Limit shapes in quantum integrable spin chains

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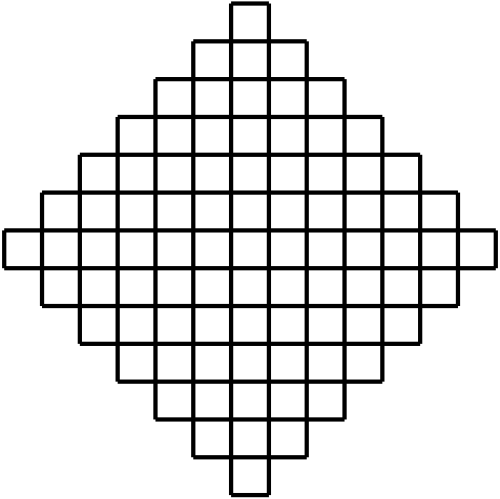
Based on [JMS, arXiv:2112.12092] [Bocini & JMS, arXiv:2007.06621] [JMS, arXiv:1707.06625] [Allegra, Dubail, JMS & Viti, arXiv:1512.02872]

Outline

Uniform dimer coverings on the Aztec diamond



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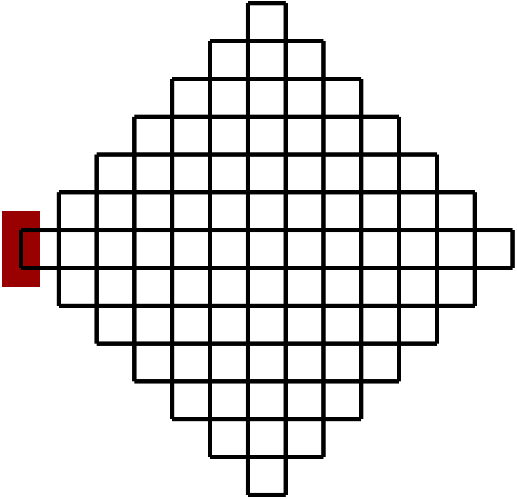

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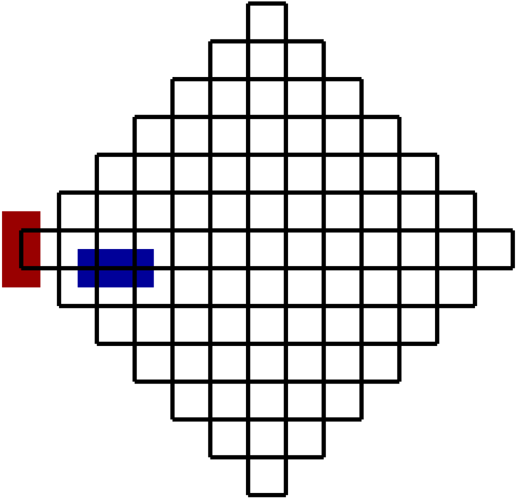

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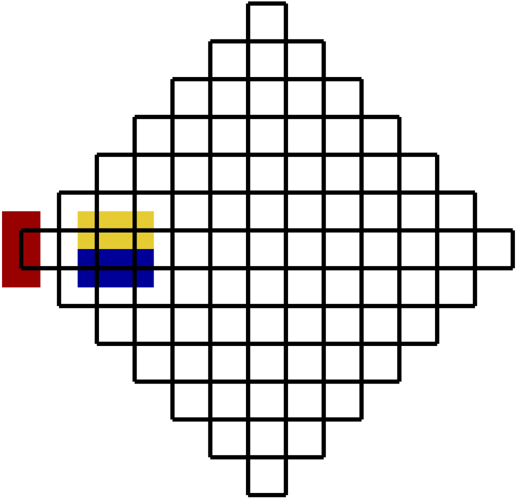

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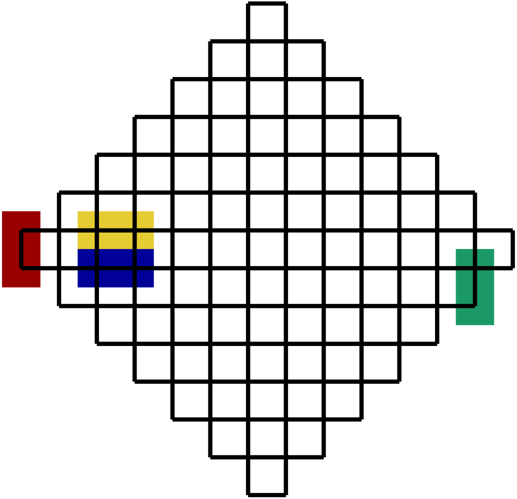

