Correlation functions in many-body systems:

Euler hydrodynamics, macroscopic fluctuation theory, and long-range correlations.

Benjamin Doyon

Department of Mathematics, King's College London

"Randomness, Integrability and Universality", Galileo Galilei Institute, Florence

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Plan

Correlation functions **at large scales of space and time**

$$\langle a_1(x_1,t_1)a_2(x_2,t_2)\cdots\rangle, \quad x_i, t_i \to \infty$$

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in one-dimensional many-body systems with short-range interactions such as **spin chains, one-dimensional gases of quantum or classical particles, field theories, etc.**

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of **local observables**

$$\text{e.g. } a(x,t)=\sigma^3_x(t):=e^{\mathrm{i}Ht}\sigma^3_x e^{-\mathrm{i}Ht} \text{ or } a(x,t)=\sum_i\delta(x-x_i(t))$$

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Using ideas of **hydrodynamics** (understood in a general sense)

Plan

- Hydrodynamic linear response for two-point functions in stationary states.

  [Spohn, BD - SciPost 2017; BD - SciPost 2018; Del Vecchio Del Vecchio, BD - 2021; BD - CMP 2021; Ampelogiannis, BD - 2022]

- Fluctuations: full counting statistics and twist field correlation functions in stationary states.

  [Myers, Bhaseen, Harris, BD - SciPost 2019; BD, Myers - AHP 2020; Del Vecchio Del Vecchio, BD - 2021]

- A general macroscopic fluctuation theory for the Euler scale: generic long-range correlations in non-stationary states of interacting models.

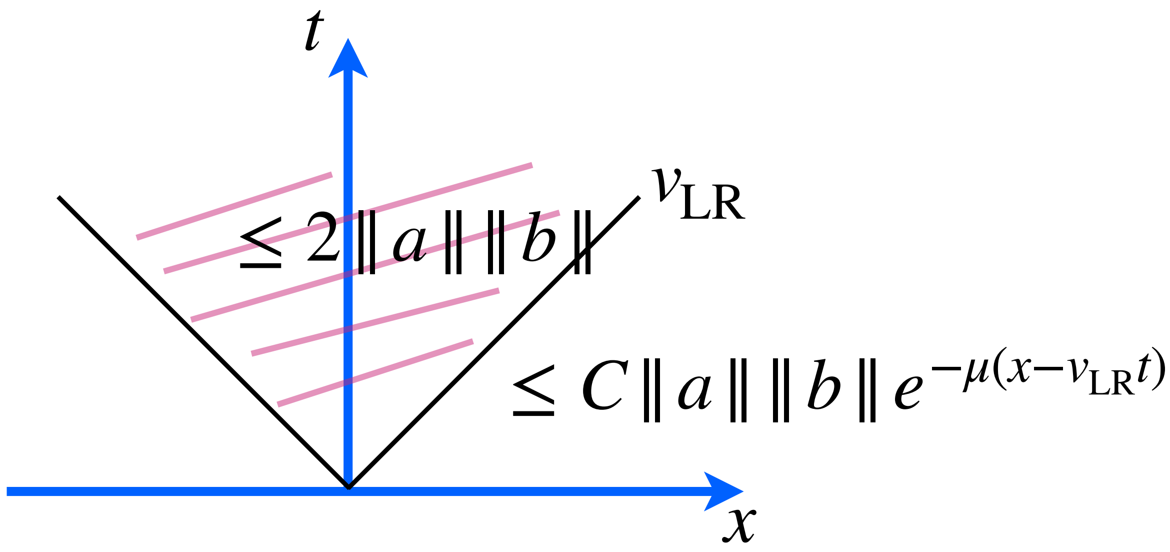
  [BD, Perfetto, Sasamoto, Yoshimura - in prep]

Basic correlation results

By Araki 1969 and Lieb & Robinson 1972:

$$\langle a(x,t)b(0,0)\rangle^{\mathrm{c}} = \langle a(x,t)b(0,0)\rangle - \langle a(x,t)\rangle\langle b(0,0)\rangle$$

In state $\langle\cdot\rangle = Z^{-1}\mathrm{Tr}\, e^{-W}\cdot$ with $W,\ H$ short range interaction



