Transition Probabilities in the Multi-species Asymmetric Exclusion Process

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Based on work with Jan de Gier and Michael Wheeler (arXiv:2109.14232)

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Outline

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1. Single-colour models
   - ASEP
   - Transition probabilities from vertex models
2. Multi-colour models
   - Transition probabilities from higher rank vertex models

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   - ASEP
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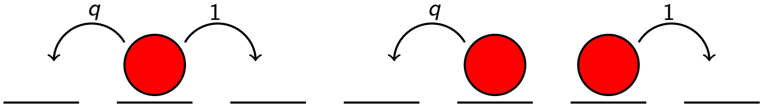
   - Transition probabilities from higher rank vertex models
3. Progress on two-colours

- Consider an ASEP on the infinite lattice $\mathbb{Z}$



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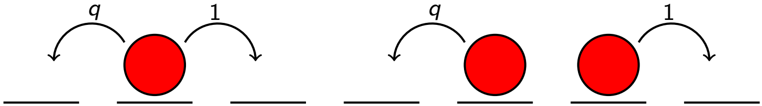

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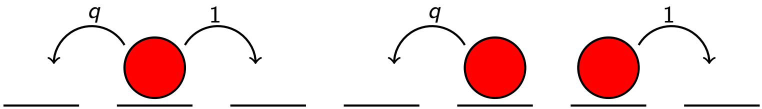

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Asymmetric simple exclusion process

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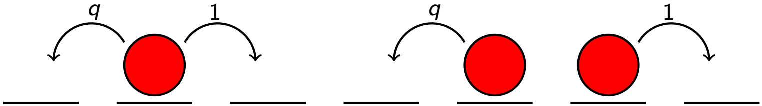
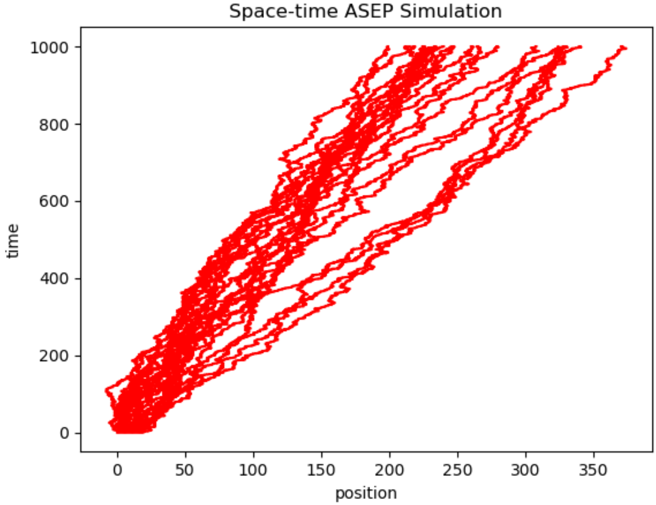


- The ASEP is
  - a continuous time Markov process
  - integrable

Asymmetric simple exclusion process



Time-evolution

- The probability satisfies the time-evolution equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{P}(\nu;t) = \sum_{\lambda\neq\nu}\mathcal{W}(\lambda\to\nu)\mathbb{P}(\lambda;t) - \sum_{\mu\neq\nu}\mathcal{W}(\nu\to\mu)\mathbb{P}(\nu;t)$$

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Theorem (Schütz (1997), Tracy and Widom (2008))

Given initial μ and final conditions ν the transition probability on $\mathbb{Z}$ is given by

$$\mathbb{P}_t^{TASEP}(\mu\to\nu)=\oint_0\prod_{i=1}^{n}\frac{\mathrm{d}z_i}{2\pi\mathrm{i}}\sum_{\pi\in S_n}(-1)^{|\pi|}\prod_{i=1}^{n}\left(\frac{1-z_i}{1-z_{\pi_i}}\right)^i e^{(z_i^{-1}-1)t}z_{\pi_i}^{\nu_i}z_i^{-\mu_i-1}$$