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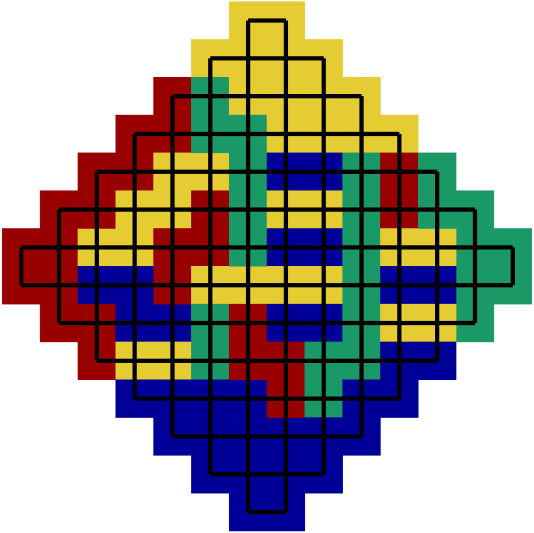
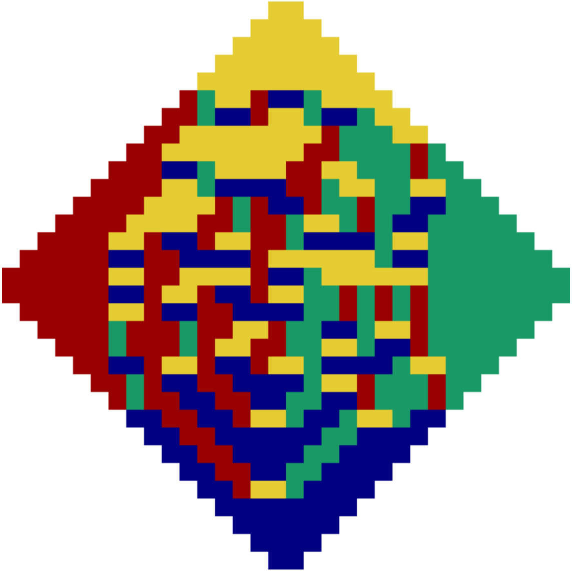
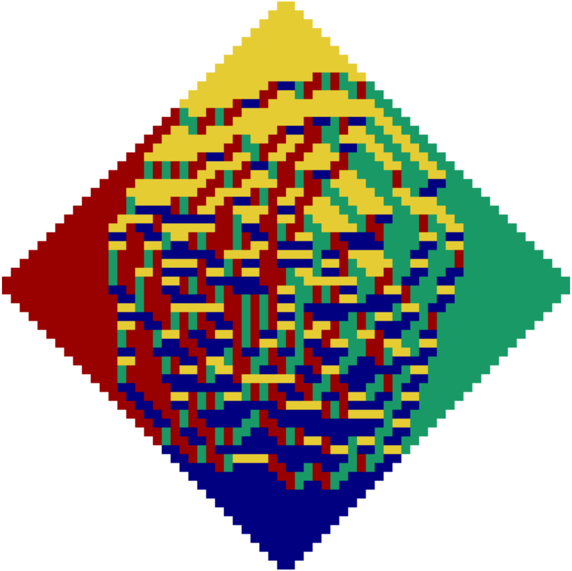
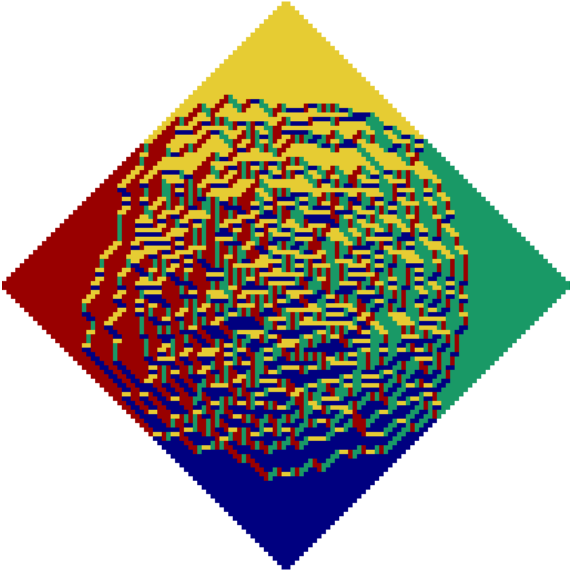
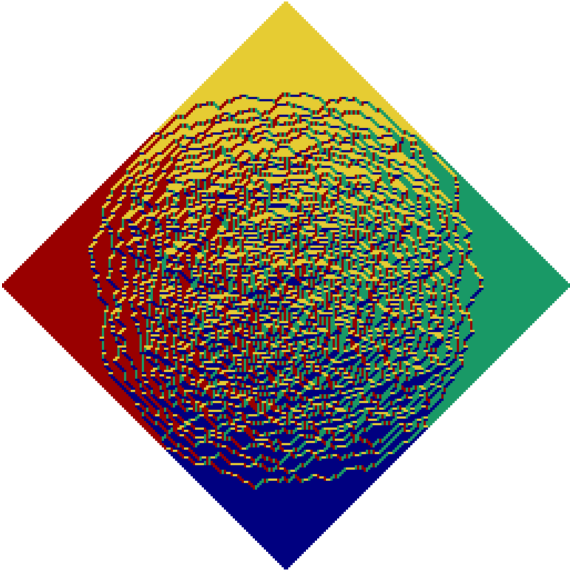
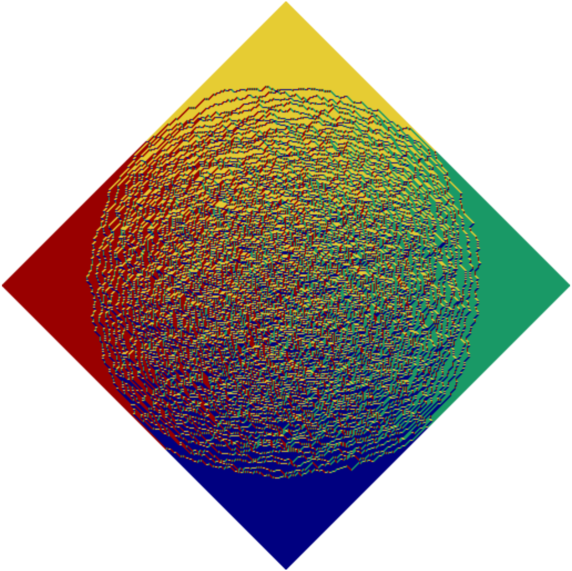


