Psi_{\lambda \rho}(\tau) = 0$ if $|\lambda| + |\rho| < \binom{n}{2}$.

**Conjecture 3**

The $\Psi_{\lambda \rho}(\tau)$’s are polynomials of defined parity.

**Conjecture 4**

The table has three involutions: ① $\Psi_{\lambda \rho}(\tau) = \Psi_{\rho \lambda}(\tau)$; ② $\Psi_{\lambda \rho}(\tau) = \Psi_{\rho' \lambda'}(\tau)$; ③ $\Psi_{\lambda \rho}(\tau) = \Psi_{\lambda \rho'}(\tau)$.

①: easily proven (Wieland + swap b/w);
②: easily corollary of Conjecture 1 (vertical reflection + swap b/w);
③: rather mysterious.

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**Conjecture 5**

The entries s.t. $|\lambda| + |\rho| = |\delta_n|$ are the Littlewood–Richardson coefficients $\Psi_{\lambda\rho}(\tau) = c_{\lambda\rho}^{\delta_n}$.

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Many new conjectures on FPL's...
Part III

Schur functions, Littlewood–Richardson coefficients and all that
Semi-Standard Young Tableaux $SSYT(\lambda, n)$:

Fillings of $\lambda$ with the integers $\{1, 2, \ldots, n\}$, repetitions allowed, satisfying $\bullet \leq \bullet$. Play a crucial role in the representation theory of the general linear group $GL(n)$.

**Remark:** $SSYT(\lambda, n) = \emptyset$ if $n < \ell(\lambda)$.

Schur polynomials are the ‘generating functions’ of SSYT's:

$$s_{\lambda}(x_1, \ldots, x_n) = \sum_{T \in SSYT(\lambda, n)} \prod_{i=1}^{n} x_i^{\#\{i \in T\}}$$

$$s_{\begin{array}{cccccc} 1 & 3 & 4 & 4 \\ 2 & 3 \\ 5 & 6 \\ 6 \end{array}}(x_1, \ldots, x_6) = \cdots + x_1^2 x_2 x_3^2 x_4^2 x_5 x_6^2 + \cdots$$

Many new conjectures on FPL's...
Many beautiful facts about Schur Functions

1. Schur polynomials are **homogeneous** of degree $|\lambda|$ and **symmetric** (seen via the Bender–Knuth involution).
   They form a basis of the algebras of symmetric polynomials

   $$\Lambda_{n,\mathbb{K}}(\vec{x}) = \left[ \text{algebra of symm. polyn. in } x_1, \ldots, x_n \right] = \text{span}_\mathbb{K} \left( s_\lambda(x_1, \ldots, x_n) \right)_{\lambda : \ell(\lambda) \leq n}$$

2. The **Weyl character formula** tells that the Schur polynomials can be written as the ratio of two determinants

   $$s_\lambda(x_1, \ldots, x_n) = \frac{1}{\Delta(\vec{x})} \det \left( (x_i^{(\lambda+\delta_n)j})_{i,j=1,\ldots,n} \right)$$

   $$\Delta(\vec{x}) = \det \left( (x_i^{(\delta_n)j})_{i,j=1,\ldots,n} \right) = \prod_{i<j} (x_i - x_j)$$

L. Cantini and A. Sportiello  Many new conjectures on FPL's
Many beautiful facts about Schur Functions

Call \[
\begin{align*}
    e_k(\vec{x}) &= \sum_{i_1 < i_2 < \ldots < i_k} x_{i_1} \cdots x_{i_k} \\
    h_k(\vec{x}) &= \sum_{i_1 \leq i_2 \leq \ldots \leq i_k} x_{i_1} \cdots x_{i_k}
\end{align*}
\]

We can write \( s_\lambda(x_1, \ldots, x_n) \) as polynomials in the \( e_k(x_1, \ldots, x_n) \)'s, or the \( h_k(x_1, \ldots, x_n) \)'s. As soon as \( n \geq \ell(\lambda) \), these expressions are given by the Jacobi–Trudi and dual Jacobi–Trudi formulas

\[
\begin{align*}
    s_\lambda &= \det \left( (h_{\lambda_i+j-i})_{i,j=1,\ldots,\ell(\lambda)} \right) \quad (JT) \\
    &= \det \left( (e_{\lambda_i'+j-i})_{i,j=1,\ldots,\lambda_1} \right) \quad (dJT)
\end{align*}
\]

In particular, they stabilise (i.e., become independent of \( n \))

This allows to define Schur functions, defined also for infinite alphabets.
Many beautiful facts about Schur Functions