which satisfies the time-evolution with initial condition

$$\mathbb{P}_0^{TASEP}(\mu\to\nu)=\prod_{i=1}^{n}\delta_{\nu_i,\mu_i}.$$

TASEP crossing probability

- We choose the step initial condition $\mu_i = i$

TASEP crossing probability

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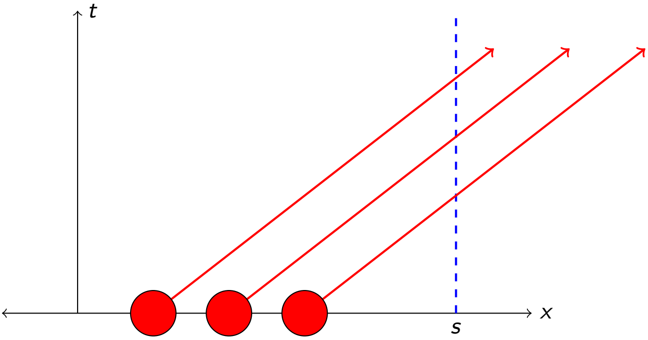
- Define probability of n particles crossing a wall at position $s \in \mathbb{N}$ as

$$P_{\text{cross}}(s) = \mathbb{P}(s \leq \nu_1 < \nu_2 < \cdots < \nu_n).$$



Limiting behaviour of TASEP

- We may find this probability as a Fredholm determinant

$$P_{\text{cross}}(s) = \det(1 - K_n(x,y))_{\ell^2(\mathbb{N})},$$

where

$$K_n(x,y) = \sum_{k=0}^{n-1} \phi_k(x)\psi_k(y).$$

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$$\phi_k(x) = \oint_1 \frac{\mathrm{d}\eta}{2\pi\mathrm{i}} \frac{\eta^{k-x} e^{-\eta t}}{(\eta - 1)^{k+1}}, \qquad \psi_k(y) = \oint_0 \frac{\mathrm{d}\zeta}{2\pi\mathrm{i}} \frac{(\zeta - 1)^k e^{\zeta t}}{\zeta^{k-y+2}}.$$

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- We change $k = vt - \kappa t^{1/3}$ for $\kappa > 0$ and through a steepest decent analysis we find

$$\lim_{t\to\infty} f_1(t)\phi_{vt-\kappa t^{1/3}}\left(\xi_1 t^{1/3}\right) = \mathrm{Ai}(\kappa + \xi_1),$$

$$\lim_{t\to\infty} f_2(t)\psi_{vt-\kappa t^{1/3}}\left(\xi_2 t^{1/3}\right) = \mathrm{Ai}(\kappa + \xi_2),$$

which converge uniformly for some unimportant functions f_1, f_2.

Limiting behaviour of TASEP

- With the change $k = vt - \kappa t^{1/3}$ and $n = vt$ as $t \to \infty$

$$K_n(x,y) = \sum_{k=0}^{n-1} \phi_k(x)\psi_k(y) \sim \int_0^{\infty} \mathrm{Ai}(\kappa + \xi_1)\,\mathrm{Ai}(\kappa + \xi_2)\mathrm{d}\kappa =: K_{\mathrm{Airy}}(\xi_1, \xi_2)$$

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- This function satisfies

$$\det(1 - K_{\mathrm{Airy}}(\xi_1, \xi_2))_{L^2(\mathbb{R}_{\geq \alpha})} = F_2(\alpha)$$

where F_2 is the Tracy-Widom distribution of the largest eigenvalue for the Gaussian unitary ensemble (GUE).

Limiting behaviour of TASEP

Theorem (Johansson, 2000)

For the step initial condition, when setting $n = vt$, we obtain the limit

$$\lim_{t\to\infty} \mathbb{P}\left(\frac{\nu_1(t) - vt}{c_0 t^{1/3}} \geq \alpha\right) = F_2(\alpha),$$