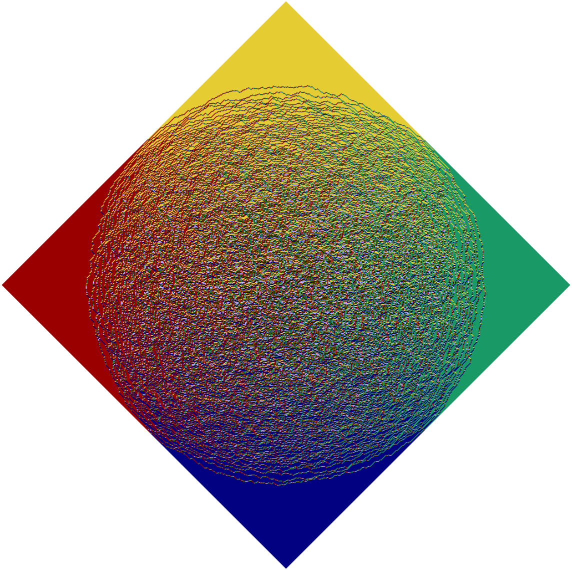


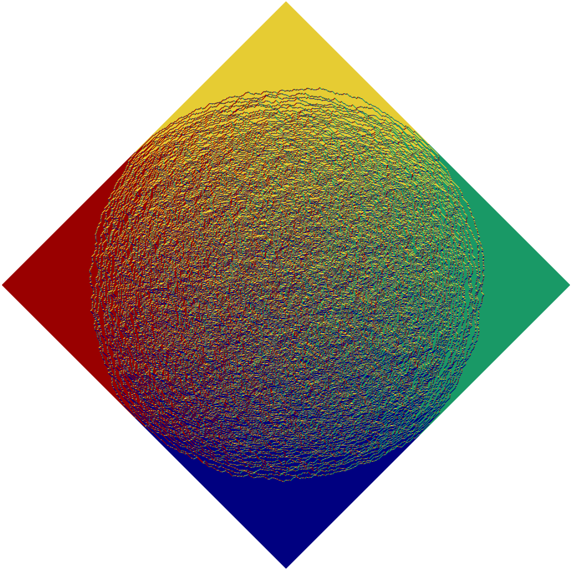










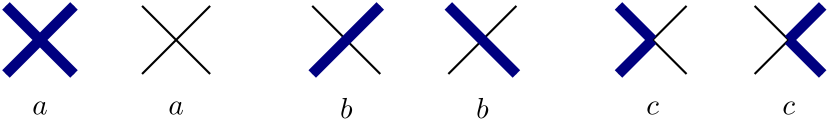


Arctic circle theorem [Jockusch, Propp and Shor 1998]