One useful class of infinite alphabets is induced by the (‘supersymmetry’) $\omega$-involution, that exchanges $e_k$’s and $h_k$’s. That is, we have Schur functions (in fact, polynomials) depending on a ‘finite supersymmetric alphabet’, $s_\lambda(x_1, \ldots, x_n|y_1, \ldots, y_m)$ (the name is ‘legitimate’ as these are super-characters of $GL(n|m)$)

These functions can be defined through the JT or dJT formulas, setting $h_k(x_1, \ldots, x_n|y_1, \ldots, y_m) = [z^k](\prod_j(1 + zy_j))/\left(\prod_i(1 - zx_i)\right)$ and $e_k(x_1, \ldots, x_n|y_1, \ldots, y_m) = [z^k](\prod_i(1 + zx_i))/\left(\prod_j(1 - zy_j)\right)$

Exchanging the ‘bosonic’ and ‘fermionic’ parts of the alphabet accounts to take the transpose Young diagrams $s_{\lambda/\mu}(x_1, \ldots, x_n|y_1, \ldots, y_m) = s_{\lambda'/\mu'}(y_1, \ldots, y_m|x_1, \ldots, x_n)$
Many beautiful facts about Schur Functions

Define the skew Schur polynomials as

\[ s_{\lambda/\mu}(x_1, \ldots, x_n) = \sum_{T \in \text{SSYT}(\lambda/\mu, n)} \prod_{i=1}^{n} x_i^{\#\{i \in T\}} \]

We have

\[ s_{\lambda/\mu} = \det \left( (h_{\lambda_i-\mu_j+j-i})_{i,j=1,\ldots,\ell(\lambda)} \right) (JT) \]
\[ = \det \left( (e_{\lambda'_i-\mu'_j+j-i})_{i,j=1,\ldots,\lambda_1} \right) (dJT) \]

In the scalar product \( \langle \cdot | \cdot \rangle \) such that the Schur basis is self-dual

\[ \langle s_{\lambda} | s_{\mu} \rangle = \delta_{\lambda\mu} \]

(this is called the Hall scalar product)

these polynomials have the property \( \langle h | s_{\lambda/\mu} \rangle = \langle h s_{\mu} | s_{\lambda} \rangle \forall h \)
Many beautiful facts about Schur Functions

It follows that

\[ s_\lambda(x_1, \ldots, x_n, x_{n+1}, \ldots, x_{n+m}) = \sum_\mu s_\mu(x_1, \ldots, x_n)s_{\lambda/\mu}(x_{n+1}, \ldots, x_{n+m}) \]

(T_1 \in SSYT(\lambda, n + m) \quad T_2 \in SSYT(\mu, n) \quad T_3 \in SSYT(\lambda/\mu, m))

This is evident for finite alphabets, but the formula

\[ s_\lambda(x \cup y) = \sum_\mu s_\mu(x)s_{\lambda/\mu}(y) \]

holds also for infinite alphabets.

The structure constants \( c_{\mu\nu}^\lambda \) of the algebra \( \Lambda = \text{span}_K(s_\lambda(\vec{x})) \) are non-negative integers known as Littlewood–Richardson coefficients

\[ s_\mu(\vec{x})s_\nu(\vec{x}) = \sum_\lambda c_{\mu\nu}^\lambda s_\lambda(\vec{x}) \quad c_{\mu\nu}^\lambda \in \mathbb{N} \]
Many beautiful facts about Schur Functions

What we said above implies that the three problems

\[
\begin{align*}
{s_\mu(\vec{x})s_\nu(\vec{x})} &= \sum_{\lambda} c_{\mu\nu}^\lambda s_\lambda(\vec{x}) \\
{s_\lambda/\mu(\vec{x})} &= \sum_{\nu} c_{\mu\nu}^{\lambda} s_\nu(\vec{x}) \\
{s_\lambda(\vec{x} \cup \vec{y})} &= \sum_{\mu, \nu} c_{\mu\nu}^{\lambda} s_\mu(\vec{x})s_\nu(\vec{y})
\end{align*}
\]

are all solved by the same Littlewood–Richardson coefficients.

Many other interesting basis of symmetric functions (Hall–Littlewood, Grothendieck, ... ) generalise the Schur case in some sense, but, if we insist on keeping the Hall \((\langle s_\lambda | s_\mu \rangle = \delta_\lambda \mu)\) scalar product, self-duality is not present in general.