Basic correlation results

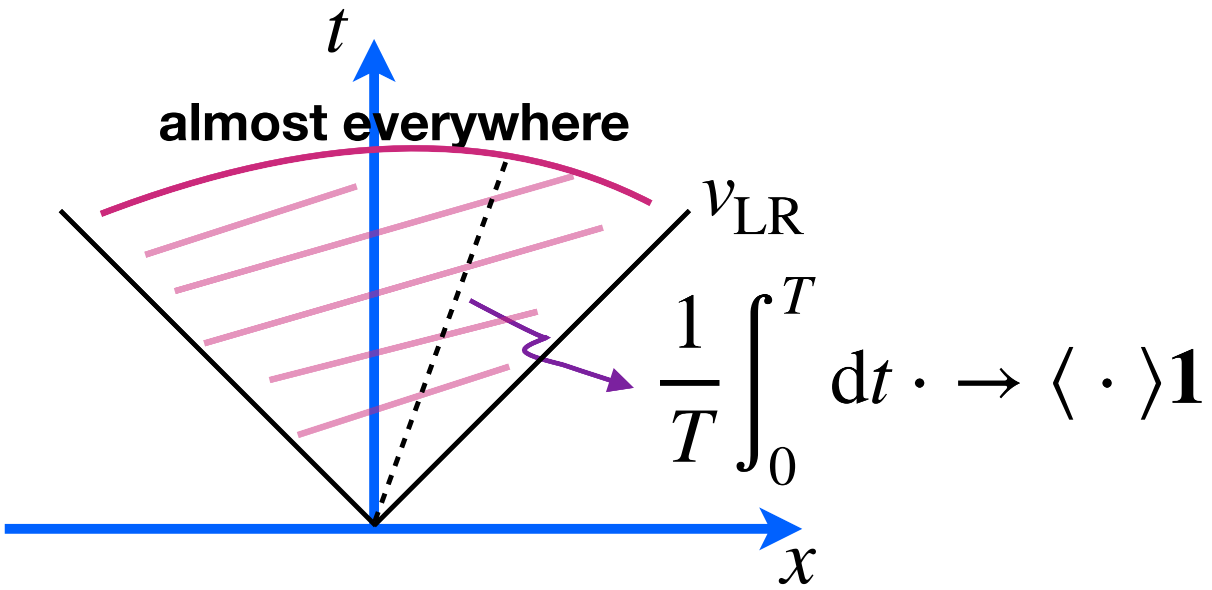
Almost-everywhere ergodicity: take stationary state and translation invariance,

$$[W, H] = 0$$

Theorem: then inside any correlation function, for any $\omega \in \mathbb{R}$,

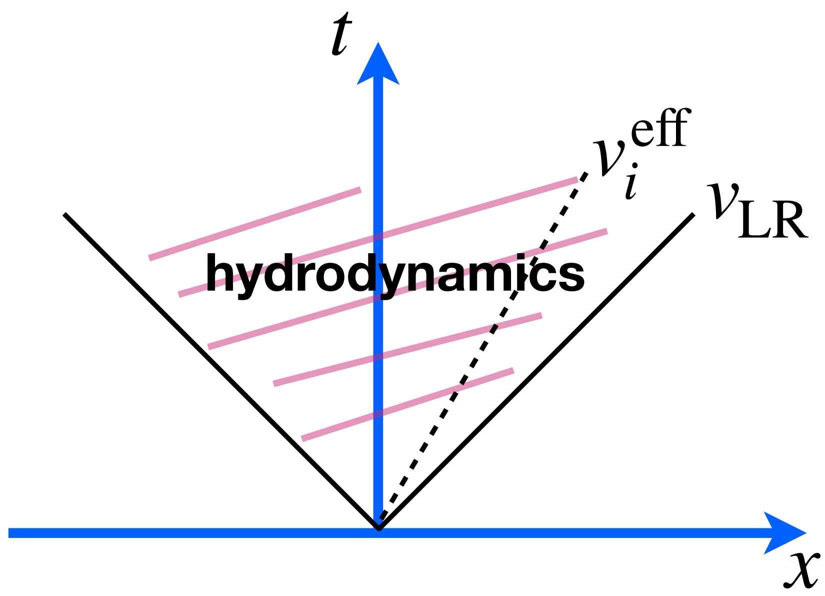
$$\lim_{T\to\infty} \frac{1}{T}\int_0^T \mathrm{d}t\, e^{\mathrm{i}\omega t} a(vt,t) = \langle a\rangle \mathbf{1}\delta_{\omega,0} \qquad \text{for almost all } v \in \mathbb{R}$$