Relations to combinatorics, probability, mathematical physics

- Statistical mechanics and crystal shapes.
- Statistics of Young diagrams and representation theory.
- Stochastic processes.
- Quantum many body, out of equilibrium and integrability.
- ...

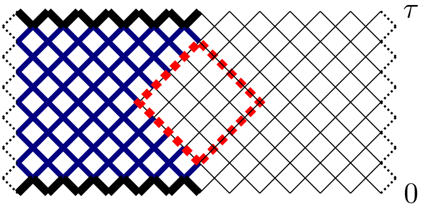
Another example: six vertex model



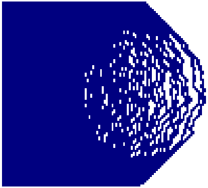
$$\Delta = \frac{a^2 + b^2 - c^2}{2ab}$$

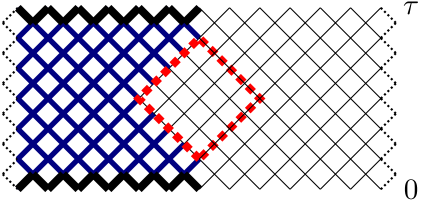
In this whole talk, $a = 1$, Δ fixed to some value.

Setup studied in this talk

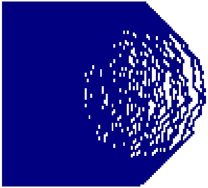


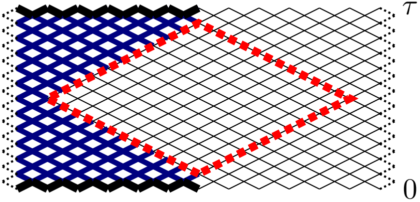
$$b = 1$$



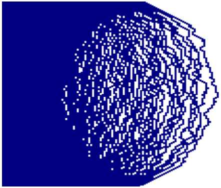


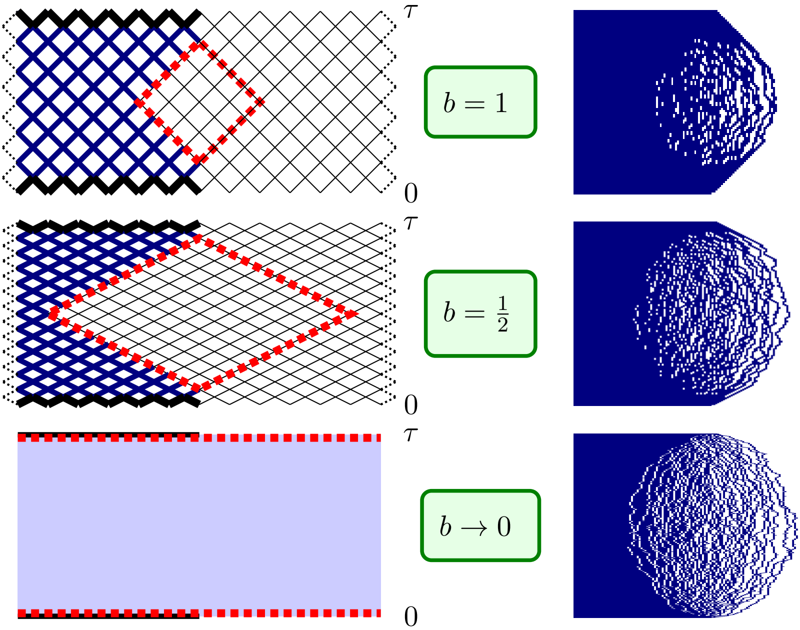
$b = 1$





$b = \frac{1}{2}$




τ
0
$b = 1$
τ
0
$b = \frac{1}{2}$
τ
0
$b \to 0$

Hamiltonian/Trotter limit of the six vertex model with domain wall boundary conditions.