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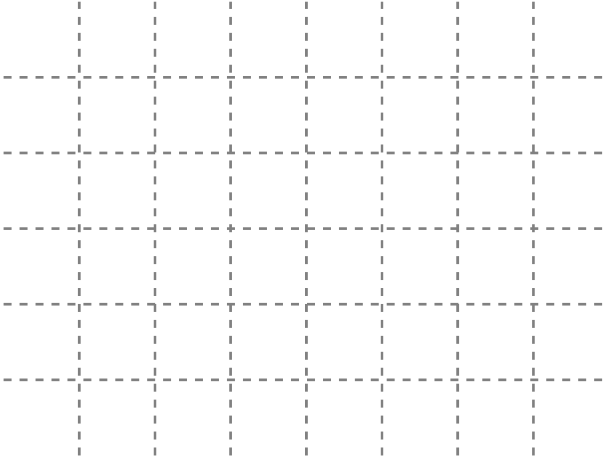
- There are very few rigorous similar results for models with distinguishable particles. Our work provides a starting point for their asymptotic analysis.

Six-vertex model

- We start with a square lattice and draw paths between vertices



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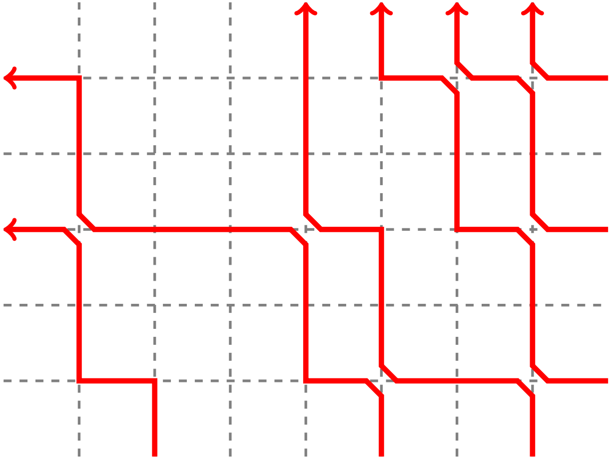

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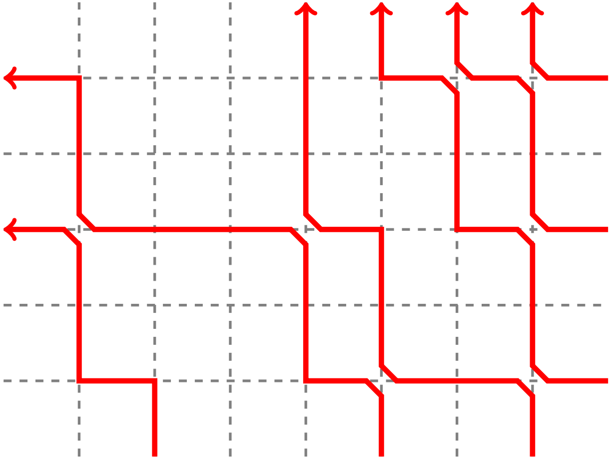

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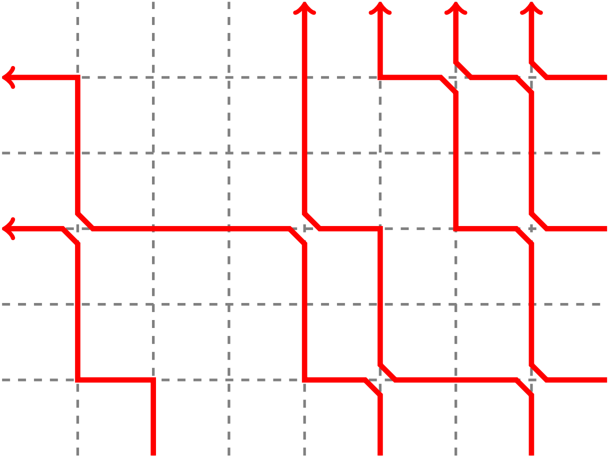
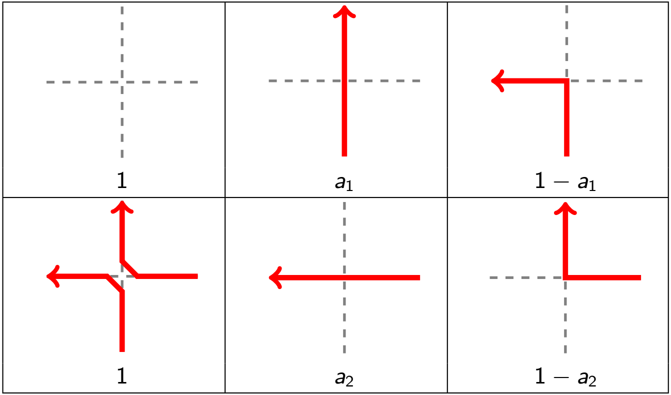


- We wish to compute the partition function of this model.
- Schütz's TASEP transition probability can be realised as the partition function of the stochastic six-vertex model.

