The multispecies totally asymmetric long-range exclusion process and Macdonald polynomials

Arvind Ayyer
Indian Institute of Science, Bangalore
(joint with G. Amir, O. Angel and J. Martin)

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This workshop

Outline

1. Multispecies ASEP
2. mTALREP
3. mTALREP with rejection
4. Observables

Single species ASEP

- (Partially) Asymmetric Simple Exclusion Process (ASEP).
- Ring of size n, with $m < n$ particles.

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- Ring of size n, with $m < n$ particles.
- Let $0 \leq t \leq 1$. Transitions are:

$$10 \xrightarrow{1} 01, \quad 01 \xrightarrow{t} 10.$$

Proposition

For any positive integers $n, m < n$, the ASEP on n sites with m particles has the uniform stationary distribution, i.e.

$$\pi(\omega) = \frac{1}{\binom{n}{m}}.$$

Multispecies ASEP

- Now suppose there are particles labelled $1, \dots, s$ withstrength order: $1 > 2 > \cdots > s$..
- Consider a ring of n sites, with particle content given by $\underline{m} = (m_1, \dots, m_s)$, where $\sum_i m_i = n$.
- The multispecies ASEP is defined by transitions

$$ij \xrightarrow{1} ji, \quad ji \xrightarrow{t} ij, \quad \text{provided } i < j.$$

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Theorem (P. Ferrari and J. Martin (*Ann. Prob.* 2007))

Consider the multispecies TASEP ($t = 0$) with content $(m_1, \ldots, m_s)$. Let $M_i = m_1 + \cdots + m_i$ for $1 \leq i \leq s$. Then the partition function is given by

$$\prod_{i=1}^{s} \binom{n}{M_i}.$$

Partition function for multispecies ASEP

Recall that $[n]_q := 1 + q + \cdots + q^{n-1}$ and $[n]_q! := [1]_q[2]_q \cdots [n]_q$.

Theorem (J. Martin (*Elec. J. Prob.* 2020))

Consider the multispecies ASEP with content $(m_1, \dots, m_s)$. Then the partition function is given by

$$Z = \prod_{i=1}^{s} \binom{n}{M_i} \frac{[M_i]_t!}{[n_i]_t!}.$$

The proofs use a multiline TASEP (with rejection) that projects to the multispecies TASEP (ASEP).

We do not know of an inhomogeneous *integrable* generalisation!

Totally Asymmetric Long-Range Exclusion Process (TALREP)

- Ring of size n with $m < n$ particles.
- From site i,

$$\cdots \underset{i}{\underline{1}}\ \underset{i+1}{\underline{1}} \cdots \underset{j-1}{\underline{1}}\ \underset{j}{\underline{0}} \cdots \longrightarrow \cdots \underset{i}{\underline{0}}\ \underset{i+1}{\underline{1}} \cdots \underset{j-1}{\underline{1}}\ \underset{j}{\underline{1}} \cdots \quad \text{with rate } \alpha_i,$$

- Also called the PushASEP and isomorphic to the Hammersley–Aldous–Diaconis (HAD) process (on $\mathbb{Z}$).

Stationary distribution

- Recall that the elementary symmetric polynomial of degree m in indeterminates $x_1, \dots, x_k$ is

$$e_m(x_1, \dots, x_k) = \sum_{1 \le i_1 < \cdots < i_m \le k} x_{i_1} \dots x_{i_k},$$

- Let $\eta = (\eta_1, \dots, \eta_n)$ be a configuration.

Proposition

The stationary probability η is

$$\frac{1}{e_m(1/\alpha_1, \dots, 1/\alpha_n)} \prod_{\substack{i=1 \\ \eta_i = 1}}^{n} \frac{1}{\alpha_i}.$$

Aside: open TALREP

- n sites with open boundaries.
- Same bulk transitions and additionally:
  - From the left boundary,

$$\underset{1}{\underline{1}}\cdots\underset{i}{\underline{1}}\ \underset{i+1}{\underline{0}}\ \cdots \longrightarrow \underset{1}{\underline{1}}\cdots\underset{i}{\underline{1}}\ \underset{i+1}{\underline{1}}\ \cdots \quad \text{with rate } \alpha_0,$$

  - From site i to outside the right boundary,

$$\cdots\underset{i}{\underline{1}}\ \underset{i+1}{\underline{1}}\ \cdots\underset{n}{\underline{1}} \longrightarrow \cdots\underset{i}{\underline{0}}\ \underset{i+1}{\underline{1}}\ \cdots\underset{n}{\underline{1}} \quad \text{with rate } \alpha_i,$$

- Here, the stationary distribution is a product measure with density $\alpha_0/(\alpha_0+\alpha_i)$ at site i, and ...
- all eigenvalues are linear in $\alpha_0,\dots,\alpha_n$ (A.–Schilling–Steinberg–Thiéry, *Comm. Math. Phys. 2015*).