We have two basis of functions, \(\{ f_\lambda \}\) and \(\{ g^\lambda \}\), such that \(\langle g^\lambda | f_\mu \rangle = \delta^\lambda_\mu\), and two different sets of structure constants

\[
f_\lambda f_\mu = \sum_{\nu} c_{\lambda\mu}^\nu f_\nu \quad g^\lambda g^\mu = \sum_{\nu} d_{\nu\mu}^{\lambda\mu} g^\nu
\]
(Skew-)Schur polynomials can be represented as partition functions of tiling models, namely as free-fermionic $U_q(\hat{sl}_2)$ Yang–Baxter integrable Vertex Models with homogeneous vertical spectral parameters, the horizontal ones determine the alphabet

$$s_{\lambda/\mu}(x_1, \ldots, x_n)$$ is described by an infinite horizontal strip, of height $n$, where all non-trivial tiles occur within a width $\lambda_1 + \ell(\lambda)$.

The partitions $\lambda$ and $\mu$ fix the top and bottom boundary conditions.
Lozenge tilings are nice, but, in order to describe in a symmetric way the ‘supersymmetric’ (skew-)Schur functions, we shall rather shear the triangular lattice into the square lattice.
Lozenge tilings are nice, but, in order to describe in a symmetric way the ‘supersymmetric’ (skew-)Schur functions, we shall rather shear the triangular lattice into the square lattice

\[ T(x_i) : \]
Lozenge tilings are nice, but, in order to describe in a symmetric way the ‘supersymmetric’ (skew-)Schur functions, we shall rather shear the triangular lattice into the square lattice.

Many new conjectures on FPL's
A supersymmetric skew Schur polynomial:

\[ T(x_i) : \]

\[ \bar{T}(y_i) : \]

\[ s (x_1, \ldots, x_6 | y_1, \ldots, y_4) = \cdots + x_1^2 x_2^3 x_3 x_4^2 x_6^2 y_1^4 y_3 y_4^3 + \cdots \]
Representation of Schur polynomials as Vertex Models

A supersymmetric skew Schur polynomial:

\[ U(x_i) : \]
\[
\begin{array}{ccc}
1 & 1 \\
& x_i & 1 \\
1 & 1 \\
\end{array}
\]

\[ \bar{U}(y_i) : \]
\[
\begin{array}{ccc}
1 & 1 \\
& y_i & 1 \\
1 & 1 \\
\end{array}
\]

\[ s_{\begin{array}{cc} \begin{array}{c} x_1, \ldots, x_6 \end{array} | \begin{array}{c} y_1, \ldots, y_4 \end{array} \end{array}} = \cdots + x_1^2 x_2^3 x_3 x_4^2 x_6^2 y_1^4 y_3 y_4^3 + \cdots \]
The operators $T(x)$ and $\bar{T}(y)$ are ‘transfer matrices’. They act on the Hilbert space indexed by integer partitions, as

$$\langle \mu | T(x) | \lambda \rangle = \begin{cases} x^{\lambda/\mu} & \mu \preceq \lambda ; \lambda/\mu \text{ is a ‘horizontal strip’ (no $\square$)} \\ 0 & \text{otherwise} \end{cases}$$

$$\langle \mu | \bar{T}(y) | \lambda \rangle = \begin{cases} y^{\lambda/\mu} & \mu \preceq \lambda ; \lambda/\mu \text{ is a ‘vertical strip’ (no $\square$)} \\ 0 & \text{otherwise} \end{cases}$$

$$s_{\lambda/\mu}(x_1, \ldots, x_n|y_1, \ldots, y_m) = \langle \mu | T(x_1) \cdots T(x_n) \bar{T}(y_1) \cdots \bar{T}(y_m) | \lambda \rangle$$

In particular $\langle \mu | T(x) | \lambda \rangle = \langle \mu' | \bar{T}(x) | \lambda' \rangle$

Of course, by definition of transpose operator,

$$\langle \mu | T^+(x) | \lambda \rangle = \langle \lambda | T(x) | \mu \rangle \text{ and } \langle \mu | \bar{T}^+(x) | \lambda \rangle = \langle \lambda | \bar{T}(x) | \mu \rangle$$
Operators $T(x)$, $\bar{T}(y)$ and their transpose form an interesting algebra

$$T(x)\langle \emptyset \rangle = \bar{T}(x)\langle \emptyset \rangle = \langle \emptyset \rangle \quad \langle \emptyset | T^+(x) = \langle \emptyset | \bar{T}^+(x) = \langle \emptyset |$$

$$[T(x), T(y)] = [\bar{T}(x), \bar{T}(y)] = [T(x), \bar{T}(y)] = 0$$

$$T(x)T^+(y) = \frac{1}{1 - xy} T^+(y)T(x) \quad \bar{T}(x)\bar{T}^+(y) = \frac{1}{1 - xy} \bar{T}^+(y)\bar{T}(x)$$

$$T(x)\bar{T}^+(y) = (1 + xy)\bar{T}^+(y)T(x) \quad \bar{T}(x)T^+(y) = (1 + xy)T^+(y)\bar{T}(x)$$

This is proven through the Yang–Baxter equation for the corresponding ‘free-fermionic 5-Vertex Model with electric fields’.

