The multispecies totally asymmetric long-range exclusion process and Macdonald polynomials

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Workshop on Randomness, Integrability and Universality,
GGI, Florence, Italy.

May 4, 2022
Multispecies ASEP

2014 Workshop on Advances in nonequilibrium stat. mech.

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This workshop

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Outline

1. Multispecies ASEP
2. mTALREP
3. mTALREP with rejection
4. Observables
Single species ASEP

- (Partially) Asymmetric Simple Exclusion Process (ASEP).
- Ring of size $n$, with $m < n$ particles.
(Partially) Asymmetric Simple Exclusion Process (ASEP).

- Ring of size \( n \), with \( m < n \) particles.
- Let \( 0 \leq t \leq 1 \). Transitions are:

\[
10 \xrightarrow{1} 01, \quad 01 \xrightarrow{t} 10.
\]

**Proposition**

For any positive integers \( n, m < n \), the ASEP on \( n \) sites with \( m \) particles has the uniform stationary distribution, i.e.

\[
\pi(\omega) = \frac{1}{\binom{n}{m}}.
\]
Now suppose there are particles labelled $1, \ldots, s$ with strength order: $1 > 2 > \cdots > s$.

Consider a ring of $n$ sites, with particle content given by $\underline{m} = (m_1, \ldots, m_s)$, where $\sum_i m_i = n$.

The multispecies ASEP is defined by transitions

$$ij \xrightarrow{1} ji, \quad ji \xrightarrow{t} ij,$$

provided $i < j$. 
Now suppose there are particles labelled 1, \ldots, s with strength order: 1 > 2 > \cdots > s.

Consider a ring of \( n \) sites, with particle content given by \( m = (m_1, \ldots, m_s) \), where \( \sum_i m_i = n \).

The multispecies ASEP is defined by transitions

\[ ij \xrightarrow{1} ji, \quad ji \xrightarrow{t} ij, \quad \text{provided } i < j. \]

**Theorem (P. Ferrari and J. Martin (Ann. Prob. 2007))**

Consider the multispecies TASEP \((t = 0)\) with content \((m_1, \ldots, m_s)\). Let \( M_i = m_1 + \cdots + m_i \) for \(1 \leq i \leq s\). Then the partition function is given by

\[
\prod_{i=1}^{s} \binom{n}{M_i}.
\]
Recall that $[n]_q := 1 + q + \cdots + q^{n-1}$ and $[n]_q! := [1]_q[2]_q\cdots[n]_q$.

**Theorem (J. Martin (Elec. J. Prob. 2020))**

Consider the multispecies ASEP with content $(m_1, \ldots, m_s)$. Then the partition function is given by

$$Z = \prod_{i=1}^s \binom{n}{M_i} \frac{[M_i]_t!}{[n_i]_t!}.$$  

The proofs use a multiline TASEP (with rejection) that projects to the multispecies TASEP (ASEP).

We do not know of an inhomogeneous integrable generalisation!

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Totally Asymmetric Long-Range Exclusion Process (TALREP)

- Ring of size $n$ with $m < n$ particles.
- From site $i$,

$$\cdots \frac{1}{i} \frac{1}{i+1} \cdots \frac{1}{j-1} \frac{0}{j} \cdots \rightarrow \cdots \frac{0}{i} \frac{1}{i+1} \cdots \frac{1}{j-1} \frac{1}{j} \cdots$$

with rate $\alpha_i$,

- Also called the PushASEP and isomorphic to the Hammersley–Aldous–Diaconis (HAD) process (on $\mathbb{Z}$).
Recall that the elementary symmetric polynomial of degree $m$ in indeterminates $x_1, \ldots, x_k$ is

$$e_m(x_1, \ldots, x_k) = \sum_{1 \leq i_1 < \cdots < i_m \leq k} x_{i_1} \cdots x_{i_k},$$

Let $\eta = (\eta_1, \ldots, \eta_n)$ be a configuration.