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- As before, the strength order of particles: $1 > 2 > \cdots > s$.

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  5. which jumps to i.
- The homogeneous version of this process is the multispecies HAD process (Ferrari and Martin, *AIHP B*, 2009).

Examples

- $\underline{m} = (2, 2, 1, 1, 1)$ so that $n = 8, s = 5$.

$$3\ 1\ 2\ 5\ 3\ 1\ 4\ 2 \xrightarrow{\alpha_2} 3\ 5\ 1\ 2\ 3\ 1\ 4\ 2$$

$$3\ 1\ 2\ 5\ 3\ 1\ 4\ 2 \xrightarrow{\alpha_4} 3\ 1\ 2\ 5\ 3\ 1\ 4\ 2$$

$$3\ 1\ 2\ 5\ 3\ 1\ 4\ 2 \xrightarrow{\alpha_5} 3\ 1\ 2\ 4\ 5\ 1\ 3\ 2$$

Basic properties

Proposition

For any $\underline{m} = (m_1, \ldots, m_s)$, the mTALREP is irreducible.

Proposition

The mTALREP is invariant under simultaneous translation (i.e. rotation) of sites, $i \to i+1$, and of parameters $\alpha_i \to \alpha_{i+1}$.

Stationary distribution

Theorem (Amir–Angel–A.–Martin, 2022+)

The stationary probability of $\eta = (\eta_1, \dots, \eta_n)$ *is given by*

$$\pi(\eta) = \frac{v(\eta)}{Z},$$

where $v(\eta) \in \mathbb{Z}[1/\alpha_1, \dots, 1/\alpha_n]$ *and* $\gcd\{v(\eta)\} = 1$.

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where $v(\eta) \in \mathbb{Z}[1/\alpha_1, \dots, 1/\alpha_n]$ *and* $\gcd\{v(\eta)\} = 1$*. Recall* $M_i = m_1 + \cdots + m_i$ *for* $1 \le i \le s$*. The partition function is*

$$Z = \prod_{i=1}^{s-1} e_{M_i}\left(\frac{1}{\alpha_1}, \dots, \frac{1}{\alpha_n}\right).$$

Connection to the multispecies TASEP

- Recall that the partition function for the multispecies TASEP is
$$\prod_{i=1}^{s} \binom{n}{M_i}.$$
- If we set $\alpha_1 = \cdots = \alpha_n = 1$ in the mTALREP, we obtain not only the same partition function, but the same stationary distribution!
- We will modify the Ferrari–Martin proof using a different multiline process.

$m = (1, 1, 1)$

- Let $\Omega = \{123, 132, 213, 231, 312, 321\}$.
- The generator is

$$M = \begin{pmatrix} -\alpha_1 - \alpha_2 & \alpha_3 & 0 & \alpha_3 & 0 & \alpha_3 \\ \alpha_2 & -\alpha_1 - \alpha_3 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\alpha_1 - \alpha_2 & 0 & \alpha_3 & 0 \\ 0 & 0 & \alpha_2 & -\alpha_1 - \alpha_3 & \alpha_2 & \alpha_2 \\ \alpha_1 & \alpha_1 & \alpha_1 & 0 & -\alpha_2 - \alpha_3 & 0 \\ 0 & 0 & 0 & \alpha_1 & 0 & -\alpha_2 - \alpha_3 \end{pmatrix}$$

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- The stationary weights turn out to be

$$v = \left(\alpha_2\alpha_3(\alpha_1 + \alpha_3), \alpha_2^2\alpha_3, \alpha_1\alpha_3^2, \alpha_1\alpha_2(\alpha_2 + \alpha_3), \alpha_1\alpha_3(\alpha_1 + \alpha_2), \alpha_1^2\alpha_2\right).$$

- $$Z = (\alpha_1 + \alpha_2 + \alpha_3)(\alpha_1\alpha_2 + \alpha_1\alpha_3 + \alpha_2\alpha_3) = e_1 e_2.$$

Jump to multiline example | Jump to ‘mTALREP with rejection’ example

Strategy of proof: Lumping

- Let S be the state space of a Markov chain with generator M.
- Suppose $\sim$ is an equivalence relation on M.
- For $s \in S$, let $[s]$ be the equivalence class of s.
- Let $M(s,[t]) = \sum_{t' \in [t]} M(s,t')$.