Six-vertex model weights

- We are allowed to have the following vertex Configurations with Boltzmann weights



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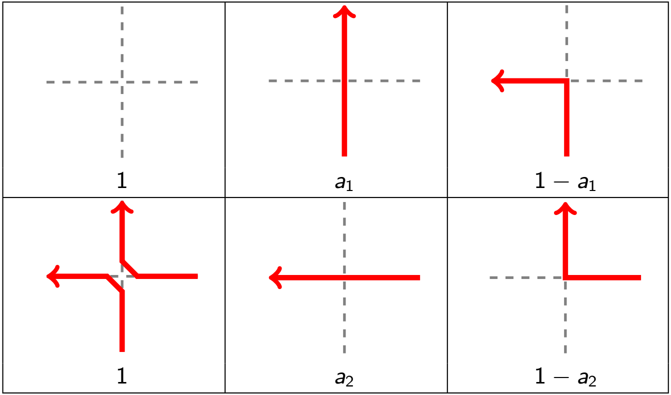

- The classical partition function can be computed by summing over connected path configurations

$$\mathcal{Z} = \sum_{\Omega} a_1^{\#}(1 - a_1)^{\#} a_2^{\#}(1 - a_2)^{\#}.$$

Six-vertex model weights

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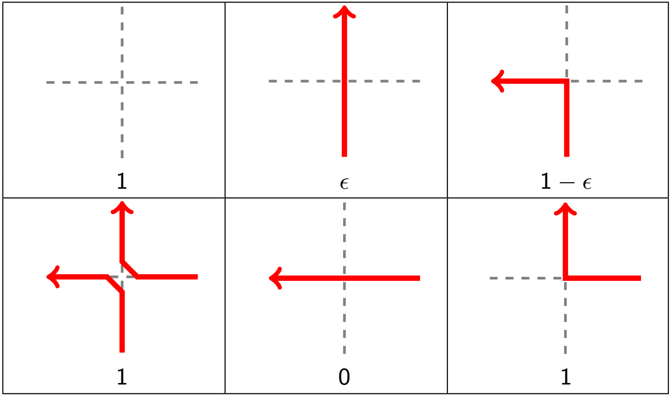
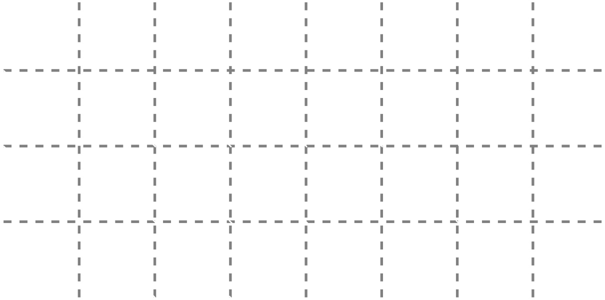
- Set $a_2 = 0$ and $a_1 = \epsilon$ which gives the weights as



Reduction to TASEP

t/ϵ rows



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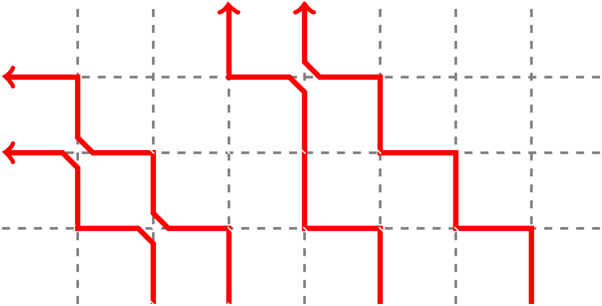