Proposition

The stationary probability $\eta$ is

$$\frac{1}{e_m(1/\alpha_1, \ldots, 1/\alpha_n)} \prod_{i=1}^{n} \frac{1}{\alpha_i}.$$

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Aside: open TALREP

- $n$ sites with open boundaries.
- Same bulk transitions and additionally:
  - From the left boundary,
    
    $\begin{array}{ccc}
    1 & \ldots & 1 \\
    1 & i & i+1 \\
    \end{array} \quad \longrightarrow \quad \begin{array}{ccc}
    1 & \ldots & 1 \\
    1 & i & i+1 \\
    \end{array}$
    with rate $\alpha_0$,
  - From site $i$ to outside the right boundary,
    
    $\begin{array}{ccc}
    \ldots & 1 & 1 \\
    i & i+1 & n \\
    \end{array} \quad \longrightarrow \quad \begin{array}{ccc}
    \ldots & 0 & 1 \\
    i & i+1 & n \\
    \end{array}$
    with rate $\alpha_i$,
- Here, the stationary distribution is a product measure with density $\alpha_0/(\alpha_0 + \alpha_i)$ at site $i$, and ... 
- all eigenvalues are linear in $\alpha_0, \ldots, \alpha_n$ 
As before, we are on the ring of $n$ sites, with particle content $\underline{m} = (m_1, \ldots, m_s)$.

As before, the strength order of particles: $1 > 2 > \cdots > s$. 
As before, we are on the ring of $n$ sites, with particle content $\bar{m} = (m_1, \ldots, m_s)$.

As before, the **strength order** of particles: $1 > 2 > \cdots > s$.

Transition when bell rings at site $i$ with rate $\alpha_i$:

1. Particle at site $i$ moves clockwise,
As before, we are on the ring of \( n \) sites, with particle content \( \mathbf{m} = (m_1, \ldots, m_s) \).

As before, the **strength order** of particles: \( 1 > 2 > \cdots > s \).

Transition when bell rings at site \( i \) with rate \( \alpha_i \):

1. Particle at site \( i \) moves clockwise,
2. finds the first weakest particle and displaces it,
As before, we are on the ring of $n$ sites, with particle content $\mathbf{m} = (m_1, \ldots, m_s)$.

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Transition when bell rings at site $i$ with rate $\alpha_i$:

1. Particle at site $i$ moves clockwise,
2. finds the first weakest particle and displaces it,
3. which in turn does the same.
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Transition when bell rings at site $i$ with rate $\alpha_i$:

1. Particle at site $i$ moves clockwise,
2. finds the first weakest particle and displaces it,
3. which in turn does the same.
4. Continue this way ending at a particle of species $s$, 

The homogeneous version of this process is the multispecies HAD process (Ferrari and Martin, AIHP B, 2009).
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Transition when bell rings at site \( i \) with rate \( \alpha_i \):

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2. finds the first weakest particle and displaces it,
3. which in turn does the same.
4. Continue this way ending at a particle of species \( s \),
5. which jumps to \( i \).

The homogeneous version of this process is the **multispecies HAD process** (Ferrari and Martin, *AIHP B*, 2009).
Examples

- $m = (2, 2, 1, 1, 1)$ so that $n = 8, s = 5$.
Basic properties

**Proposition**

For any $m = (m_1, \ldots, m_s)$, the mTALREP is irreducible.

**Proposition**

The mTALREP is invariant under simultaneous translation (i.e. rotation) of sites, $i \to i + 1$, and of parameters $\alpha_i \to \alpha_{i+1}$. 
Theorem (Amir–Angel–A.–Martin, 2022+)

The stationary probability of $\eta = (\eta_1, \ldots, \eta_n)$ is given by

$$\pi(\eta) = \frac{v(\eta)}{Z},$$

where $v(\eta) \in \mathbb{Z}[1/\alpha_1, \ldots, 1/\alpha_n]$ and $\gcd\{v(\eta)\} = 1.$
Theorem (Amir–Angel–A.–Martin, 2022+)