Definition

If $M(s,[t]) = M(s',[t])$ for all $s, s', t \in S$, then the projected process on $\{[s] \mid s \in S\}$ is a Markov chain, known as the lumping of the original chain.

We will construct a Markov chain whose lumping is the mTALREP.

Configuration space

- As before, let $m = (m_1, \dots, m_s)$ with $n = \sum_i m_i$.
- Configurations live on a discrete cylinder with $s - 1$ rows and n columns.
- Each site is either vacant or occupied by a particle, ...
- such that the i'th row as M_i particles.

Multiline TALREP

- With rate α_i, the site $(s-1, i)$ will ring.
- If no particle there, go to site $(s-2, i)$.

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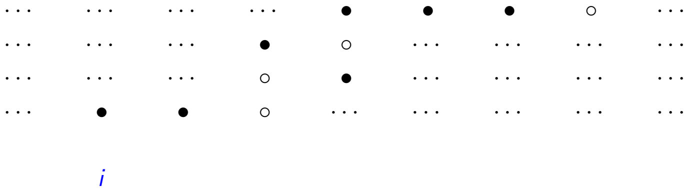
- If no particle there, go to site $(s-2, i)$.
- If there is a particle, it performs a TALREP move to site i_2, say. Now go to site $(s-2, i_2)$.
- Repeat these steps at row $s-2$, and continue this way until we reach row 1.

Illustration



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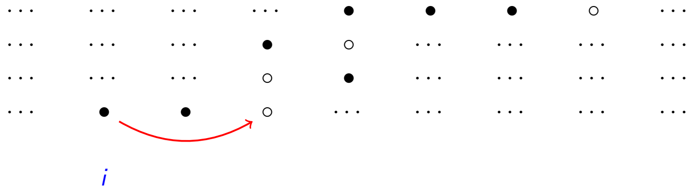

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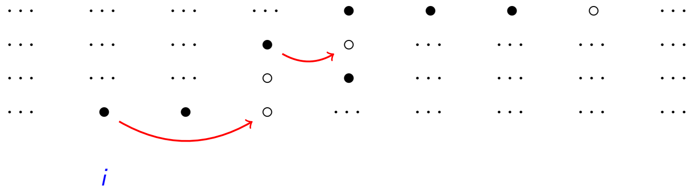

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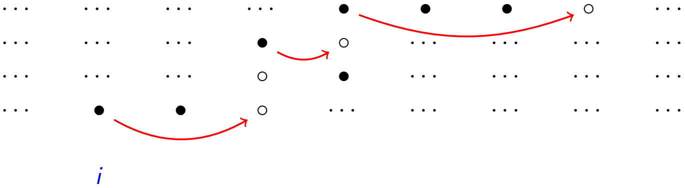

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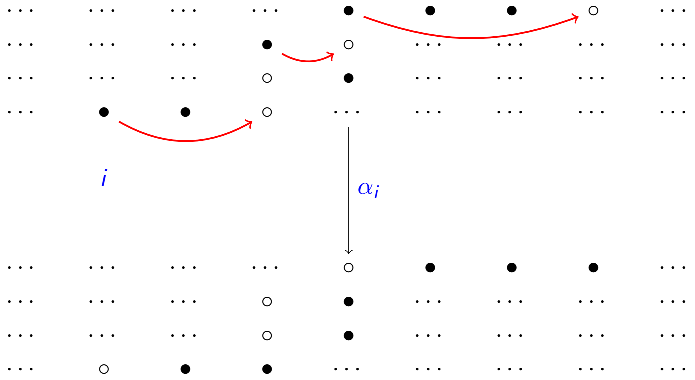

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Let $\underline{m} = (m_1, \dots, m_s)$ and $n = \sum_i m_i$.
Let $c_j(\hat{\eta})$ be the number of 1's in the j'th column of $\hat{\eta}$.

