Transition Probabilities in the Multi-species Asymmetric Exclusion Process

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Outline

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   - Transition probabilities from higher rank vertex models
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   - Transition probabilities from higher rank vertex models
3. Progress on two-colours
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The ASEP is
Asymmetric simple exclusion process

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- a continuous time Markov process
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- a continuous time Markov process
- integrable
Asymmetric simple exclusion process
The probability satisfies the time-evolution equation

$$\frac{d}{dt} P(\nu; t) = \sum_{\lambda \neq \nu} \mathcal{W}(\lambda \rightarrow \nu) P(\lambda; t) - \sum_{\mu \neq \nu} \mathcal{W}(\nu \rightarrow \mu) P(\nu; t)$$
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\]

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Theorem (Schütz (1997), Tracy and Widom (2008))

Given initial \(\mu\) and final conditions \(\nu\) the transition probability on \(\mathbb{Z}\) is given by

\[
P_{t}^{TASEP}(\mu \rightarrow \nu) = \int_{0}^{\infty} \prod_{i=1}^{n} \frac{dz_{i}}{2\pi i} \sum_{\pi \in S_{n}} (-1)^{|\pi|} \prod_{i=1}^{n} \left( \frac{1 - z_{i}}{1 - z_{\pi_{i}}} \right)^{i} e^{(z_{i}^{-1} - 1)t} z_{\nu_{i}}^{\nu_{i}} z_{i}^{-\mu_{i} - 1}
\]

which satisfies the time-evolution with initial condition

\[
P_{0}^{TASEP}(\mu \rightarrow \nu) = \prod_{i=1}^{n} \delta_{\nu_{i}, \mu_{i}}.
\]
We choose the step initial condition $\mu_i = i$
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Define probability of $n$ particles crossing a wall at position $s \in \mathbb{N}$ as

$$P_{\text{cross}}(s) = \mathbb{P}(s \leq \nu_1 < \nu_2 < \cdots < \nu_n).$$
We may find this probability as a Fredholm determinant

\[ P_{\text{cross}}(s) = \det(1 - K_n(x, y))_{\ell^2(\mathbb{N})}, \]

where

\[ K_n(x, y) = \sum_{k=0}^{n-1} \phi_k(x)\psi_k(y). \]
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The functions \( \phi_k, \psi_k \) are defined as contour integrals

\[ \phi_k(x) = \oint_1 \frac{d\eta}{2\pi i} \frac{\eta^{k-x} e^{-\eta t}}{(\eta - 1)^{k+1}}, \quad \psi_k(y) = \oint_0 \frac{d\zeta}{2\pi i} \frac{(\zeta - 1)^k e^{\zeta t}}{\zeta^{k-y+2}}. \]
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We change \( k = vt - \kappa t^{1/3} \) for \( \kappa > 0 \) and through a steepest decent analysis we find

\[ \lim_{t \to \infty} f_1(t) \phi_{vt - \kappa t^{1/3}} \left( \xi_1 t^{1/3} \right) = \text{Ai}(\kappa + \xi_1), \]

\[ \lim_{t \to \infty} f_2(t) \psi_{vt - \kappa t^{1/3}} \left( \xi_2 t^{1/3} \right) = \text{Ai}(\kappa + \xi_2), \]

which converge uniformly for some unimportant functions \( f_1, f_2 \).
Limiting behaviour of TASEP

With the change \( k = \nu t - \kappa t^{1/3} \) and \( n = \nu t \) as \( t \to \infty \)

\[
K_n(x, y) = \sum_{k=0}^{n-1} \phi_k(x) \psi_k(y) \sim \int_0^\infty \text{Ai}(\kappa + \xi_1) \text{Ai}(\kappa + \xi_2) d\kappa =: K_{\text{Airy}}(\xi_1, \xi_2)
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With the change $k = vt - \kappa t^{1/3}$ and $n = vt$ as $t \to \infty$

$$K_n(x, y) = \sum_{k=0}^{n-1} \phi_k(x) \psi_k(y) \sim \int_0^\infty Ai(\kappa + \xi_1) Ai(\kappa + \xi_2) d\kappa =: K_{\text{Airy}}(\xi_1, \xi_2)$$

This function satisfies

$$\det(1 - K_{\text{Airy}}(\xi_1, \xi_2))_{L^2(\mathbb{R} \geq \alpha)} = F_2(\alpha)$$

where $F_2$ is the Tracy-Widom distribution of the largest eigenvalue for the Gaussian unitary ensemble (GUE).
Theorem (Johansson, 2000)

For the step initial condition, when setting $n = vt$, we obtain the limit

$$\lim_{t \to \infty} P \left( \frac{\nu_1(t) - vt}{c_0 t^{1/3}} \geq \alpha \right) = F_2(\alpha),$$

for some constant $c_0$. 