[BD - CMP 2021; Ampelogiannis, BD - 2022]



Hydrodynamic linear response

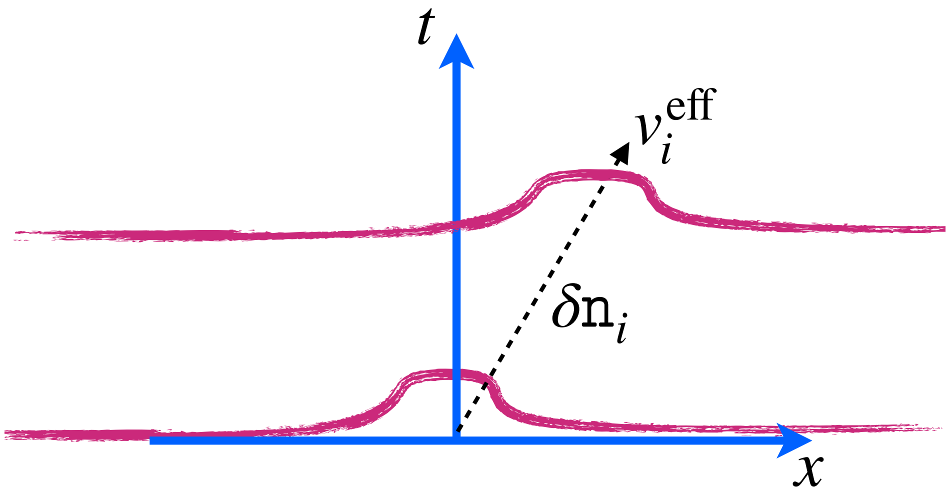
This is enough to give rise to the hydrodynamic structure inside the LR cone: the true relevant velocities for correlations are the hydrodynamic velocities $v_i^{\rm eff}$



Hydrodynamic linear response

Hydrodynamic linear response:

leading large-scale correlations are due to travelling waves



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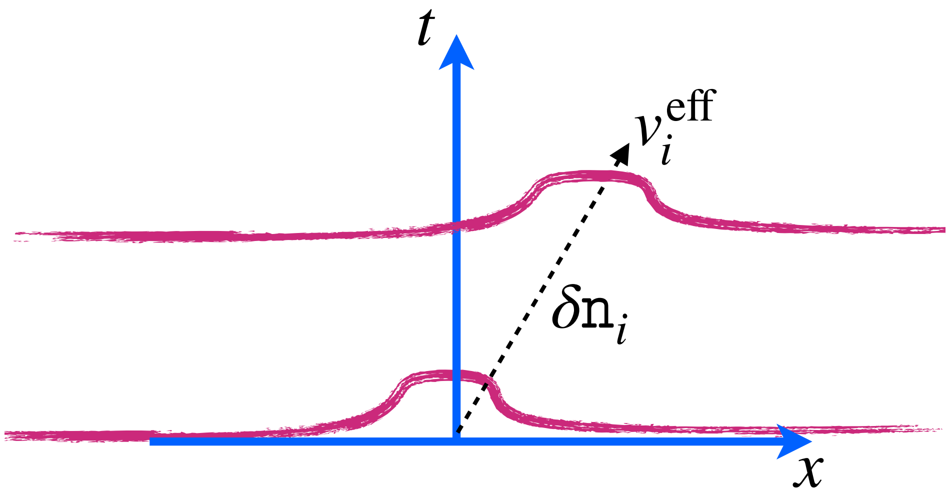


Conserved densities (∂_x is "discrete derivative" in the case of quantum chains)

$$\partial_t q_i + \partial_x j_i = 0$$

e.g. in XX model: $q_0(x) = \sigma^3_x, \quad q_1(x) = \sigma^1_x \sigma^1_{x+1} + \sigma^2_x \sigma^2_{x+1}$ etc.

and their currents: $j_0(x) = 2(\sigma^1_{x-1}\sigma^2_x - \sigma^2_{x-1}\sigma^1_x),$ etc.