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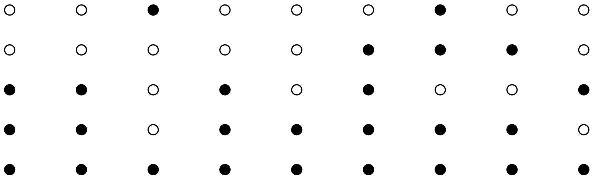
Let $c_j(\hat{\eta})$ be the number of 1's in the j'th column of $\hat{\eta}$.
Then the stationary probability of $\hat{\eta} = (\hat{\eta}_1, \dots, \hat{\eta}_n)$ is given by

$$\pi(\hat{\eta}) = \frac{1}{Z} \prod_{i=1}^{n} \alpha_i^{-c_i(\hat{\eta})}.$$

Recall $M_i = m_1 + \cdots + m_i$ for $1 \le i \le s$. Then clearly

$$Z = \prod_{i=1}^{s} e_{M_i}\left(\frac{1}{\alpha_1}, \dots, \frac{1}{\alpha_n}\right).$$

Example



Weight: $$\frac{1}{\alpha_1^3\alpha_2^3\alpha_3^2\alpha_4^3\alpha_5^2\alpha_6^4\alpha_7^4\alpha_8^3\alpha_9^2}$$

Idea of proof

- Follow the strategy of P. Ferrari and J. Martin (*Ann. Prob.* 2007) for the multispecies TASEP.
- Construct a time-reversed process at stationarity.
- This is related to the notion of pairwise balance for Markov chains (Schütz, Ramaswamy, Barma, *J. Phys. A.* 1996).
- Fix $s \in S$. For every $s' \neq s$ such that

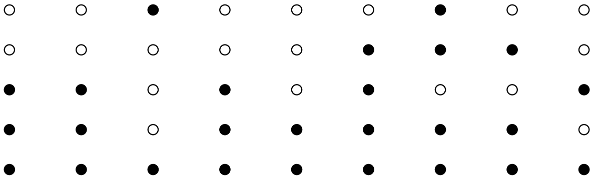
$$s \longrightarrow s',$$

  we find a weight-preserving $s'' \neq s$

$$s'' \longrightarrow s \longrightarrow s'.$$

- If $s'' = s'$ for all $s \in S$, then the chain is reversible.

Lumping via bully paths



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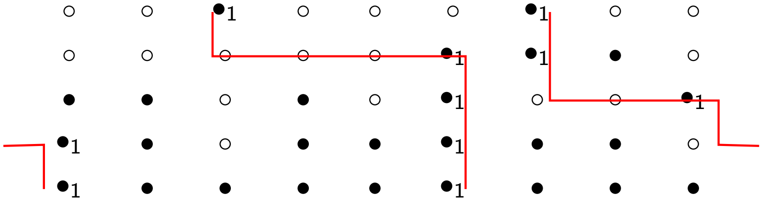

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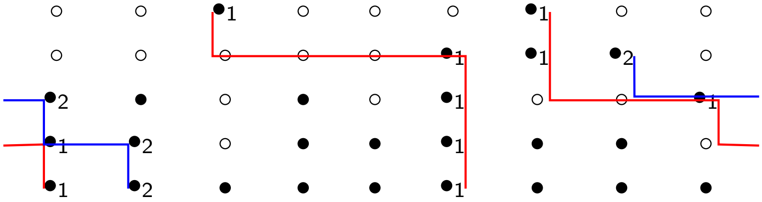

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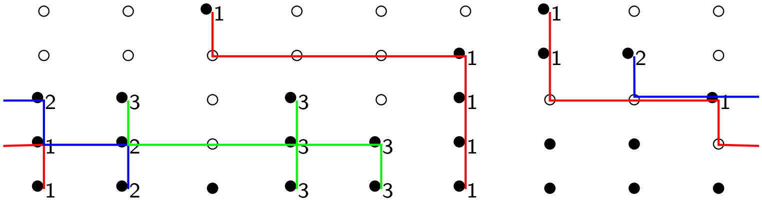