Partition functions and correlation functions of several dimer models (lozenges, domino tilings, ...) can be calculated in this way.

Remarkably, also the Littlewood–Richardson coefficients are described by an integrable Vertex Model, this time of square-triangle tilings, with underlying $U_q(\hat{sl}_3)$ symmetry.


The key idea is to express the two sides of the coproduct identity $s_\lambda(\vec{x}|\vec{y}) = \sum_{\mu,\nu} c^\lambda_{\mu\nu} s_\mu(\vec{x})s^\prime_\nu(\vec{y})$ as partition functions in a rank-2 model (i.e., with particles of three colours).

The three Schur terms, $s_\lambda(\vec{x}|\vec{y})$, $s_\mu(\vec{x})$ and $s^\prime_\nu(\vec{y})$, are realised within the three possible embeddings of $\hat{sl}_2$ in $\hat{sl}_3$ that is, the three choices of two colours among three

The identity is a consequence of commutation of transfer matrices, which in turns comes from the Yang–Baxter equation of the rank-2 model.
Littlewood–Richardson coefficients as a Vertex Model

\[
s_\lambda(\vec{x}|\vec{y}) = \sum_{\mu,\nu} c^\lambda_{\mu\nu} s_\mu(\vec{x}) s_{\nu'}(\vec{y})
\]
Littlewood–Richardson coefficients as a Vertex Model

\[ s_\lambda(\vec{x} | \vec{y}) = \sum_{\mu, \nu} c_\mu^\lambda s_\mu(\vec{x}) s_\nu'(\vec{y}) \]

L. Cantini and A. Sportiello

Many new conjectures on FPL's
A property of the Littlewood–Richardson coefficients

Let us come back to our “many new conjectures”…

**Conjecture 4**

1. $\Psi_{\lambda \rho} = \Psi_{\rho \lambda}$;  
2. $\Psi_{\lambda \rho} = \Psi_{\rho' \lambda'}$;  
3. $\Psi_{\lambda \rho} = \Psi_{\lambda \rho'}$.

**Conjecture 5**

When $|\lambda| + |\rho| = |\delta_n|$ we have $\Psi_{\lambda \rho} = c^\delta_{\lambda \rho}$ (Littlewood–Richardson)

Are these two conjectures even compatible?

Indeed, 1 and 2 are simple symmetries of LR coeffs
(with 2 using the fact $\delta_n = (\delta_n)'$),
but why on Earth should we have $c^\lambda_{\mu \nu} = c^\lambda_{\mu \nu'}$?

Call $T = \{\delta_n\}_{n \geq 1}$ and $M = \{\lambda | c^\lambda_{\mu \nu} = c^\lambda_{\mu \nu'}, \forall \mu, \nu\}$

**Lemma**

$T = M$
Lemma

\( T = \{ \delta_n \}_{n \geq 1} \) and \( M = \{ \lambda \mid c^\lambda_{\mu\nu} = c^\lambda_{\mu\nu}', \forall \mu, \nu \} \) coincide.

Proof. The implication \( \lambda \notin T \Rightarrow \lambda \notin M \) is easy (recognise that \( \lambda \notin T \iff \lambda = [\alpha \circ \circ \bullet \beta] \) or \( \lambda = [\alpha \circ \bullet \bullet \beta] \), call \( \mu = [\alpha \bullet \circ \circ \beta] \) or \( \mu = [\alpha \bullet \bullet \circ \beta] \), and evaluate \( c^\lambda_{\mu (2)} \), \( c^\lambda_{\mu (1,1)} \))

The implication \( \lambda \in T \Rightarrow \lambda \in M \) is interesting.

The crucial observation is that \( T(x) | \delta_n \rangle = \bar{T}(x) | \delta_n \rangle \)

that, using the commutation of \( T \)'s and \( \bar{T} \)'s, implies on supersymmetric skew Schur functions

\( s_{\delta_n/\mu}(\vec{x} | \vec{y}) = s_{\delta_n/\mu}(\vec{y} | \vec{x}) \)

by the coproduct definition of LR's:

\[
\sum_\nu c^\delta_{\mu\nu} s_\nu(\vec{x} | \vec{y}) = s_{\delta_n/\mu}(\vec{x} | \vec{y}) = s_{\delta_n/\mu}(\vec{y} | \vec{x}) = \sum_\nu c^\delta_{\mu\nu} s_\nu(\vec{y} | \vec{x}) = \sum_\nu c^\delta_{\mu\nu} s_\nu(\vec{x} | \vec{y}) = \sum_\nu c^\delta_{\mu\nu} s_\nu(\vec{y} | \vec{x}).
\]

By the linear independence of Schur functions \( c^\delta_{\mu\nu} = c^\delta_{\mu\nu}' \).
A mystery plot

We have mentioned that there exists several deformations of Schur functions (Grothendieck, Hall–Littlewood, . . . ), many of them allow for a representation as an integrable Vertex Model, and even some representation à la Zinn-Justin of the corresponding structure constants (i.e., with the trick “sl$_2$ embeds into sl$_3$ in three ways”).