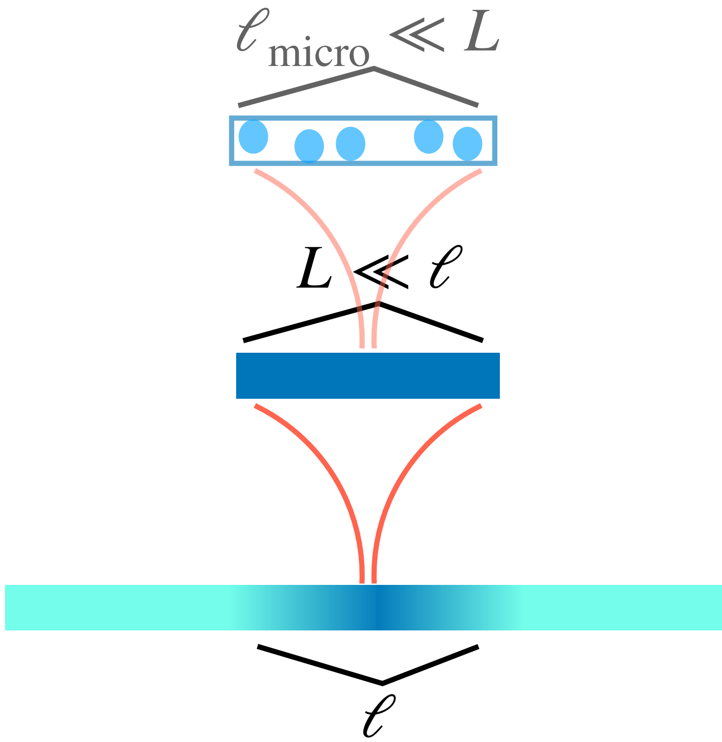
Hydrodynamic linear response

Hydrodynamics: from initial **inhomogeneous states** of large wavelength ℓ, e.g.

$$\rho \propto e^{-\int \mathrm{d}x\, \beta^i(x/\ell) q_i(x)}$$

in “fluid cells” of mesoscopic sizes L much greater than microscopic lengths ℓ_{micro},

$$\ell_{\mathrm{micro}} \ll L \ll \ell$$



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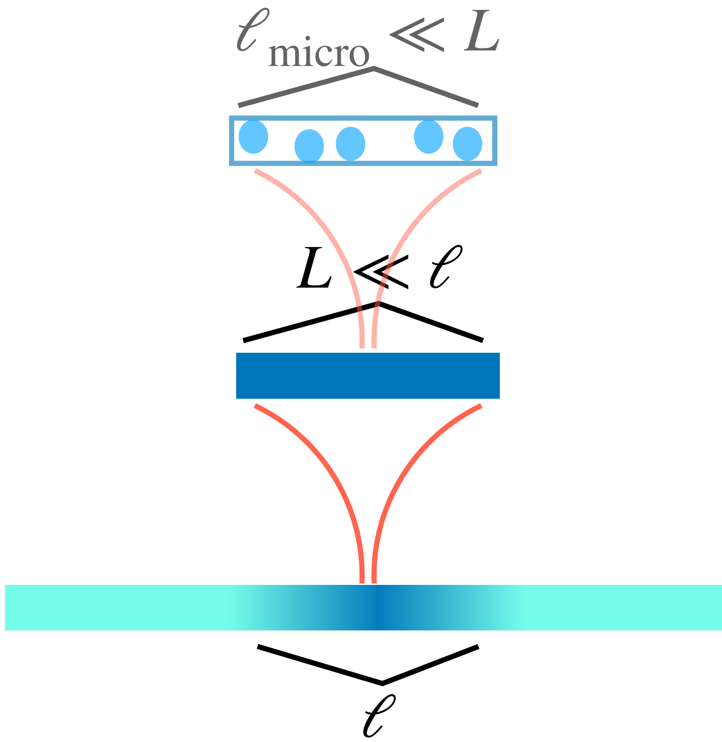