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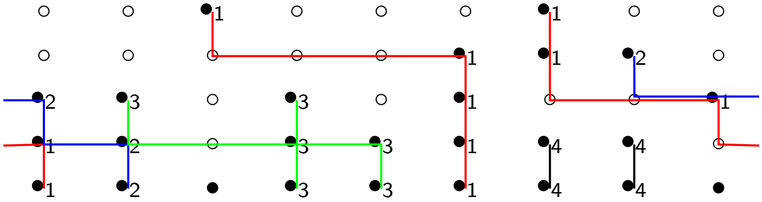

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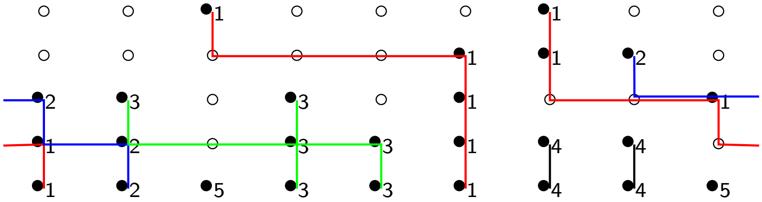

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Consequences

Proposition

- *The marginal process of each row of the multiline TALREP is the single-species TALREP.*
- *The law of the lumped process at the i'th row is the mTALREP with content* $(m_1, \dots, m_i, m_{i+1} + \cdots + m_s)$.

This proves the theorem on the stationary distribution of the mTALREP.

Example: $\underline{m} = (1,1,1)$

- Back to mTALREP example
- Up to translation, only 2 configurations in Ω.
- $v(312) = \alpha_1\alpha_3(\alpha_1+\alpha_2)$:

$$\begin{array}{ccc} 0 & 1 & 0 \\ 0 & 1 & 1 \\ \hline 3 & 1 & 2 \\ & \alpha_1^2\alpha_3 & \end{array} \qquad \begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 1 \\ \hline 3 & 1 & 2 \\ & \alpha_1\alpha_2\alpha_3 & \end{array}$$

- $v(321) = \alpha_1^2\alpha_2$:

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Multispecies TALREP with rejection

- Let $0 \leq t \leq 1$.
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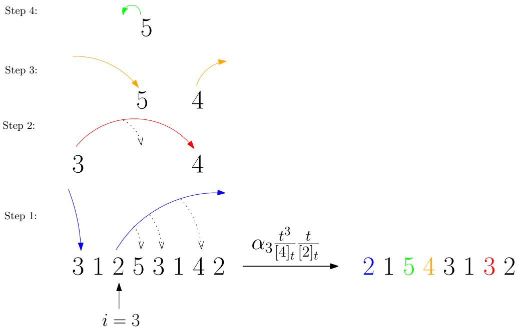

Partition function

Theorem (Amir–Angel–A.–Martin, 2022+)

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where $v(\eta) \in \mathbb{Z}[1/\alpha_1, \ldots, 1/\alpha_n, t]$.

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$$Z = \prod_{i=1}^{s} e_{M_i}\left(\frac{1}{\alpha_1}, \dots, \frac{1}{\alpha_n}\right) \frac{[M_i]_t!}{[n_i]_t!}.$$

is the partition function.

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- Let $\Omega = \{123, 132, 213, 231, 312, 321\}$.
- The generator is

$$M = \begin{pmatrix} -\alpha_1-\alpha_2 & \alpha_3 & 0 & \alpha_3/[2]_t & 0 & \alpha_3/[2]_t \\ \alpha_2 & -\alpha_1-\alpha_3 & \alpha_2 t/[2]_t & 0 & \alpha_2 t/[2]_t & 0 \\ 0 & 0 & -\alpha_1-\alpha_2 & \alpha_3 t/[2]_t & \alpha_3 & \alpha_3/[2]_t \\ 0 & 0 & \alpha_2/[2]_t & -\alpha_1-\alpha_3 & \alpha_2/[2]_t & \alpha_2 \\ \alpha_1/[2]_t & \alpha_1/[2]_t & \alpha_1 & 0 & -\alpha_2-\alpha_3 & 0 \\ \alpha_1 t/[2]_t & \alpha_1 t/[2]_t & 0 & \alpha_1 & 0 & -\alpha_2-\alpha_3 \end{pmatrix}$$

- The stationary weights turn out to be

$$v = \Big(\alpha_2\alpha_3(\alpha_1 + (1+t)\alpha_3), \alpha_2\alpha_3(t\alpha_1 + (1+t)\alpha_2), \alpha_1\alpha_3(t\alpha_2 + (1+t)\alpha_3),$$
$$\alpha_1\alpha_2((1+t)\alpha_2 + \alpha_3), \alpha_1\alpha_3((1+t)\alpha_1 + \alpha_2), \alpha_1\alpha_2((1+t)\alpha_1 + t\alpha_3)\Big).$$

- $Z = (1+t)(\alpha_1+\alpha_2+\alpha_3)(\alpha_1\alpha_2+\alpha_1\alpha_3+\alpha_2\alpha_3)$.