Finite dimensional linear algebra. N is integer ≥ 2. Hilbert space with an orthonornal basis labelled by binary words of length N. ● $= 1$ is a particle, **O** $= 0$ is a hole. Label sites $j \in \{1, \ldots, N\}$.

There are 2^N basis states, one of which is shown below ($N = 8$):

|○○●●○●○○⟩

The allowed states $|v\rangle$ (column vectors) are linear combinations of basis states with complex coefficients.

$\langle v| := (|v\rangle)^H = (|v\rangle)^\dagger$ is corresponding line vector.

$\langle u|v\rangle := \langle u|\,|v\rangle$ scalar product between the vectors $|u\rangle$ and $|v\rangle$.

Left and right "hopping" operators

- R_j: if there is a particle at site j and a hole at site $j+1$, moves the particle from j to $j+1$. Otherwise returns 0.
- L_j: if there is a particle at site $j+1$ and a hole at site j, moves the particle from $j+1$ to j. Otherwise returns 0.

$$R_3 \left| \circ \circ \bullet \bullet \circ \bullet \circ \circ \right\rangle = 0$$

$$R_4 \left| \circ \circ \bullet \bullet \circ \bullet \circ \circ \right\rangle = \left| \circ \circ \bullet \circ \bullet \bullet \circ \circ \right\rangle$$

$$L_2 \left| \circ \circ \bullet \bullet \circ \bullet \circ \circ \right\rangle = \left| \circ \bullet \circ \bullet \circ \bullet \circ \circ \right\rangle$$

$$L_3 \left| \circ \circ \bullet \bullet \circ \bullet \circ \circ \right\rangle = 0$$

Counting the interfaces between particles and holes

$$D\,|\circ\circ\bullet\bullet\circ\bullet\circ\circ\rangle = 4\,|\circ\circ\bullet\bullet\circ\bullet\circ\circ\rangle$$

D is a diagonal $2^L \times 2^L$ matrix.

Can also make local versions D_j counting whether there is one interface beween j and $j+1$. Then

$$D = \sum_{j=1}^{N-1} D_j$$

The XXZ Hamiltonian

For some anisotropy parameter $\Delta \in \mathbb{R}$, define

$$H = -\Delta D + \sum_{j=1}^{N-1} (L_j + R_j)$$

Remarks

- Quantum integrable, related to the six vertex model.
- The term proportional to D is sometimes called *interactions*.
- In case $\Delta = 1$, H coincides with the generator for SSEP.

Infinite Hamiltonian with domain wall boundary conditions

Reference state $|\psi\rangle$, with all sites filled for $j \leq 0$, empty for $j > 0$.

$$|\psi\rangle = \Big| \cdots \bullet \bullet \bullet \bullet \circ \circ \circ \circ \cdots \Big\rangle$$

$|\psi_{x_l,\dots,x_1}\rangle$ obtained from $|\psi\rangle$ by moving particles at positions $-l+1,\dots,0$ to positions $-l+1 \leq x_l < \dots < x_1$. For example

$$|\psi_{02}\rangle = \Big| \cdots \bullet \bullet \circ \bullet \circ \bullet \circ \circ \cdots \Big\rangle$$

Can make sense of

$$H = -\Delta D + \sum_{j \in \mathbb{Z}} (L_j + R_j)$$

over the space of states spanned by the $|\psi_{x_l,\ldots,x_1}\rangle$ for $l \geq 0$. For example

$$H|\psi\rangle = H\left|\cdots \bullet\bullet\bullet\bullet\circ\circ\circ\circ\cdots\right\rangle$$

$$= -\Delta|\psi\rangle + |\psi_1\rangle$$

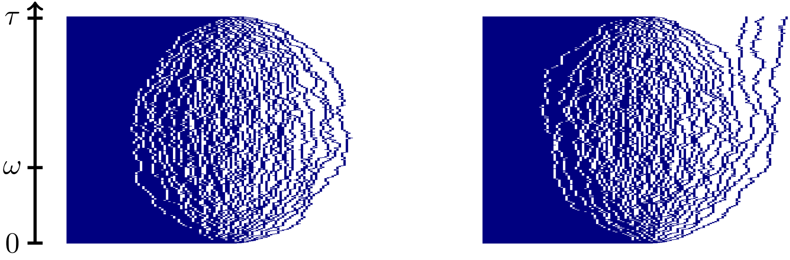
$$H^2|\psi\rangle \quad = (1+\Delta^2)|\psi\rangle - 4\Delta|\psi_1\rangle + |\psi_2\rangle + |\psi_{01}\rangle$$

and more generally, objects such as $e^{\tau H}|\psi\rangle$ for any $\tau \in \mathbb{C}$.