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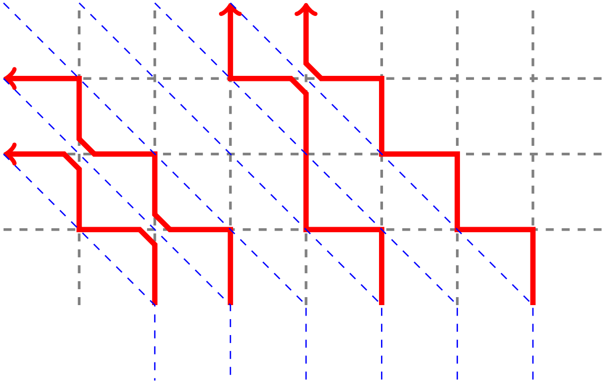

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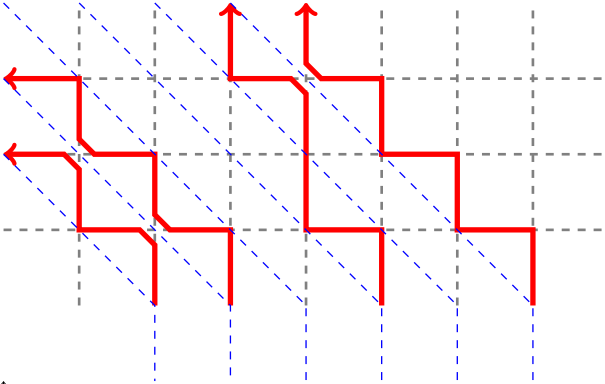

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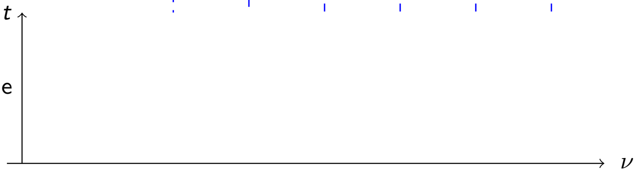
TASEP space-time diagram



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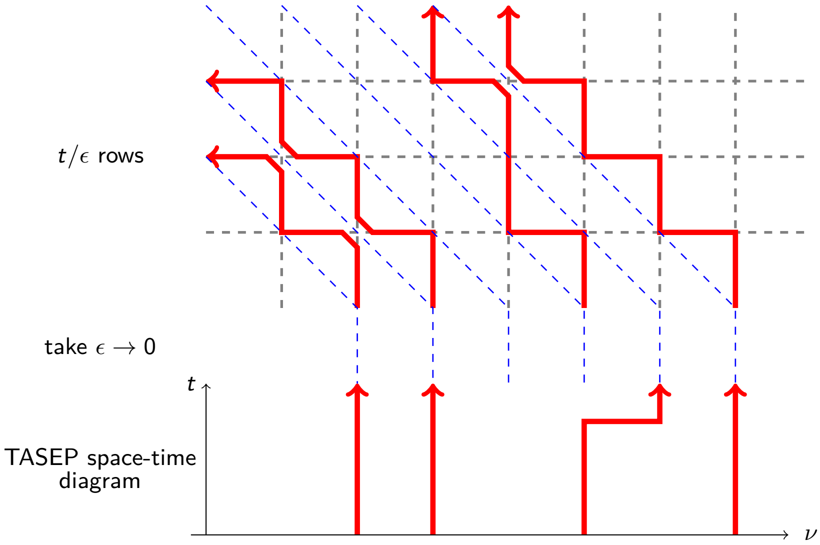

Reduction to TASEP

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Proposition

$$\lim_{\epsilon \to 0} \mathbb{P}^{6VM}[\mu \to \nu - (t/\epsilon)^n]\Big|_{\ell = t/\epsilon, a_1 = \epsilon, a_2 = 0} = \mathbb{P}_t^{TASEP}(\mu \to \nu)$$

Multi-species TASEP

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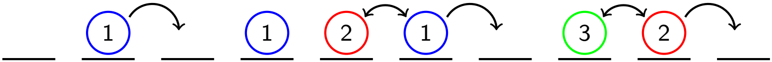

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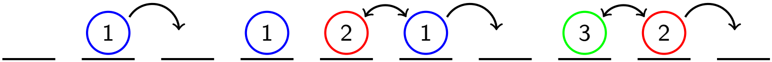

- We aim to recover transition probabilities for the r-TASEP from a vertex model.