Back to mTALREP example

Proof strategy

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- We give (now t-dependent) weights to multiline configurations.

Proof strategy

- We do not have a multiline TALREP with rejection (yet)!
- We give (now t-dependent) weights to multiline configurations.
- In `arXiv:1811.01024`, Corteel, Mandelshtam and Williams give a combinatorial formula for the nonsymmetric Macdonald polynomial $E_\lambda(x_1, \ldots, x_n; q, t)$ and the permuted basement Macdonald polynomials (Ferreira 2011, Alexandersson 2016) $E^\sigma_\alpha(x_1, \ldots, x_n; q, t)$.
- Both involves a sum over weights of these multiline configurations with the same projection map.

Proof strategy

- This further leads to a combinatorial formula for $P_\lambda(x_1, \ldots, x_n; q, t)$.
- Our weights match theirs when $q = 1$.

Proof strategy

- This further leads to a combinatorial formula for $P_\lambda(x_1, \dots, x_n; q, t)$.
- Our weights match theirs when $q = 1$.
- Alexandersson and Sawhney (*Ann. Comb.* 2019) proved a certain factorisation property for $E_\alpha^\sigma(x_1, \dots, x_n; q, t)$ which gives our result.
- Note that this is a very indirect proof.

Symmetric functions

- Let $x_1, x_2, \ldots$ be a family of commuting indeterminates.
- Let $\Lambda \equiv \Lambda(q,t)$ be the algebra of symmetric functions in these indeterminates with coefficients in $\mathbb{Q}(q,t)$.
- There are several natural bases of $\Lambda(\mathbb{Q})$ indexed by partitions λ, e.g. Schur functions s_λ.
- The Macdonald polynomials are an amazing two-parameter family of symmetric polynomials $P_\lambda(x;q,t)$ (I. Macdonald, *Sem. Loth. Comb.*, 1988).
- A simultaneous generalisation of many known families of symmetric functions.

Specialisations

- $q = t$:
$$P_\lambda(x; t, t) = s_\lambda(x).$$
- $t = 1$:
$$P_\lambda(x; q, 1) = m_\lambda(x).$$
- $q = 1$:
$$P_\lambda(x; 1, t) = e_{\lambda'}(x) = \prod_{i \geq 1} e_{\lambda'_i}(x).$$

Where are the Macdonald polynomials?

- Recall that the particle content is given by $(m_1, \ldots, m_s)$ and $n = \sum_i m_i$.
- Construct the partition $\lambda = \langle (s-1)^{m_1}, \ldots, 0^{m_s} \rangle$.

Where are the Macdonald polynomials?

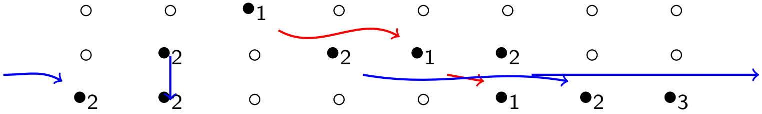
- Recall that the particle content is given by $(m_1, \dots, m_s)$ and $n = \sum_i m_i$.
- Construct the partition $\lambda = \langle (s-1)^{m_1}, \dots, 0^{m_s} \rangle$.
- Then we have

$$Z = P_\lambda(1/\alpha_1, \dots, 1/\alpha_n; 1, t) = \prod_{i=1}^{s-1} e_{M_i}\left(\frac{1}{\alpha_1}, \dots, \frac{1}{\alpha_n}\right).$$

More evidence for Macdonald polynomials: I

- Recall Martin's formula for the stationary distribution of the multispecies ASEP.
- The prefactor in Z involving t-factorials is the same as the one found by Martin.
- His proof used multiline queues with rejection.

