Limit shapes in quantum integrable spin chains

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Arctic circle theorem  
[Jockusch, Propp and Shor 1998]
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Main results

Relations to combinatorics, probability, mathematical physics

- Statistical mechanics and crystal shapes.
- Statistics of Young diagrams and representation theory.
- Stochastic processes.
- Quantum many body, out of equilibrium and integrability.
- ...
Another example: six vertex model

\[ \Delta = \frac{a^2 + b^2 - c^2}{2ab} \]

In this whole talk, \( a = 1 \), \( \Delta \) fixed to some value.
Setup studied in this talk
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Main results

$b = 1$
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Main results

\[ b = \frac{1}{2} \]

\[ b = 1 \]
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Main results

\[ b = 1 \]

\[ b = \frac{1}{2} \]

\[ b \rightarrow 0 \]
Hamiltonian/Trotter limit of the six vertex model with domain wall boundary conditions.
Finite dimensional linear algebra. \( N \) is integer \( \geq 2 \). Hilbert space with an orthonormal basis labelled by binary words of length \( N \).

\( \bullet = 1 \) is a particle, \( \circ = 0 \) is a hole. Label sites \( j \in \{1, \ldots, N\} \).

There are \( 2^N \) basis states, one of which is shown below \((N = 8)\):

\[
|\bullet \circ \bullet \circ \bullet \circ \bullet \circ \circ \circ \rangle
\]

The allowed states \( |v\rangle \) (column vectors) are linear combinations of basis states with complex coefficients.

\[
\langle v | := (|v\rangle)^H = (|v\rangle)^\dagger
\]

is corresponding line vector.

\[
\langle u | v \rangle := \langle u | v \rangle \text{ scalar product between the vectors } |u\rangle \text{ and } |v\rangle.
\]
Left and right “hopping” operators

- $R_j$: if there is a particle at site $j$ and a hole at site $j + 1$, moves the particle from $j$ to $j + 1$. Otherwise returns 0.
- $L_j$: if there is a particle at site $j + 1$ and a hole at site $j$, moves the particle from $j + 1$ to $j$. Otherwise returns 0.

\[ R_3 |\circ \circ \bullet \bullet \circ \circ \circ \circ \circ \rangle = 0 \]
\[ R_4 |\circ \circ \bullet \bullet \circ \circ \circ \circ \circ \rangle = |\circ \circ \bullet \bullet \circ \circ \circ \circ \circ \rangle \]
\[ L_2 |\circ \circ \bullet \bullet \circ \circ \circ \circ \circ \rangle = |\circ \circ \bullet \bullet \circ \circ \circ \circ \circ \rangle \]
\[ L_3 |\circ \circ \bullet \bullet \circ \circ \circ \circ \circ \rangle = 0 \]
Counting the interfaces between particles and holes

\[ D \left| \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet \right\rangle = 4 \left| \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet \right\rangle \]

\(D\) is a diagonal \(2^L \times 2^L\) matrix.

Can also make local versions \(D_j\) counting whether there is one interface between \(j\) and \(j+1\). Then

\[ D = \sum_{j=1}^{N-1} D_j \]
The XXZ Hamiltonian

For some anisotropy parameter $\Delta \in \mathbb{R}$, define

$$H = -\Delta D + \sum_{j=1}^{N-1} (L_j + R_j)$$

Remarks

- Quantum integrable, related to the six vertex model.
- The term proportional to $D$ is sometimes called \textit{interactions}.
- In case $\Delta = 1$, $H$ coincides with the generator for SSEP.
Reference state $|\psi\rangle$, with all sites filled for $j \leq 0$, empty for $j > 0$.

\[ |\psi\rangle = |\cdots \bullet \bullet \bullet \bullet o o o o o \cdots \rangle \]

$|\psi_{x_l,\ldots,x_1}\rangle$ obtained from $|\psi\rangle$ by moving particles at positions $-l + 1, \ldots, 0$ to positions $-l + 1 \leq x_l < \ldots < x_1$. For example

\[ |\psi_{02}\rangle = |\cdots \bullet \bullet o \bullet o o o o o \cdots \rangle \]
Can make sense of

\[ H = -\Delta D + \sum_{j \in \mathbb{Z}} (L_j + R_j) \]

over the space of states spanned by the \( |\psi_{x_l, \ldots, x_1}\rangle \) for \( l \geq 0 \). For example

\[
H |\psi\rangle = H \left| \cdots \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet \cdots \right\rangle \\
= -\Delta |\psi\rangle + |\psi_1\rangle
\]