Maybe there exists a basis/dual-basis of symmetric functions \( \{ f_{\lambda} \}, \{ g^{\lambda} \} \), which are a \( \tau \)-deformation of Schur fns., such that

\[
\Psi_{\lambda \rho}(\tau) = c^{\delta_n}_{\lambda \rho} \quad \text{or} \quad \Psi_{\lambda \rho}(\tau) = d^{\lambda \rho}_{\delta_n}, \quad \text{for all pairs } \lambda, \rho \leq \delta_n?
\]

Maybe we will have a result of the form

\[
\Psi_{\lambda \rho}(\tau) = \sum_{P \in P_{\lambda, \rho, \delta_n}} \tau^{x(P)}
\]

with \( P_{\lambda, \rho, \delta_n} \) some variant of Knutson–Tao puzzles, and \( x(P) \) the number of tiles of some kind?
A mystery plot: collecting the hints

We shall suppose that these new functions exist, are still described by an integrable Vertex Model, and are given by a ‘minimal’ deformation of $T(x)$ and $\bar{T}(y)$ operators.

Which properties shall we reproduce?
1. The degree condition (and its corollary on which $\Psi_{\lambda\rho}$ do vanish)
2. Polynomials of defined parity
3. The mysterious extra symmetry $\Psi_{\lambda\rho} = \Psi_{\lambda\rho}'$
4. The new $T$ and $\bar{T}$ must still constitute a commuting family
5. $\langle \mu | T(x) | \lambda \rangle$ well-defined on infinite strings $\cdots \bullet \bullet \bullet [\cdots] \circ \circ \circ \cdots$

Which generalisations we do not want?
1. We do not “change $\delta_n$” (e.g., try $\Psi_{\lambda\rho}(\tau) = \sum_{\theta \subseteq \delta_n} c_{\lambda\rho}^\theta \tau^{\theta/\delta_n}$)
2. We only investigate Vertex Models with “spin $\frac{1}{2}$” horizontal and vertical spaces

The reason is that we want our proof of $c_{\lambda\rho}^{\delta_n} = c_{\lambda\rho}^{\delta_n}'$ to extend to $\Psi_{\lambda\rho}(\tau)$ almost verbatim.
The standard technique from Integrable Systems is to construct a \( RLL = LLR \) relation (a version of Yang–Baxter when the spaces are not all equal), that is, for \( L \) the tile-weights appearing in the transfer matrices \( T \) and \( \bar{T} \), devise a matrix \( R \) such that

\[
R(x, y) L(x) L(y) = L(y) L(x) R(x, y)
\]

L. Cantini and A. Sportiello Many new conjectures on FPL's
For the (non–free-fermionic) 5-Vertex Model, this can be done unambiguously, once we take into account:

1. weight well-defined on infinite strings;
2. gauge invariance;
3. covariance under reparametrisation;

\[ T(x_i) \]

\[ \bar{T}(y_i) \]

Many new conjectures on FPL's
For the (non–free-fermionic) 5-Vertex Model, this can be done unambiguously, once we take into account:

1. weight well-defined on infinite strings;
2. gauge invariance;
3. covariance under reparametrisation;

\[
T (x_i) \quad 1 \quad x_i \quad 1
\]

\[
\bar{T} (y_i) \quad y_i \quad 1 \quad y_i
\]
For the (non–free-fermionic) 5-Vertex Model, this can be done unambiguously, once we take into account:

1. weight well-defined on infinite strings;
2. gauge invariance;
3. covariance under reparametrisation;

\[
T(x_i) = 1 \quad 1 \quad x_i \quad 1
\]
\[
\bar{T}(y_i) = 1 \quad y_i \quad 1 \quad y_i
\]
For the (non–free-fermionic) 5-Vertex Model, this can be done unambiguously, once we take into account:

1. weight well-defined on infinite strings;
2. gauge invariance;
3. covariance under reparametrisation;

\[
T(x_i) \quad 1 \quad \bar{T}(y_i) \quad b(y_i) \quad 1
\]

\[
1 \quad a(x_i) \quad 1 \quad 1 \quad y_i
\]

\[
a(x_1) - x_1 = a(x_2) - x_2 = b(y_1) - y_1 = b(y_2) - y_2
\]
For the (non–free-fermionic) 5-Vertex Model, this can be done \textit{unambiguously}, once we take into account:
\begin{enumerate}
  \item weight well-defined on infinite strings;
  \item gauge invariance;
  \item covariance under reparametrisation;
\end{enumerate}
Non-FF 5VM and dual Canonical Grothendieck polynomials