Higher Rank Stochastic Vertex Model

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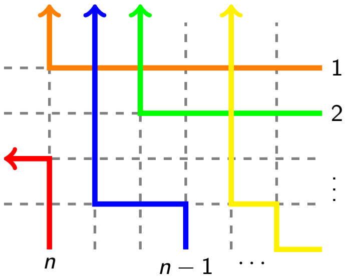

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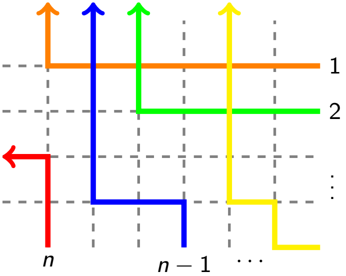

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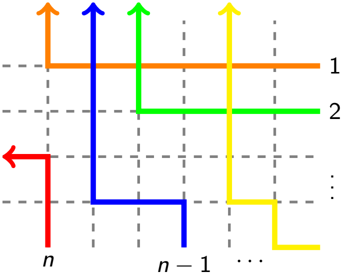

- We can also reduce the multi-coloured partition function to the rainbow TASEP.
- This rainbow TASEP can be partially symmetrized into the form of a general r-species TASEP.

$$\mathbb{P}_t^{r\text{-TASEP}}(\mu \to \nu)$$

The Two-species Model

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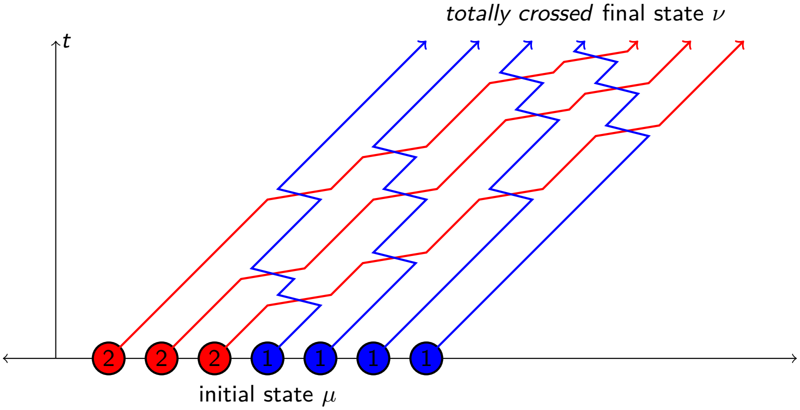

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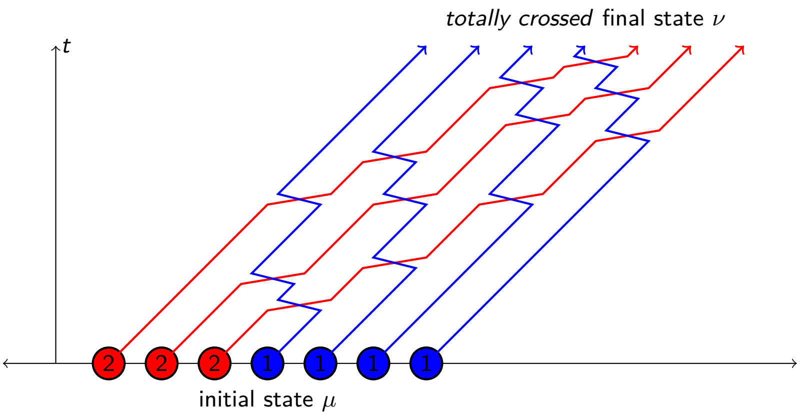

- We consider n total particles with m of them being type 2.