The stationary probability of \( \eta = (\eta_1, \ldots, \eta_n) \) is given by

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\pi(\eta) = \frac{v(\eta)}{Z},
\]

where \( v(\eta) \in \mathbb{Z}[1/\alpha_1, \ldots, 1/\alpha_n] \) and \( \gcd\{v(\eta)\} = 1 \). Recall \( M_i = m_1 + \cdots + m_i \) for \( 1 \leq i \leq s \). The partition function is

\[
Z = \prod_{i=1}^{s-1} e^{M_i} \left( \frac{1}{\alpha_1}, \ldots, \frac{1}{\alpha_n} \right).
\]
Recall that the partition function for the multispecies TASEP is

$$\prod_{i=1}^{s} \binom{n}{M_i}.$$  

If we set $\alpha_1 = \cdots = \alpha_n = 1$ in the mTALREP, we obtain not only the same partition function, but the same stationary distribution!

We will modify the Ferrari–Martin proof using a different multiline process.
Let $\Omega = \{123, 132, 213, 231, 312, 321\}$.

The generator is

$$M = \begin{pmatrix}
-a_1 - a_2 & a_3 & 0 & a_3 & 0 & a_3 \\
 a_2 & -a_1 - a_3 & 0 & 0 & 0 & 0 \\
 0 & 0 & -a_1 - a_2 & 0 & a_2 & 0 \\
 0 & 0 & a_2 & -a_1 - a_3 & a_2 & a_2 \\
a_1 & a_1 & a_1 & 0 & -a_2 - a_3 & 0 \\
 0 & 0 & 0 & a_1 & 0 & -a_2 - a_3 \\
\end{pmatrix}$$
\(m = (1, 1, 1)\)

- Let \(\Omega = \{123, 132, 213, 231, 312, 321\}\).

- The generator is

\[
M = \begin{pmatrix}
-\alpha_1 - \alpha_2 & \alpha_3 & 0 & \alpha_3 & 0 & \alpha_3 \\
\alpha_2 & -\alpha_1 - \alpha_3 & 0 & 0 & 0 & 0 \\
0 & 0 & -\alpha_1 - \alpha_2 & 0 & \alpha_3 & 0 \\
0 & 0 & \alpha_2 & -\alpha_1 - \alpha_3 & \alpha_2 & \alpha_2 \\
\alpha_1 & \alpha_1 & \alpha_1 & 0 & -\alpha_2 - \alpha_3 & 0 \\
0 & 0 & 0 & \alpha_1 & 0 & -\alpha_2 - \alpha_3
\end{pmatrix}
\]

- The stationary weights turn out to be

\(v = (\alpha_2 \alpha_3 (\alpha_1 + \alpha_3), \alpha_2^2 \alpha_3, \alpha_1 \alpha_3^2, \alpha_1 \alpha_2 (\alpha_2 + \alpha_3), \alpha_1 \alpha_3 (\alpha_1 + \alpha_2), \alpha_1^2 \alpha_2)\).

- \(Z = (\alpha_1 + \alpha_2 + \alpha_3)(\alpha_1 \alpha_2 + \alpha_1 \alpha_3 + \alpha_2 \alpha_3) = e_1 e_2\).
Let $S$ be the state space of a Markov chain with generator $M$.

Suppose $\sim$ is an equivalence relation on $M$.

For $s \in S$, let $[s]$ be the equivalence class of $s$.

Let $M(s, [t]) = \sum_{t' \in [t]} M(s, t')$.

Definition

If $M(s, [t]) = M(s', [t])$ for all $s, s', t \in S$, then the projected process on $\{[s] \mid s \in S\}$ is a Markov chain, known as the lumping of the original chain.

We will construct a Markov chain whose lumping is the mTALREP.
As before, let $m = (m_1, \ldots, m_s)$ with $n = \sum_i m_i$.

Configurations live on a discrete cylinder with $s - 1$ rows and $n$ columns.