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Theorem (Johansson, 2000)

For the step initial condition, when setting $n = vt$, we obtain the limit

$$\lim_{t \to \infty} \mathbb{P}\left( \frac{v_1(t) - vt}{c_0 t^{1/3}} \geq \alpha \right) = F_2(\alpha),$$

for some constant $c_0$.

- The TASEP lies within the KPZ universality class for the case of step initial condition.

- There are very few rigorous similar results for models with distinguishable particles. Our work provides a starting point for their asymptotic analysis.
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Schütz’s TASEP transition probability can be realised as the partition function of the stochastic six-vertex model.
We are allowed to have the following vertex Configurations with Boltzmann weights:

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</table>
We are allowed to have the following vertex configurations with Boltzmann weights:

The classical partition function can be computed by summing over connected path configurations:

\[
\mathcal{Z} = \sum_{\Omega} a_1^\# (1 - a_1)^\# a_2^\# (1 - a_2)^\#. 
\]
Six-vertex model weights

- Introduce a small parameter $\epsilon > 0$
Introduce a small parameter $\epsilon > 0$

Set $a_2 = 0$ and $a_1 = \epsilon$ which gives the weights as

\[
\begin{array}{ccc}
1 & \epsilon & 1 - \epsilon \\
\epsilon & 0 & 1 \\
1 & 1 & 1 \\
\end{array}
\]
Reduction to TASEP

$t/\epsilon$ rows
Reduction to TASEP

t/ε rows
Reduction to TASEP

t/\epsilon \text{ rows}

take \epsilon \to 0
Reduction to TASEP

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TASEP space-time diagram
Reduction to TASEP

t/\epsilon\) rows

take \(\epsilon \rightarrow 0\)

TASEP space-time diagram
The TASEP transition probability can be realised as the partition function of the stochastic six-vertex model.
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**Proposition**

\[
\lim_{\epsilon \to 0} \mathbb{P}^{6VM}[\mu \to \nu - (t/\epsilon)^n]\bigg|_{\ell=t/\epsilon, a_1=\epsilon, a_2=0} = \mathbb{P}_t^{TASEP}(\mu \to \nu)
\]
We investigate a multi-species version of the TASEP.
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We have many distinguishable particle species, where higher particle species have priority over lower species.
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We aim to recover transition probabilities for the $r$-TASEP from a vertex model.
We also consider a multi-coloured higher rank version of the stochastic six vertex model with $U_q \left( \hat{\mathfrak{sl}}_{n+1} \right)$ symmetry.
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![Diagram of a multi-coloured stochastic vertex model with arrows and lines indicating transitions between states.](image)

We can also reduce the multi-coloured partition function to the rainbow TASEP.

$$P_r \text{-TASEP}(\mu \rightarrow \nu)$$
We also consider a multi-coloured higher rank version of the stochastic six vertex model with $U_q \left( \hat{sl}_{n+1} \right)$ symmetry.

It will have a partition function with appropriate weights represented by

$$\frac{1}{2} \cdots \frac{1}{n}$$

We can also reduce the multi-coloured partition function to the rainbow TASEP.