2-TASEP Crossing Probability

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Proposition

$$\mathbb{P}_t^{\text{2-TASEP}}(\mu \to \nu) = \oint \prod_{i=1}^{m} \frac{\mathrm{d}z_i}{2\pi \mathrm{i}} \prod_{j=1}^{n-m} \frac{\mathrm{d}w_j}{2\pi \mathrm{i}}$$

$$\times \prod_{i=1}^{m} \frac{e^{(z_i^{-1}-1)t}}{(1-z_i)^{n-m}} \prod_{i=1}^{n-m} e^{(w_i^{-1}-1)t} \prod_{i=1}^{m} \prod_{j=1}^{n-m} (w_j - z_i)$$

$$\times \det\left(z_i^{\nu_{n-m+j}-\mu_i-1} (1-z_i)^{i-j} \right)_{1\le i,j\le m} \det\left(w_i^{\nu_j-\mu_{m+i}-1} (1-w_i)^{i-j} \right)_{1\le i,j\le n-m}$$

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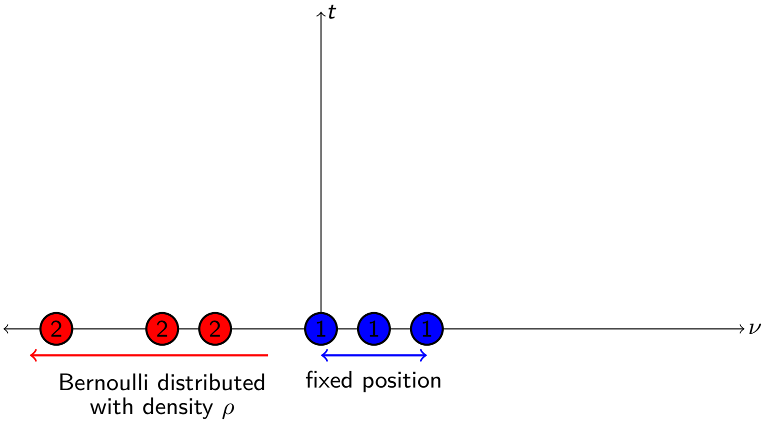
- This result generalises to r-species using the vertex model approach.

2-TASEP Crossing Probability



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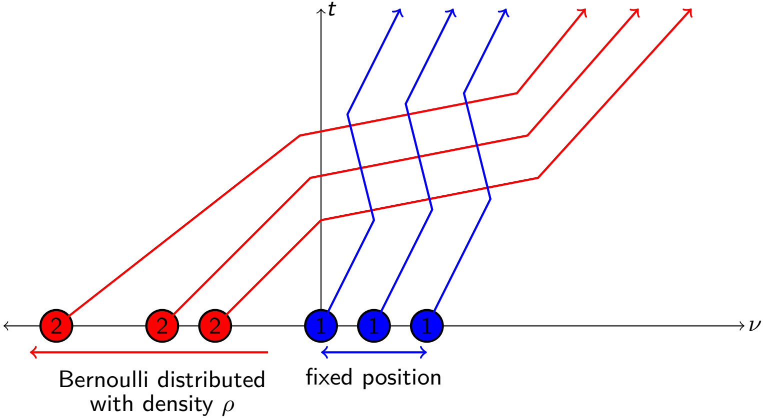

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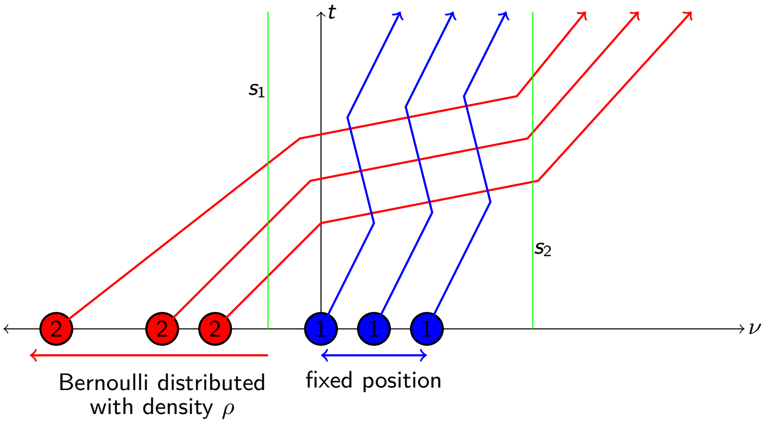