Each site is either vacant or occupied by a particle, . . .

such that the $i$'th row as $M_i$ particles.
• With rate $\alpha_i$, the site $(s - 1, i)$ will ring.
• If no particle there, go to site $(s - 2, i)$.
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If no particle there, go to site $(s - 2, i)$.

If there is a particle, it performs a TALREP move to site $i_2$, say. Now go to site $(s - 2, i_2)$.

Repeat these steps at row $s - 2$, and continue this way until we reach row 1.
Illustration

\[ \ldots \ldots \ldots \ldots \ldots \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \ldots \ldots \ldots \ldots \ldots \ldots \]

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\[ i \]
Illustration

\[ \ldots \ldots \ldots \ldots \ldots \ldots \bullet \bullet \bullet \circ \circ \circ \ldots \]

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Illustration

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Illustration
Basic properties

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**Proposition**

The multiline TALREP is invariant under simultaneous translation (i.e. rotation) of sites, $i \rightarrow i + 1$, and of parameters $\alpha_i \rightarrow \alpha_{i+1}$. 

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Theorem (Amir–Angel–A.–Martin, 2022+)

Let \( m = (m_1, \ldots, m_s) \) and \( n = \sum_i m_i \).
Let \( c_j(\hat{\eta}) \) be the number of 1’s in the \( j \)’th column of \( \hat{\eta} \).
Stationary distribution

Theorem (Amir–Angel–A.–Martin, 2022+)

Let \( m = (m_1, \ldots, m_s) \) and \( n = \sum_i m_i \).

Let \( c_j(\hat{\eta}) \) be the number of 1’s in the \( j \)'th column of \( \hat{\eta} \). Then the stationary probability of \( \hat{\eta} = (\hat{\eta}_1, \ldots, \hat{\eta}_n) \) is given by

\[
\pi(\hat{\eta}) = \frac{1}{Z} \prod_{i=1}^n \alpha_i^{-c_i(\hat{\eta})}.
\]

Recall \( M_i = m_1 + \cdots + m_i \) for \( 1 \leq i \leq s \). Then clearly

\[
Z = \prod_{i=1}^s e_{M_i} \left( \frac{1}{\alpha_1}, \ldots, \frac{1}{\alpha_n} \right).
\]
Example

Weight: \( \frac{1}{\alpha_1^3 \alpha_2^3 \alpha_3^2 \alpha_4^3 \alpha_5^2 \alpha_6^4 \alpha_7^4 \alpha_8^3 \alpha_9^2} \)
Idea of proof

- Follow the strategy of P. Ferrari and J. Martin (Ann. Prob. 2007) for the multispecies TASEP.
- Construct a time-reversed process at stationarity.
- This is related to the notion of pairwise balance for Markov chains (Schütz, Ramaswamy, Barma, J. Phys. A. 1996).
- Fix $s \in S$. For every $s' \neq s$ such that $s \rightarrow s'$,

we find a weight-preserving $s'' \neq s$

$$s'' \rightarrow s \rightarrow s'.$$

- If $s'' = s'$ for all $s \in S$, then the chain is reversible.
Lumping via bully paths

Weight: \( \frac{1}{\alpha_1^3 \alpha_2^3 \alpha_3^2 \alpha_4^3 \alpha_5^2 \alpha_6^4 \alpha_7^4 \alpha_8^3 \alpha_9^2} \)
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Weight: \( \frac{1}{\alpha_1^3 \alpha_2^3 \alpha_3^3 \alpha_4^2 \alpha_5^4 \alpha_6^4 \alpha_7^3 \alpha_8^2} \)
Proposition

- The marginal process of each row of the multiline TALREP is the single-species TALREP.
- The law of the lumped process at the i’th row is the mTALREP with content \((m_1, \ldots, m_i, m_{i+1} + \cdots + m_s)\).

This proves the theorem on the stationary distribution of the mTALREP.
Example: \( \mathbf{m} = (1, 1, 1) \)

- **Back to mTALREP example**

- Up to translation, only 2 configurations in \( \Omega \).