More evidence for Macdonald polynomials: II



$$\frac{1}{\alpha_1\alpha_2^2\alpha_3\alpha_4\alpha_5\alpha_6^2\alpha_7\alpha_8}\,\frac{qt^3(1-t)^4}{(1-qt^2)(1-qt^3)(1-qt^4)(1-q^2t^5)}$$

- Upon setting $\alpha_1 = \cdots = \alpha_n = q = 1$ in the combinatorial formula for the nonsymmetric Macdonald polynomial, Corteel, Mandelshtam and Williams (`arXiv:1811.01024`) recover the results of Martin.

What about the full Macdonald polynomial?

- $P_\lambda(x; q, t)$ does not factorise in general.
- From `arXiv:1811.01024`, the intuition is that q should be a parameter in the transition involving sites n and 1.
- Therefore, we lose translation invariance.
- We do not have either a generalised mTALREP with rejection or a generalised multiline TALREP whose partition function is the Macdonald polynomial.
- We believe insights from integrable models can play a key role in defining such a model.

Current

- The current of particles of species j across any edge is the number of such particles traversing that edge per unit time in the long-time limit.
- Because of particle conservation, this is independent of the edge.

Theorem

For the multispecies TALREP with content $(m_1, \dots, m_s)$ on n sites, the current of species j is given by

$$\frac{s_{\langle 2^{M_{j+1}}, 1^{m_j - 1} \rangle}(1/\alpha_1, \dots, 1/\alpha_L)}{e_{M_j}(1/\alpha_1, \dots, 1/\alpha_L) e_{M_{j+1}}(1/\alpha_1, \dots, 1/\alpha_L)}.$$

Density

- The density of particles of species j on a site is the probability of such particle occupying that site in the long-time limit.
- By symmetry, it is enough to consider the density at site 1.

Theorem

For the multispecies TALREP with content $(m_1, \dots, m_s)$ on n sites, the density of species j at the first site is given by

$$\frac{1}{\alpha_1} \frac{s_{\langle 2^{M_{j-1}}, 1^{m_j - 1}\rangle}(1/\alpha_2, \dots, 1/\alpha_L)}{e_{N_j}(1/\alpha_1, \dots, 1/\alpha_L) e_{N_{j-1}}(1/\alpha_1, \dots, 1/\alpha_L)}.$$

FPSAC 2022


FPSAC 2022 July 18-22
Indian Institute of Science, Bengaluru, India
34th International Conference on Formal Power Series & Algebraic Combinatorics
A satellite conference of ICM 2022
Topics include all aspects of combinatorics and their relation to other parts of mathematics, physics, computer science, chemistry and biology
INVITED SPEAKERS
Apoorva Khare
Indian Institute of Science
Sen-Peng Eu
National Taiwan Normal University
Piotr Sniady
Polish Academy of Sciences
Cynthia Vinzant
North Carolina State University
Hugh Thomas
L'Université du Québec à Montréal
Rekha Thomas
University of Washington
Anton Mellit
University of Vienna
Omer Angel
University of British Columbia
Daniela Kuhn
University of Birmingham
The Suci Prastāra (Needle Progression) for computing binomial coefficients was described by the Indian prosodist Pingala in the Chandahśāstra (200 BCE)
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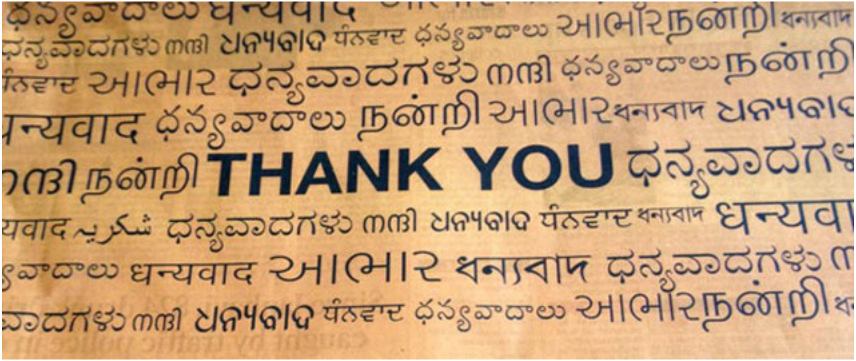
http://math.iisc.ac.in/fpsac2022


THANK YOU