The FF 5VM operators $T$ and $\bar{T}$ act on integer partitions as

$$\langle \mu | T (x) | \lambda \rangle = \begin{cases} x^{\lambda/\mu} & \mu \preceq \lambda; \lambda/\mu \text{ hor. strip} \\ 0 & \text{otherwise} \end{cases}$$

$$\langle \mu | \bar{T} (y) | \lambda \rangle = \begin{cases} y^{\lambda/\mu} & \mu \preceq \lambda; \lambda/\mu \text{ vert. strip} \\ 0 & \text{otherwise} \end{cases}$$

$$s_{\lambda/\mu}(x_1, \ldots, x_n | y_1, \ldots, y_m) = \langle \mu | T (x_1) \cdots T (x_n) \bar{T} (y_1) \cdots \bar{T} (y_m) | \lambda \rangle$$

$$\begin{array}{cccccc}
1 & 1 & 3 & 4 & 4 & 4 \\
2 & 3 & & & & \\
4 & 6 & & & & \\
6 & & & & & \\
\end{array}$$

$$x_1^2 x_2 x_3^2 x_4 x_6^2$$

Many new conjectures on FPL's
The non-FF 5VM operators $T$ and $\bar{T}$ act on integer partitions as

$$\langle \mu | T^{5v}(x) | \lambda \rangle = \begin{cases} x^K(\lambda/\mu)(x - \tau)^{|\lambda/\mu| - K(\lambda/\mu)} & \mu \preceq \lambda; \lambda/\mu \text{ hor. strip} \\ 0 & \text{otherwise} \end{cases}$$

$$\langle \mu | \bar{T}^{5v}(y) | \lambda \rangle = \begin{cases} y^K(\lambda/\mu)(y - \tau)^{|\lambda/\mu| - K(\lambda/\mu)} & \mu \preceq \lambda; \lambda/\mu \text{ vert. strip} \\ 0 & \text{otherwise} \end{cases}$$

$$f_{\lambda/\mu}(x_1, \ldots, x_n | y_1, \ldots, y_m) = \langle \mu | T^{5v}(x_1) \cdots T^{5v}(x_n) \bar{T}^{5v}(y_1) \cdots \bar{T}^{5v}(y_m) | \lambda \rangle$$

$$x_1 (x_1 - \tau) x_2 x_3^2 x_4^2 (x_4 - \tau)^2 x_6^2$$
Schur vs. $f_{\lambda}$: an example

A supersymmetric skew Schur function:

$$s_{\alpha}(x_1, \ldots, x_6 | y_1, \ldots, y_4) = \cdots + x_1^2 x_2^3 x_3 x_4^2 x_6^2 y_1^4 y_3 y_4^3 + \cdots$$
Schur vs. $f_\lambda$: an example

A supersymmetric skew $f_\lambda$ function:

$$s_T(x_1, \ldots, x_6 | y_1, \ldots, y_4) = \cdots + x_1^2 x_2^2 x_3 x_4 x_6^2 y_1^3 y_3 y_4^2 \cdot (x_2 - \tau)(x_4 - \tau)(y_1 - \tau)(y_4 - \tau) + \cdots$$
How $f_\lambda$’s could possibly relate to our $\Psi_{\pi_b, \pi_w}(\tau)$’s

*Remark:* $f_{\lambda/\mu}(\vec{x}|\vec{y})$ are homogeneous of degree $|\lambda/\mu|$ in $x_i$’s, $y_j$’s and $\tau$ (so that in fact only the cases $\tau = 0$ (Schur) and $\tau = 1$ do matter)

As a result, we cannot hope that the structure constants $c^\nu_{\lambda\mu}$ or $d^\lambda_{\nu\mu}$ of the $f_\lambda$’s are *tout court* our $\Psi_{\lambda, \rho}(\tau)$’s. Our best hope is that they reproduce the leading coefficient of the polynomials, i.e. the coefficient of degree $|\lambda| + |\rho| - \binom{n}{2}$ in $\tau$.

Indeed, we have some preliminary results, that go beyond this limitation, and involve some 6-Vertex Model generalisation of our transfer matrices *(but this is not really working so far, so I do not show you this...)*
Towards an expansion of $f_\lambda$'s over Schur functions

It is easily seen that $f_\lambda = \sum_{\mu : |\mu| \leq |\lambda|} \tau^{|\lambda| - |\mu|} s_\mu B^\mu_\lambda$, with $B^\mu_\lambda \in \mathbb{Z}$.