Main results

Probabilities of particle occupancies in the Hamiltonian limit of the six vertex model. For some $\tau > 0$ and $\omega \in [0, \tau]$

$$P_{\omega,\tau}(x_l, \ldots, x_1) = \frac{\mathcal{A}_{x_l,\ldots,x_1}(\omega)\mathcal{A}_{x_l,\ldots,x_1}(\tau - \omega)}{\mathcal{A}(\tau)}$$



- $\mathcal{A}(\tau) = \langle\psi|e^{\tau H}|\psi\rangle$ as a Fredholm determinant. [JMS 2017]
- Formulas for all $\mathcal{A}_{x_l,\ldots,x_1}(\tau) = \langle\psi_{x_l,\ldots,x_1}|e^{\tau H}|\psi\rangle$. [JMS 2021]

For $\tau \geq 0$ all amplitudes are positive, so this defines a legitimate probabilistic model. Limit shapes in the scaling limit.

Another very interesting problem (for $t \geq 0$):

$$P_{x_l,\dots,x_1}(t) = |\mathcal{A}_{x_l,\dots,x_1}(it)|^2$$

Real-time evolution of the quantum system, with initial state $|\psi\rangle$. Conjectures from generalized hydrodynamics in the scaling limit. [Castro-Alvaredo, Doyon, Yoshimura 2016] [Bertini, Collura, De Nardis, Fagotti 2016].

Simple analytical solution for the density profile studied here ($|\Delta| < 1$) [Collura, De Luca, Viti 2017]

Transfer matrix for the six vertex model [Allegra, Dubail, JMS, Viti 2016]

$$T_{\mathrm{e}} = \prod_{j\,\mathrm{even}} \left[a + b(L_j + R_j) + (c-a)D_j\right]$$

$$T_{\mathrm{o}} = \prod_{j\,\mathrm{odd}} \left[a + b(L_j + R_j) + (c-a)D_j\right]$$

Remember $a = 1$, $\Delta = \cos\gamma$ fixed.

$$T(b) = T_{\mathrm{e}} T_{\mathrm{o}}$$

Can show using the Lie-Trotter formula

$$\lim_{n\to\infty} \left[T\left(\frac{\tau \sin\gamma}{n}\right)\right]^n = e^{\tau H}$$

$$\mathcal{A}\left(\frac{\tau}{\sin\gamma}\right) = e^{-\frac{\tau^2}{6}} \exp\left(\sum_{n\geq 1} \frac{1}{n} \int_{\mathbb{R}^n} V(x_1, x_2)\dots V(x_n, x_1) dx_1 \dots dx_n\right)$$

$$V(x,y) = \frac{\sqrt{\tau y} J_0(2\sqrt{\tau x}) J_0'(2\sqrt{\tau y}) - \sqrt{\tau x} J_0(2\sqrt{\tau y}) J_0'(2\sqrt{\tau x})}{2(x-y)} [\Theta(y) - w_0(y)]$$

with

$$w_0(y) = \frac{1 - e^{-\gamma y}}{1 - e^{-\pi y}}$$

follows from a result of [Slavnov 2003].

$h(\tau|z) = \frac{1}{\mathcal{A}(\tau)} \sum_{x\geq 0} \mathcal{A}_x(\tau) z^x$ satisfies the exact PDE [JMS 2021]

$$\left(\tau\partial_\tau^2 + \left[1 - 2\tau\left(\frac{1}{z} - \Delta\right)\right]\partial_\tau + Q(\tau) - z + \Delta\right) h = \left(1 - 2\Delta z + z^2\right)\partial_z h$$

where $Q(\tau) = 2\tau \frac{d^2 \log \mathcal{A}(\tau)}{d\tau^2} + \frac{d \log \mathcal{A}(\tau)}{d\tau}$.

All amplitudes are given by

$$\frac{\mathcal{A}_{x_l,\dots,x_1}(\tau)}{\mathcal{A}(\tau)} = \oint_{\mathcal{C}^l} \prod_{j=1}^{l} \frac{dz_j}{2\mathrm{i}\pi z_j^{x_j+l}} \frac{\det\limits_{1\leq j,k\leq l}\left(z_k^{l-j}\left[1 - z_k\partial_\tau\right]^{j-1} h(\tau|z_k)\right)}{\prod\limits_{1\leq j<k\leq l}(z_j z_k - 2\Delta z_k + 1)}$$

Free/determinantal case $\Delta = 0$

Partition function $Z(\tau) = \mathcal{A}(\tau) = e^{\tau^2/2}$.