2-TASEP Crossing Probability

Proposition

The Bernoulli crossing probability is given by

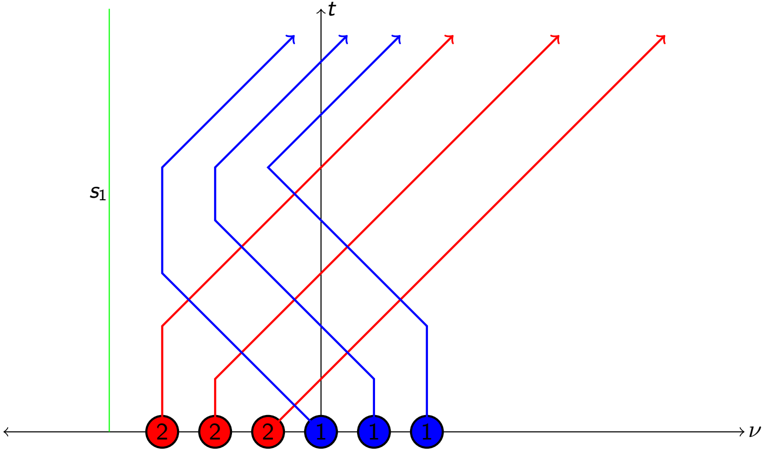
$$\mathbb{P}_t^{B\text{-}cross}(s_1, s_2) = \frac{\rho^m}{m!} \oint_{0,1,1-\rho} \prod_{i=1}^{m} \frac{\mathrm{d}z_i}{2\pi\mathrm{i}} \prod_{i \neq j}(z_j - z_i) \prod_{i=1}^{m} \frac{e^{(z_i-1)t} z_i^{-s_2-m+1}}{(z_i-1)^n (z_i - 1 + \rho)}$$

$$\times \oint_{0,1} \prod_{i=1}^{n-m} \frac{\mathrm{d}w_i}{2\pi\mathrm{i}} \prod_{i=1}^{m} \prod_{j=1}^{n-m} (z_i - w_j) \prod_{i=1}^{n-m} \frac{-e^{(w_i-1)t} w_i^{-s_1-m}}{(1-w_i)^{n-m-i+1}}$$

$$\times \det\left(w_i^{j-1} - w_i^{n-m+s_1-s_2-1}\right)_{1 \leq i,j \leq n-m}.$$

2-TASEP Crossing Probability

What if we take $s_1 < -m$?



- Since the type 1 particles move backwards when overtaken, all possible total crossing configurations contribute towards $\mathbb{P}_t^{\text{B-cross}}(s_1, s_2)$.

2-TASEP Crossing Probability

Proposition

When $s_1 \leq -m$ the $(n-m)$-fold integral over type 1 particles collapses into 1 integral

$$\mathbb{P}_t^{B\text{-}cross}(s_1, s_2) = \rho^m \oint_0 \frac{\mathrm{d}w}{2\pi\mathrm{i}} \frac{e^{(w-1)t} w^{n-2m-s_2-1}}{w-1} \times \det\left(\oint_{0,1,1-\rho} \frac{\mathrm{d}z}{2\pi\mathrm{i}} \frac{e^{(z-1)t} z^{i+j-s_2-m-1}}{(z-1)^{m+1}(z-1+\rho)} (w-z) \right)_{1 \leq i,j \leq m}.$$

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- The focus of future work will investigate the KPZ style asymptotic analysis of multi-coloured models.
    - This has only been investigated in very limited circumstances.
    - Recent work (Nejjar, 2020) investigates the asymptotics with one second-class particle, which we expect to recover.

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- The initial TASEP results in this talk exist in an extended form with left-hopping included (general ASEP).
- The focus of future work will investigate the KPZ style asymptotic analysis of multi-coloured models.
    - This has only been investigated in very limited circumstances.
    - Recent work (Nejjar, 2020) investigates the asymptotics with one second-class particle, which we expect to recover.
- These multi-species transition probabilities are useful for constructing higher-rank stochastic dualities and their expectations (work in progress).