Some more work shows that (call $\ell = \ell(\lambda)$)

1. $B^\mu_\lambda \neq 0$ only if $\ell(\lambda) = \ell(\mu)$ and $|\mu| \preceq |\lambda|$ (where $\preceq$ is the inclusion order)

2. $\prod_{i=1}^{\ell} x_i$ divides $f_\lambda(x_1, \ldots, x_\ell)$

3. If $\lambda_\ell \geq 2$, then $f_\lambda(x_1, \ldots, x_\ell) = f_{\lambda^\diamond}(x_1, \ldots, x_\ell) \prod_{i=1}^{\ell} (x_i - \tau)$, with $\lambda^\diamond = (\lambda_1 - 1, \ldots, \lambda_\ell - 1)$

4. If $\lambda_\ell = 1$, then $f_\lambda(x_1, \ldots, x_\ell) = x_\ell f_{\lambda^\diamond}(x_1, \ldots, x_{\ell-1}) + O(x_\ell^2)$, with $\lambda^\diamond = (\lambda_1, \ldots, \lambda_{\ell-1})$

This leads to a heuristic formula for $B^\mu_\lambda$, valid for a generic minimal alphabet $(x_1, \ldots, x_\ell)$, that you can then prove right by determining the inverse $(B^{-1})^\mu_\lambda$ and showing that (at $\tau = 1$)

$$T^{5v}(x) = B^{-1} T(x) B$$
Expansion of $f_\lambda$'s and $g_\lambda$'s over Schur functions

$$f_\lambda = \sum_{\mu \preceq \lambda, \ell(\mu) = \ell(\lambda)} \tau^{\lambda/\mu} s_\mu B_\lambda^\mu$$

$$g_\nu = \sum_{\mu \succeq \nu, \ell(\mu) = \ell(\nu)} \tau^{\mu/\nu} (B^{-1})^\nu_\mu s_\mu$$

$$B_\lambda^\mu = (-1)^{\lambda/\mu} \det \left[ \binom{\lambda_j - 1}{\mu_i - i + j - 1} \right]_{i,j=1,\ldots,\ell}$$

$$\left(B^{-1}\right)_\lambda^\mu = \det \left[ \binom{\lambda_i - i + j - 1}{\mu_j - 1} \right]_{i,j=1,\ldots,\ell}$$

Many new conjectures on FPL's
Expansion of $f_{\lambda}$’s and $g^\lambda$’s over Schur functions

$$f_{\lambda} = \sum_{\mu \preceq \lambda, \ell(\mu) = \ell(\lambda)} \tau^{\lambda/\mu} s_{\mu} B^\mu_{\lambda}$$

$$g^\nu = \sum_{\mu \succeq \nu, \ell(\mu) = \ell(\nu)} \tau^{\mu/\nu} (B^{-1})^\nu_{\mu} s_{\mu}$$

$$B^\mu_{\lambda} = (-1)^{\lambda/\mu} \det \begin{bmatrix} \lambda_j - 1 \\ \mu_i - i + j - 1 \end{bmatrix}_{i,j=1,...,\ell}$$

$$(B^{-1})^\lambda_{\mu} = \det \begin{bmatrix} \lambda_i - i + j - 1 \\ \mu_j - 1 \end{bmatrix}_{i,j=1,...,\ell}$$
Determinantal formulas for the $f_\lambda$'s

**Weyl-type determinantal formula for $f_\lambda$**

$$f_\lambda(x_1, \ldots, x_n) = \frac{1}{\Delta(\vec{x})} \det \left[ (x_j - \tau)^{\lambda_i - 1} (x_j^{n-i+1} - \tau^{n-i+1}) \delta_{\lambda_j,0} \right]_{i,j=1,\ldots,n}$$

**Jacobi–Trudi-type determinantal formula for $f_{\lambda/\mu}$**

$$f_{\lambda/\mu}(\vec{x'}) = \det \left( (h_{[\lambda_i-\mu_j-1, j-i+1]})_{i,j=1,\ldots,\ell(\lambda)} \right)$$

$$h_{[a,b]} := \sum_{c=0}^{a} \binom{a}{c} (-\tau)^c h_{a+b-c} = [z^{a+b}](1 - \tau z)^a \prod \frac{1}{1 - zx_i}$$

The Jacobi–Trudi-type formula indeed generalises the one for Schur, recalling that $s_{\lambda/\mu} = \det \left( (h_{\lambda_i-\mu_j+j-i})_{i,j=1,\ldots,\ell(\lambda)} \right)$ and observing that $h_{[a,b]} = h_{a+b}$ when $\tau = 0$.