Solution to the PDE: $h(\tau|z) = e^{\tau z}$.

Yields after some manipulations

$$\frac{\mathcal{A}_{x_l,\ldots,x_1}(\tau)}{\mathcal{A}(\tau)} = \det_{1\leq j,k\leq l}\left(\oint_{\mathcal{C}} \frac{dz\, e^{\tau z}}{2\mathrm{i}\pi z^{x_j+k}}\right)$$

Relation to PNG droplet model [Praehoffer Spohn 2001], Poissonized Plancherel measures [Baik, Deift, Johansson 1998], Gross-Witten-Wadia matrix model, ...

Partition function and one particle asymptotics

Use [Korepin, Zinn-Justin 2000] [Bleher, Fokin 2006] to show as $\tau \to +\infty$

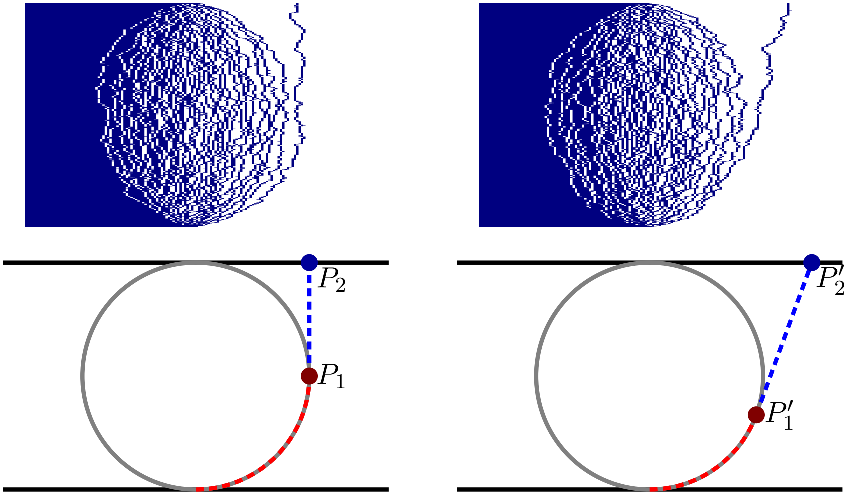
$$\mathcal{A}(\tau) = \exp\left(\frac{1}{6}\left[\frac{\pi^2}{(\pi-\gamma)^2} - 1\right](\tau \sin\gamma)^2\right)\tau^{\kappa} O(1)$$

$$h(\tau|z) = e^{\tau F(z)} O(1)$$

allows to compute the energy function

$$\frac{1}{\tau}\log \mathcal{A}_{X\tau}(\tau) \to G(X)$$

Can reconstruct the full arctic curve using the tangent method [Colomo, Sportiello 2016]


P_2
P_1
P_2'
P_1'

Real time conjectures

$\mathcal{R}(t) = |\mathcal{A}(it)|^2$ gives the exact return probability. [JMS 2017]

- $\Delta = \cos\gamma$, $\gamma = \frac{\pi p}{q}$

$$-\log \mathcal{R}(t) = \left(\frac{q^2}{(q-1)^2} - 1 \right) \frac{(t \sin\gamma)^2}{12} + o(t)$$

- $\Delta = \cos\gamma$, $\frac{\gamma}{\pi} \notin \mathbb{Q}$

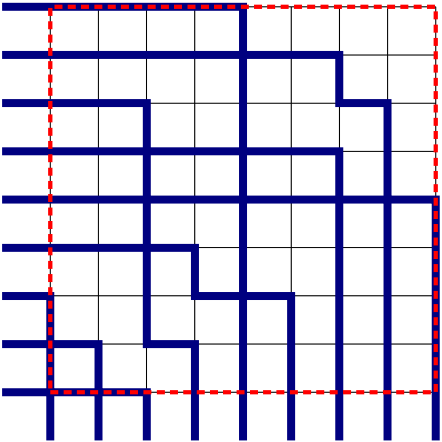
$$-\log \mathcal{R}(t) = (\sin\gamma) t + o(t)$$

- $|\Delta| = 1$

$$-\log \mathcal{R}(t) = \zeta(3/2)\sqrt{t/\pi} - \frac{1}{2}\log t + o(\log t)$$