This rainbow TASEP can be partially symmetrized into the form of a general $r$-species TASEP.

$$\mathbb{P}^r_{t} - \text{TASEP} (\mu \rightarrow \nu)$$
The Two-species Model

- The simplest multi-species model is the 2-TASEP
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totaly crossed final state $\nu$
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- We wish to study total crossing events of the 2-TASEP

We consider $n$ total particles with $m$ of them being type 2.
The 2-TASEP transition probability simplifies under total crossing events.
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**Proposition**

\[
\mathbb{P}_{t}^{2-TASEP}(\mu \rightarrow \nu) = \oint \prod_{i=1}^{m} \frac{dz_i}{2\pi i} \prod_{j=1}^{n-m} \frac{dw_j}{2\pi i} \\
\times \prod_{i=1}^{m} \left(\frac{e^{(z_i^{-1}-1)t}}{(1-z_i)^{n-m}}\right) \prod_{i=1}^{n-m} \left(\frac{e^{(w_i^{-1}-1)t}}{(1-w_i)^{n-m}}\right) \prod_{i=1}^{m} \prod_{j=1}^{n-m} (w_j - z_i) \\
\times \det \left(\binom{\nu_{n-m+j}-\mu_{i-1}}{(1-z_i)^{i-j}}\right)_{1 \leq i, j \leq m} \det \left(\binom{\nu_{j+m-i-1}}{(1-w_i)^{i-j}}\right)_{1 \leq i, j \leq n-m}
\]
The 2-TASEP transition probability simplifies under total crossing events.

**Proposition**

\[
\mathbb{P}^{2\text{-TASEP}}_t (\mu \rightarrow \nu) = \oint \prod_{i=1}^{m} \frac{dz_i}{2\pi i} \prod_{j=1}^{n-m} \frac{dw_j}{2\pi i} \\
\times \prod_{i=1}^{m} \frac{e^{(z_i^{-1}-1)t}}{(1-z_i)^{n-m}} \prod_{i=1}^{n-m} e^{(w_i^{-1}-1)t} \prod_{i=1}^{m} \prod_{j=1}^{n-m} (w_j - z_i) \\
\times \text{det} \left( z_i^{\nu_j - \mu_i + 1} (1 - z_i)^{i-j} \right)_{1 \leq i,j \leq m} \text{det} \left( w_i^{\nu_j - \mu_m + i - 1} (1 - w_i)^{i-j} \right)_{1 \leq i,j \leq n-m}
\]

This result generalises to \( r \)-species using the vertex model approach.
2-TASEP Crossing Probability

Bernoulli distributed with density $\rho$

fixed position
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$s_1$

$s_2$
The Bernoulli crossing probability is given by

\[
P_{B-cross}^{t}(s_1, s_2) = \frac{\rho^m}{m!} \oint_{0,1,1-\rho} \prod_{i=1}^{m} \frac{dz_i}{2\pi i} \prod_{i \neq j} (z_j - z_i) \prod_{i=1}^{m} \frac{e^{(z_i-1)t} z_i^{-s_2-m+1}}{(z_i-1)^n(z_i-1+\rho)}
\]

\[
\times \oint_{0,1} \prod_{i=1}^{n-m} \frac{dw_i}{2\pi i} \prod_{i=1}^{m} \prod_{j=1}^{n-m} (z_i - w_j) \prod_{i=1}^{n-m} \frac{-e^{(w_i-1)t} w_i^{-s_1-m}}{(1-w_i)^{n-m-i+1}}
\]

\[
\times \det \left( w_i^{j-1} - w_i^{n-m+s_1-s_2-1} \right)_{1 \leq i, j \leq n-m}.
\]
What if we take $s_1 < -m$?

Since the type 1 particles move backwards when overtaken, all possible total crossing configurations contribute towards $\mathbb{P}_t^{B-cross}(s_1, s_2)$.
Proposition

When $s_1 \leq -m$ the $(n - m)$-fold integral over type 1 particles collapses into 1 integral

\[ P^t_{B-cross}(s_1, s_2) = \rho^m \oint_0^{2\pi i} \frac{dw}{w - 1} e^{(w-1)t} w^{n-2m-s_2-1} \]
\[ \times \det \left( \oint_{0,1,1-\rho} \frac{dz}{2\pi i} \frac{e^{(z-1)t} z^{i+j-s_2-m-1}}{(z-1)^{m+1}(z-1+\rho)} (w - z) \right)_{1 \leq i,j \leq m}. \]
The initial TASEP results in this talk exist in an extended form with left-hopping included (general ASEP).

Recent work (Nejjar, 2020) investigates the asymptotics with one second-class particle, which we expect to recover. These multi-species transition probabilities are useful for constructing higher-rank stochastic dualities and their expectations (work in progress).
Final Thoughts

- The initial TASEP results in this talk exist in an extended form with left-hopping included (general ASEP).

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