- \( \nu(312) = \alpha_1 \alpha_3 (\alpha_1 + \alpha_2) \):
  
  \[
  \begin{array}{ccc}
  0 & 1 & 0 \\
  0 & 1 & 1 \\
  3 & 1 & 2 \\
  \end{array}
  \quad
  \begin{array}{ccc}
  1 & 0 & 0 \\
  0 & 1 & 1 \\
  3 & 1 & 2 \\
  \end{array}
  \]

  \( \alpha_1^2 \alpha_3 \) \quad \( \alpha_1 \alpha_2 \alpha_3 \)

- \( \nu(321) = \alpha_1^2 \alpha_2 \):
  
  \[
  \begin{array}{ccc}
  0 & 0 & 1 \\
  0 & 1 & 1 \\
  3 & 1 & 2 \\
  \end{array}
  \]

  \( \alpha_1^2 \alpha_2 \)
Let $0 \leq t \leq 1$.

As for the mTALREP, $m = (m_1, \ldots, m_s)$.

Transition when bell rings at site $i$ with rate $\alpha_i$:

1. Particle at site $i$ moves to the right,
Let $0 \leq t \leq 1$.

As for the mTALREP, $\underline{m} = (m_1, \ldots, m_s)$.

Transition when bell rings at site $i$ with rate $\alpha_i$:

1. Particle at site $i$ moves to the right,
2. displaces the $j$'th weakest particle with probability $t^{j-1}/[m]_t$, where there are $m$ particles with labels larger than it.
Let $0 \leq t \leq 1$.

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3. The displaced particle does the same.
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4. Continue this way ending at a particle of species $s$,
Let $0 \leq t \leq 1$.

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\[ \underline{m} = (2, 2, 1, 1, 1) \] so that \( n = 8, s = 5. \)
Theorem (Amir–Angel–A.–Martin, 2022+)

Let \( \mathbf{m} = (m_1, \ldots, m_s) \) and \( n = \sum_i m_i \). Then the stationary probability of \( \eta = (\eta_1, \ldots, \eta_n) \) in the multispecies TALREP with rejection is given by

\[
\pi(\eta) = \frac{v(\eta)}{Z},
\]

where \( v(\eta) \in \mathbb{Z}[1/\alpha_1, \ldots, 1/\alpha_n, t] \).
Theorem (Amir–Angel–A.–Martin, 2022+)

Let $m = (m_1, \ldots, m_s)$ and $n = \sum_i m_i$. Then the stationary probability of $\eta = (\eta_1, \ldots, \eta_n)$ in the multispecies TALREP with rejection is given by

$$\pi(\eta) = \frac{v(\eta)}{Z},$$

where $v(\eta) \in \mathbb{Z}[1/\alpha_1, \ldots, 1/\alpha_n, t]$. Recall $M_i = m_1 + \cdots + m_i$ for $1 \leq i \leq s$. Then

$$Z = \prod_{i=1}^{s} e^{M_i} \left( \frac{1}{\alpha_1}, \ldots, \frac{1}{\alpha_n} \right) \frac{[M_i]_t!}{[n_i]_t!}.$$ 

is the partition function.
Let $\Omega = \{123, 132, 213, 231, 312, 321\}$.

The generator is

$$M = \begin{pmatrix}
-\alpha_1 - \alpha_2 & \alpha_3 & 0 & \alpha_3/[2]_t & 0 & \alpha_3/[2]_t \\
\alpha_2 & -\alpha_1 - \alpha_3 & \alpha_2 t/[2]_t & 0 & \alpha_2 t/[2]_t & 0 \\
0 & 0 & -\alpha_1 - \alpha_2 & \alpha_3 t/[2]_t & \alpha_3 & \alpha_3/[2]_t \\
0 & 0 & \alpha_2/[2]_t & -\alpha_1 - \alpha_3 & \alpha_2/[2]_t & \alpha_2 \\
\alpha_1/[2]_t & \alpha_1/[2]_t & \alpha_1 & 0 & -\alpha_2 - \alpha_3 & 0 \\
\alpha_1 t/[2]_t & \alpha_1 t/[2]_t & 0 & \alpha_1 & 0 & -\alpha_2 - \alpha_3
\end{pmatrix}$$
Let $\Omega = \{123, 132, 213, 231, 312, 321\}$.