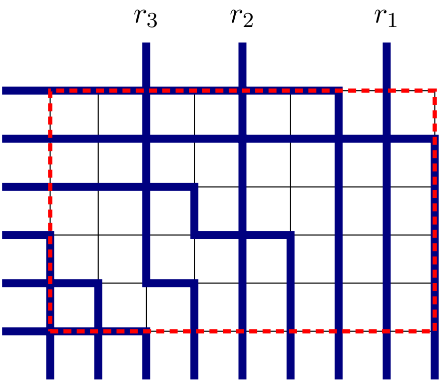
Log-enhanced diffusion. [Gamayun, Miao, Ilievski 2019]

Some steps in the derivation.



Partition function: Izergin-Korepin $n \times n$ determinant.

Hamiltonian limit: Fredholm determinant.



[Colomo, Pronko 2007] [Cantini, Colomo, Pronko 2019] [Colomo, Di Giulio, Pronko 2021].

Requires the knowledge of polynomials $p(n, \epsilon|x)$ which are orthonormal wrt the weights

$$w_\epsilon(x) = e^{-\epsilon x} w_0(x) \qquad , \qquad w_0(x) = \frac{1 - e^{-\gamma x}}{1 - e^{-\pi x}}$$

Orthogonal polynomials

The limit

$$q(\alpha|x) = \lim_{n\to\infty} \sqrt{\frac{n}{\alpha}}\, p(n, \alpha/n|x)$$

satisfies the ODE

$$\left[\alpha\partial_\alpha^2 + \partial_\alpha + f(\alpha) + x\right] q(\alpha|x) = 0$$

where

$$f(\alpha) = 2\alpha \frac{d^2 \log \mathcal{Y}(\alpha)}{d\alpha^2} + \frac{d \log \mathcal{Y}(\alpha)}{d\alpha}$$

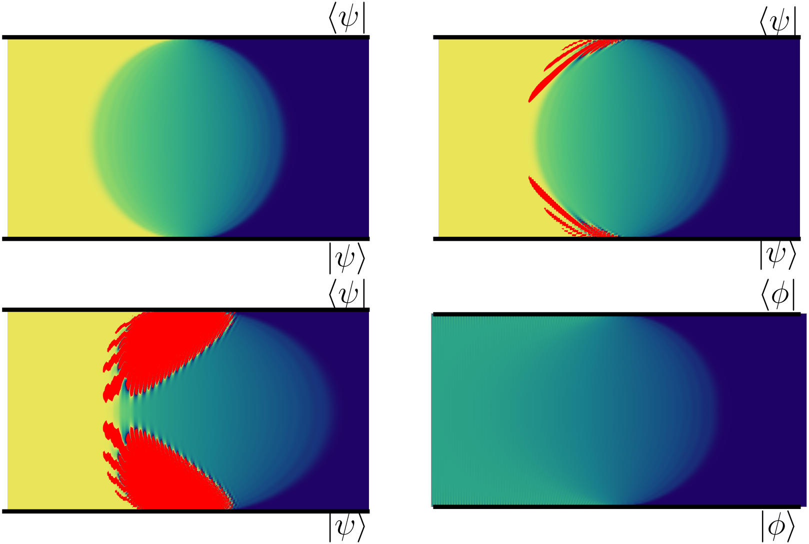
is given in terms of the Fredholm determinant

$$\mathcal{Y}(\alpha) = \det(I - V)_{L^2(\mathbb{R})}$$

$$V(x,y) = \frac{\sqrt{y}J_0(2\sqrt{x})J_0'(2\sqrt{y}) - \sqrt{x}J_0(2\sqrt{y})J_0'(2\sqrt{x})}{(x-y)} \left[\Theta(y) - w_0\left(\frac{y}{\alpha}\right)\right]$$

- More can be extracted from those formulas in imaginary time.
- Real time asymptotics and connection to hydrodynamics?
- Exact formula for the emptiness formation probability, gives access to the distribution of the rightmost particle. KPZ or not KPZ.
- Alternative methods to compute the amplitudes from coordinate Bethe Ansatz [Saenz, Tracy Widom 2022] or the F basis [Feher, Pozsgay 2019].
- Hamiltonian limits of models with $2-$periodic weights?

Adding other charges to H, e.g. $\tilde{H} = H + \alpha Q_2$ [Bocini, JMS 2020].



Higher order edge kernels [Betea, Bouttier, Walsh 2020]

Thank you!