we have maximisation of entropy with respect to all available conservation laws

$$\mathrm{Tr}_{\mathbb{R}\setminus[\ell x - L/2, \ell x + L/2]} \rho(\ell t) \approx e^{-\sum_i \beta^i(x,t) Q_i^{(L)}} : \rho_{\boldsymbol{\beta}(x,t)}, \quad Q_i^{(L)} = \int_0^L \mathrm{d}x\, q_i(x)$$

Hydrodynamic linear response

Then averages of densities and currents

$$\mathtt{q}_i(x,t) = \mathrm{Tr}\,\rho_{\boldsymbol{\beta}(x,t)} q_i, \quad \mathtt{j}_i(x,t) = \mathrm{Tr}\,\rho_{\boldsymbol{\beta}(x,t)} j_i$$

satisfy continuity equation

$$\partial_t \mathtt{q}_i(x,t) + \partial_x \mathtt{j}_i(x,t) = 0$$

This is an equation for $\mathtt{q}_i$ using the bijection $\{\mathtt{q}_i\} \leftrightarrow \{\beta^i\}$.

Hydrodynamic linear response

Linear response theory: take a state that is nearly stationary, $\rho \propto e^{-W-\int \mathrm{d}x\, \delta\beta^i(x) q_i(x)}$, then small disturbance propagates according to linearised hydrodynamics

$$\partial_t \delta \mathtt{q}_i(x,t) + \mathsf{A}_i^{\ j} \partial_x \delta \mathtt{q}_j(x,t) = 0, \quad \mathsf{A}_i^{\ j} = \frac{\delta \mathtt{j}_i}{\delta \mathtt{q}_j}\Big|_{\text{stationary } e^{-W}}$$

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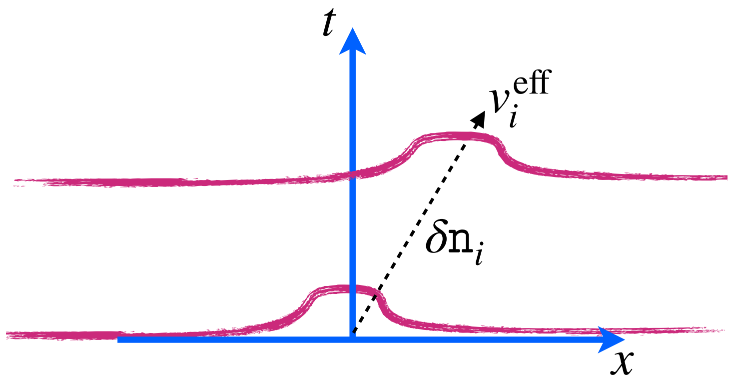
which you can diagonalise to normal modes $\delta\mathtt{n}_i(x,t)$ with velocities $v_i^{\text{efff}} \in \text{spec}(\mathsf{A}) \subset \mathbb{R}$, i.e. $\partial_t \delta\mathtt{n}_i + v_i^{\text{eff}} \partial_x \delta\mathtt{n}_i = 0$



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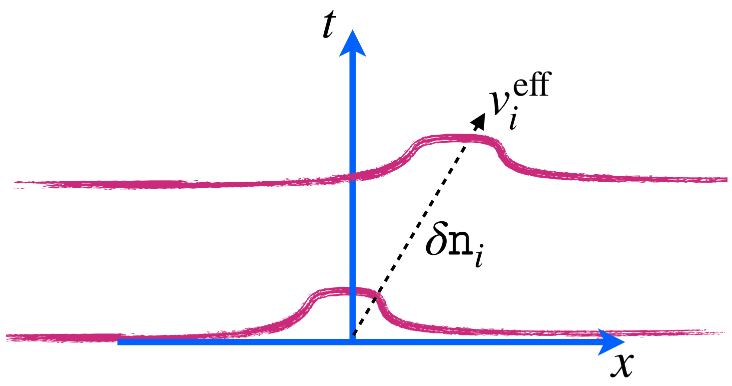