\[
H^2 |\psi\rangle = (1 + \Delta^2) |\psi\rangle - 4\Delta |\psi_1\rangle + |\psi_2\rangle + |\psi_{01}\rangle
\]

and more generally, objects such as \( e^{\tau H} |\psi\rangle \) for any \( \tau \in \mathbb{C} \).
Main results
Probabilities of particle occupancies in the Hamiltonian limit of the six vertex model. For some \( \tau > 0 \) and \( \omega \in [0, \tau] \)

\[
P_{\omega, \tau}(x_l, \ldots, x_1) = \frac{A_{x_l, \ldots, x_1}(\omega)A_{x_l, \ldots, x_1}(\tau - \omega)}{A(\tau)}
\]

- \( A(\tau) = \langle \psi | e^{\tau H} | \psi \rangle \) as a Fredholm determinant. [JMS 2017]
- Formulas for all \( A_{x_l, \ldots, x_1}(\tau) = \langle \psi_{x_l, \ldots, x_1} | e^{\tau H} | \psi \rangle \). [JMS 2021]
For $\tau \geq 0$ all amplitudes are positive, so this defines a legitimate probabilistic model. Limit shapes in the scaling limit.

Another very interesting problem (for $t \geq 0$):

$$P_{x_l,...,x_1}(t) = |A_{x_l,...,x_1}(it)|^2$$

Real-time evolution of the quantum system, with initial state $|\psi\rangle$. Conjectures from generalized hydrodynamics in the scaling limit. [Castro-Alvaredo, Doyon, Yoshimura 2016] [Bertini, Collura, De Nardis, Fagotti 2016].

Simple analytical solution for the density profile studied here ($|\Delta| < 1$) [Collura, De Luca, Viti 2017]
Transfer matrix for the six vertex model [Allegra, Dubail, JMS, Viti 2016]

\[ T_e = \prod_{j \text{ even}} [a + b(L_j + R_j) + (c - a)D_j] \]

\[ T_o = \prod_{j \text{ odd}} [a + b(L_j + R_j) + (c - a)D_j] \]

Remember \( a = 1, \ \Delta = \cos \gamma \) fixed.

\[ T(b) = T_e T_o \]

Can show using the Lie-Trotter formula

\[ \lim_{n \to \infty} \left[ T \left( \frac{\tau \sin \gamma}{n} \right) \right]^n = e^{\tau H} \]
\[ A \left( \frac{\tau}{\sin \gamma} \right) = e^{-\frac{\tau^2}{6}} \exp \left( \sum_{n \geq 1} \frac{1}{n} \int_{\mathbb{R}^n} V(x_1, x_2) \ldots V(x_n, x_1) \, dx_1 \ldots dx_n \right) \]

\[ V(x, y) = \frac{\sqrt{\tau y} J_0(2\sqrt{\tau x}) J'_0(2\sqrt{\tau y}) - \sqrt{\tau x} J_0(2\sqrt{\tau y}) J'_0(2\sqrt{\tau x})}{2(x - y)} \Theta(y) - w_0(y) \]

with

\[ w_0(y) = \frac{1 - e^{-\gamma y}}{1 - e^{-\pi y}} \]

follows from a result of [Slavnov 2003].
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Main results

\[ h(\tau|z) = \frac{1}{A(\tau)} \sum_{x \geq 0} A_x(\tau) z^x \] satisfies the exact PDE [JMS 2021]

\[
\left( \tau \partial^2_{\tau} + \left[ 1 - 2\tau \left( \frac{1}{z} - \Delta \right) \right] \partial_{\tau} + Q(\tau) - z + \Delta \right) h = (1 - 2\Delta z + z^2) \partial_z h
\]

where \( Q(\tau) = 2\tau \frac{d^2 \log A(\tau)}{d\tau^2} + \frac{d \log A(\tau)}{d\tau} \).

All amplitudes are given by

\[
\frac{A_{x_l, \ldots, x_1}(\tau)}{A(\tau)} = \oint_{C_l} \prod_{j=1}^{l} \frac{dz_j}{2i\pi z_j^{x_j+l}} \frac{\det_{1 \leq j, k \leq l} \left( z_k^{l-j} \left[ 1 - z_k \partial_{\tau} \right]^{j-1} h(\tau|z_k) \right)}{\prod_{1 \leq j < k \leq l} (z_j z_k - 2\Delta z_k + 1)}
\]
Free/determinantal case $\Delta = 0$

Partition function $Z(\tau) = A(\tau) = e^{\tau^2/2}$.