Also, it is **stable**, i.e. you can take matrices of dimension $d \geq \ell(\lambda)$
The $f_\lambda$ are $(\alpha, \beta) = (-1, 0)$ Canonical Grothendieck poly’s

All these results allow to identify the $f_\lambda$’s with functions that have already arised in various places in the literature


- K. Motegi and T. Scrimshaw, *Refined Dual Grothendieck Polynomials, Integrability, and the Schur Measure*, SLC 85 (2021) [ex. 3.7, with $t_i \rightarrow \tau$, identified by the formula for $B^\mu_\lambda$]


Note that in these papers the $f_\lambda$’s arise from a bosonic Vertex Model!
What about the $g^{\lambda}$'s?

Now that we have our favourite $f^{\lambda}$'s, how can we determine the duals $g^{\lambda}$'s?

(1) you feel lucky, and search for a $\tau$-deformation of $U(x)$ and $\bar{U}(y)$;

(2) you go the safe way, and evaluate the branching rule of the $g^{\lambda}$'s, that is

$$U^{5v}(x) = B T(x) B^{-1} \quad (\tau = 1)$$

Remark: $g^{\lambda/\mu}(\vec{x} | \vec{y})$ are polynomials in $x_i$'s, $y_j$'s and $\tau$, and homogeneous of degree $|\lambda/\mu|$ in $x_i$'s, $y_j$'s and $\tau^{-1}$
Determinantal formulas for the $g^\lambda$'s

Weyl-type determinantal formula for $g^\lambda$

$$g^\lambda(x_1, \ldots, x_n) = \frac{1}{\Delta(\vec{x})} \det \left[ \left( \frac{x_j}{1 - \tau x_j} \right)^{\lambda_i} x_j^{n-i} \right]_{i,j=1,...,n}$$

Jacobi–Trudi-type determinantal formula for $g^{\lambda/\mu}$

$$g^{\lambda/\mu}(\vec{x}) = \det \left( \left( h_{\{\lambda_i - \mu_j - 1, j-i+1\}} \right)_{i,j=1,...,\ell(\lambda)} \right)$$

$$h_{\{a,b\}} := \sum_{c \geq 0} \binom{a+c}{a} \tau^c h_{a+b+c} = [z^{a+b}](1 - \tau/z)^{-a-1} \prod \frac{1}{1-zx_i}$$

Again, the Jacobi–Trudi-type formula generalises the one for Schur, because also $h_{\{a,b\}} = h_{a+b}$ when $\tau = 0$. 
Our best conjecture so far...

So, we had hopes that the structure constants of our new basis \( \{ f_\lambda \} \) may be related to our UASM enumeration vectors, but, due to the homogeneity in \( \deg(\vec{x}) + \deg(\tau) \), only for the leading coefficient of the enumeration polynomials, namely

**Conjecture 6**

\[
f_\mu(\vec{x})f_\nu(\vec{x}) = \sum_\lambda c^\lambda_{\mu \nu} f_\lambda(\vec{x}) \quad [\tau |\lambda|+|\rho|-(\begin{array}{c} n \\ 2 \end{array})] \psi_{\lambda \rho}(\tau) = c^\delta_n_{\lambda \rho}
\]

This conjecture indeed holds up to \( n = 5 \)

Recall that consistency with our conjectures requires

\[
[\tau |\lambda|+|\rho|-(\begin{array}{c} n \\ 2 \end{array})](\psi_{\lambda \rho}(\tau) - \psi_{\lambda \rho'}(\tau)) = c^\delta_n_{\lambda \rho} - c^\delta_n_{\lambda \rho'} = 0
\]

Indeed our proof works out of the box for the coproduct coefficients, i.e., starting from \( g^\lambda(\vec{x} \cup \vec{y}) := \sum_{\mu,\nu} c^\lambda_{\mu \nu} g^\mu(\vec{x})g^\nu(\vec{y}) \), and establishing \( U(x)|\delta_n\rangle = \bar{U}(x)|\delta_n\rangle \), which implies a “triangular=magic” lemma also in this case.
This is clearly a work in progress, with many things going on... I recall you our three main open questions:

▶ How can we prove our conjectures on the $\Psi_{\lambda \rho}(\tau)$ enumerations?
▶ There is any hope for a conjecture of the form $\Psi_{\lambda \rho}(\tau) = c^{\delta_n}_{\lambda \rho}$, for some family of functions?
▶ There is a puzzle description of the $c^\lambda_{\mu \nu}$ and $d^\mu_{\nu \lambda}$ structure constants for the canonical Grothendieck polynomials?  
[see also the work in progress of A. Gunna and P. Zinn-Justin]

Thank you for listening!