The generator is

$$M = \begin{pmatrix}
-a_1 - a_2 & a_3 & 0 & a_3/2t & 0 & a_3/2t \\
2 & -a_1 - a_3 & a_2 t/2t & 0 & a_2 t/2t & 0 \\
0 & 0 & -a_1 - a_2 & a_3 t/2t & a_3 & a_3/2t \\
0 & 0 & a_2/2t & -a_1 - a_3 & a_2/2t & a_2 \\
a_1/2t & a_1/2t & a_1 & 0 & -a_2 - a_3 & 0 \\
a_1 t/2t & a_1 t/2t & 0 & a_1 & 0 & -a_2 - a_3
\end{pmatrix}$$

The stationary weights turn out to be

$$\nu = 
\left( a_2 a_3 (a_1 + (1 + t)a_3), a_2 a_3 (t a_1 + (1 + t)a_2), a_1 a_3 (t a_2 + (1 + t)a_3),
\right.
\left. a_1 a_2 ((1 + t)a_2 + a_3), a_1 a_3 ((1 + t)a_1 + a_2), a_1 a_2 ((1 + t)a_1 + t a_3) \right).$$

$$Z = (1 + t)(a_1 + a_2 + a_3)(a_1 a_2 + a_1 a_3 + a_2 a_3).$$
Proof strategy

- We do not have a multiline TALREP with rejection (yet)!
- We give (now $t$-dependent) weights to multiline configurations.
Proof strategy

- We do not have a multiline TALREP with rejection (yet)!
- We give (now $t$-dependent) weights to multiline configurations.
- In arXiv:1811.01024, Corteel, Mandelshtam and Williams give a combinatorial formula for the nonsymmetric Macdonald polynomial $E_\lambda(x_1, \ldots, x_n; q, t)$ and the permuted basement Macdonald polynomials (Ferreira 2011, Alexandersson 2016) $E_\sigma^\alpha(x_1, \ldots, x_n; q, t)$.
- Both involves a sum over weights of these multiline configurations with the same projection map.
Proof strategy

- This further leads to a combinatorial formula for \( P_\lambda(x_1, \ldots, x_n; q, t) \).
- Our weights match theirs when \( q = 1 \).
Proof strategy

- This further leads to a **combinatorial formula** for $P_\lambda(x_1, \ldots, x_n; q, t)$.
- Our weights match theirs when $q = 1$.
- Alexandersson and Sawhney (Ann. Comb. 2019) proved a certain **factorisation property** for $E_\alpha(x_1, \ldots, x_n; q, t)$ which gives our result.
- Note that this is a very indirect proof.
Symmetric functions

- Let $x_1, x_2, \ldots$ be a family of commuting indeterminates.
- Let $\Lambda \equiv \Lambda(q, t)$ be the algebra of symmetric functions in these indeterminates with coefficients in $\mathbb{Q}(q, t)$.
- There are several natural bases of $\Lambda(\mathbb{Q})$ indexed by partitions $\lambda$, e.g. Schur functions $s_\lambda$.
- A simultaneous generalisation of many known families of symmetric functions.
Specialisations

- $q = t$: 
  \[ P_\lambda(x; t, t) = s_\lambda(x). \]

- $t = 1$: 
  \[ P_\lambda(x; q, 1) = m_\lambda(x). \]

- $q = 1$: 
  \[ P_\lambda(x; 1, t) = e_\lambda'(x) = \prod_{i \geq 1} e_{\lambda'_i}(x). \]
Where are the Macdonald polynomials?