Solution to the PDE: $h(\tau | z) = e^{\tau z}$.

Yields after some manipulations

$$\frac{A_{x_l,\ldots,x_1}(\tau)}{A(\tau)} = \det_{1 \leq j,k \leq l} \left( \int_C \frac{dz}{2i\pi} e^{\tau z} z^{x_j+k} \right)$$

Relation to PNG droplet model [Praehoffer Spohn 2001], Poissonized Plancherel measures [Baik, Deift, Johansson 1998], Gross-Witten-Wadia matrix model, ....
Partition function and one particle asymptotics

Use [Korepin, Zinn-Justin 2000] [Bleher, Fokin 2006] to show as $\tau \to +\infty$

\[ A(\tau) = \exp \left( \frac{1}{6} \left[ \frac{\pi^2}{(\pi - \gamma)^2} - 1 \right] (\tau \sin \gamma)^2 \right) \tau^\kappa O(1) \]

\[ h(\tau|z) = e^{\tau F(z)} O(1) \]

allows to compute the energy function

\[ \frac{1}{\tau} \log A_{X\tau}(\tau) \to G(X) \]

Can reconstruct the full arctic curve using the tangent method

[Colomo, Sportiello 2016]
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Main results
Real time conjectures

\[ R(t) = |A(it)|^2 \] gives the exact return probability. \[\text{[JMS 2017]}\]

- \( \Delta = \cos \gamma, \ \gamma = \frac{\pi p}{q} \)

\[
- \log R(t) = \left( \frac{q^2}{(q - 1)^2} - 1 \right) \frac{(t \sin \gamma)^2}{12} + o(t)
\]

- \( \Delta = \cos \gamma, \ \frac{\gamma}{\pi} \notin \mathbb{Q} \)

\[
- \log R(t) = (\sin \gamma) t + o(t)
\]

- \( |\Delta| = 1 \)

\[
- \log R(t) = \zeta(3/2) \sqrt{t/\pi} - \frac{1}{2} \log t + o(\log t)
\]

Log-enhanced diffusion. \[\text{[Gamayun, Miao, Ilievski 2019]}\]
Some steps in the derivation.
Partition function: Izergin-Korepin $n \times n$ determinant.

Hamiltonian limit: Fredholm determinant.
Requires the knowledge of polynomials $p(n, \epsilon|x)$ which are orthonormal wrt the weights

$$w_{\epsilon}(x) = e^{-\epsilon x} w_0(x) \quad , \quad w_0(x) = \frac{1 - e^{-\gamma x}}{1 - e^{-\pi x}}$$
Orthogonal polynomials

The limit
\[ q(\alpha|x) = \lim_{n \to \infty} \sqrt{\frac{n}{\alpha}} p(n, \alpha/n|x) \]
satisfies the ODE
\[ \left[ \alpha \partial^2_{\alpha} + \partial_{\alpha} + f(\alpha) + x \right] q(\alpha|x) = 0 \]
where
\[ f(\alpha) = 2\alpha \frac{d^2 \log \mathcal{Y}(\alpha)}{d\alpha^2} + \frac{d \log \mathcal{Y}(\alpha)}{d\alpha} \]
is given in terms of the Fredholm determinant
\[ \mathcal{Y}(\alpha) = \det(I - V)_{L^2(\mathbb{R})} \]
\[ V(x, y) = \frac{\sqrt{y} J_0(2\sqrt{x}) J'_0(2\sqrt{y}) - \sqrt{x} J_0(2\sqrt{y}) J'_0(2\sqrt{x})}{(x - y)} \left[ \Theta(y) - w_0 \left( \frac{y}{\alpha} \right) \right] \]
More can be extracted from those formulas in imaginary time.

Real time asymptotics and connection to hydrodynamics?

Exact formula for the emptiness formation probability, gives access to the distribution of the rightmost particle. KPZ or not KPZ.

Alternative methods to compute the amplitudes from coordinate Bethe Ansatz [Saenz, Tracy Widom 2022] or the $F$ basis [Feher, Pozsgay 2019].

Hamiltonian limits of models with 2–periodic weights?
Adding other charges to $H$, e.g. $\tilde{H} = H + \alpha Q_2$ [Bocini, JMS 2020].

Higher order edge kernels [Betea, Bouttier, Walsh 2020]
Thank you!