- Recall that the particle content is given by \((m_1, \ldots, m_s)\) and \(n = \sum_i m_i\).
- Construct the partition \(\lambda = \langle (s-1)^{m_1}, \ldots, 0^{m_s} \rangle\).
Where are the Macdonald polynomials?

- Recall that the particle content is given by \((m_1, \ldots, m_s)\) and 
  \[n = \sum_i m_i.\]
- Construct the partition \(\lambda = \langle (s - 1)^{m_1}, \ldots, 0^{m_s} \rangle.\)
- Then we have

\[
Z = P_\lambda(1/\alpha_1, \ldots, 1/\alpha_n; 1, t) = \prod_{i=1}^{s-1} e_{M_i} \left( \frac{1}{\alpha_1}, \ldots, \frac{1}{\alpha_n} \right).
\]
More evidence for Macdonald polynomials: I

- Recall Martin’s formula for the stationary distribution of the multispecies ASEP.
- The prefactor in $Z$ involving $t$-factorials is the same as the one found by Martin.
- His proof used multiline queues with rejection.
More evidence for Macdonald polynomials: II

Upon setting $\alpha_1 = \cdots = \alpha_n = q = 1$ in the combinatorial formula for the nonsymmetric Macdonald polynomial, Corteel, Mandelshtam and Williams (arXiv:1811.01024) recover the results of Martin.
What about the full Macdonald polynomial?

- $P_\lambda(x; q, t)$ does not factorise in general.
- From arXiv:1811.01024, the intuition is that $q$ should be a parameter in the transition involving sites $n$ and 1.
- Therefore, we lose translation invariance.
- We do not have either a generalised mTALREP with rejection or a generalised multiline TALREP whose partition function is the Macdonald polynomial.
- We believe insights from integrable models can play a key role in defining such a model.
The current of particles of species $j$ across any edge is the number of such particles traversing that edge per unit time in the long-time limit.

Because of particle conservation, this is independent of the edge.

**Theorem**

For the multispecies TALREP with content $(m_1, \ldots, m_s)$ on $n$ sites, the current of species $j$ is given by

$$s_{\langle 2^{M_j+1}, 1^{m_j-1} \rangle}(1/\alpha_1, \ldots, 1/\alpha_L) \frac{e_{M_j}(1/\alpha_1, \ldots, 1/\alpha_L)e_{M_{j+1}}(1/\alpha_1, \ldots, 1/\alpha_L)}{e_{M_{j+1}}(1/\alpha_1, \ldots, 1/\alpha_L)}. $$
Density

- The density of particles of species $j$ on a site is the probability of such particle occupying that site in the long-time limit.
- By symmetry, it is enough to consider the density at site 1.

Theorem

For the multispecies TALREP with content $(m_1, \ldots, m_s)$ on $n$ sites, the density of species $j$ at the first site is given by

$$\frac{1}{\alpha_1} \frac{s_{2^{M_j-1},1^{m_j-1}}(1/\alpha_2, \ldots, 1/\alpha_L)}{e_{N_j}(1/\alpha_1, \ldots, 1/\alpha_L) e_{N_{j-1}}(1/\alpha_1, \ldots, 1/\alpha_L)}.$$
FPSAC 2022

Indian Institute of Science, Bengaluru, India
34th International Conference on Formal Power Series & Algebraic Combinatorics
A satellite conference of ICM 2022

Topics include all aspects of combinatorics and their relation to other parts of mathematics, physics, computer science, chemistry and biology.

INVITED SPEAKERS

Apoorva Khare
Indian Institute of Science

Sen-Peng Eu
National Taiwan Normal University

Piotr Sniady
Polish Academy of Sciences

Cynthia Vinzant
North Carolina State University

Hugh Thomas
L'Université du Québec à Montréal

Rakha Thomas
University of Washington

Anton Mellit
University of Vienna

Omer Angel
University of British Columbia

Daniela Kuhn
University of Birmingham

The Sule Gorreida (Needle Progression) for composing baccal coefficients was described by the Indian prosopist Pulpaka in the Gilgalaibara (260 CE)

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THANK YOU