This translates into **a linear equation for correlation functions**: reduction of degrees of freedom for calculating correlation functions!

$$\partial_t \langle q_i(x,t) q_j(0,0)\rangle^{\text{c}} + \mathsf{A}_i^{\ k} \partial_x \langle q_k(x,t) q_j(0,0)\rangle^{\text{c}} = 0$$

Hydrodynamic linear response

In integrable systems of fermionic type: use GHD to get [BD, Spohn - SciPost 2017; BD - SciPost 2018]

$$\langle q_i(\xi t,t)q_j(0,0)\rangle^{\rm c} \sim \frac{1}{t}\,\frac{\rho_{\rm p}(p)(1-n(p))h_i^{\rm dr}(p)h_j^{\rm dr}(p)}{|v^{\rm eff}{}'(p)|}\Big|_{v^{\rm eff}(p)=\xi}$$

where $\rho_{\rm p}(p)$ is the Bethe root density, $n(p)$ is the occupation function, ${}^{\rm dr}$ is the TBA dressing operation, and $h_i(p)$ is the one-particle eigenvalue of the charge Q_i.

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- the GHD effective velocities $v^{\rm eff}(p)$ are identified with the hydrodynamic mode velocities
- the formula has been shown using / reproduces calculations from finite-density form factors in integrable models [De Nardis, Panfil - JSTAT 2018; Cortés Cubero, Panfil 2019]
- in the XX model, free fermions, $v^{\rm eff}(p) = 4\sin p$ is the group velocity from the dispersion relation, and the dressing is trivial.

For instance for $q_0(x) = \sigma^3_x$, we take $h_0^{\rm dr}(p) = h_0(p) = 1$.

Hydrodynamic linear response

The need for fluid-cell averaging:

But wait, in the XX model, by Wick's theorem and saddle point analysis ($\xi \in (-4,4)$)

$$\langle \sigma^3_{\xi t}(t)\sigma^3_0(0)\rangle^{\rm c} \sim \frac{2}{\pi|t|\,\sqrt{16-\xi^2}}\sum_{a=\pm}\times$$

$$\times\ \ n_a\Big(1-n_a+a{\rm i}\,(1-n_{-a})(-1)^x e^{-2a{\rm i}(x\,\arcsin(\xi/4)+t\sqrt{16-\xi^2})}\Big)$$

where

$$n_\pm = \frac{1}{1+\exp\Big[\pm\beta\sqrt{16-\xi^2}\Big]}$$

There is an oscillatory term!

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Hydrodynamic results for correlation functions are valid under fluid cell averaging:

$$\langle \overline{a}(\ell x,\ell t)\cdots\rangle^{\rm c}\quad \text{for } \overline{a}(\ell x,\ell t) = \frac{1}{LT}\int_{-L/2}^{L/2}{\rm d}y\int_{-T/2}^{T/2}{\rm d}s\, a(\ell x+y,\ell t+s)$$

Hydrodynamic linear response

Rigorous result: the general form of linearised Euler equation holds for every translation invariant quantum chains with short range interactions.

Fluid-cell mean: time average, space average

$$S_{a,b}(\kappa) = \lim_{T\to\infty} \frac{1}{T-T_0} \int_{T_0}^{T} \mathrm{d}t \sum_{x\in\mathbb{Z}} e^{\mathrm{i}\kappa x/t} \langle a(x,t) b(0,0)\rangle^{\mathrm{c}}$$

Theorem: then linearised Euler equation holds

$$\frac{\mathrm{d}}{\mathrm{d}\kappa} S_{q_i,q_j}(\kappa) = \mathrm{i}\mathsf{A}_i^{\;k} S_{q_k,q_j}(\kappa)$$

[BD - CMP 2021]

Hydrodynamic linear response

Equal-time total connected correlator:

$$(a, b) := \sum_{x \in \mathbb{Z}} \langle a(x) b(0) \rangle^{\rm c}$$

Positive semidefinite $(a, a) \geq 0 \rightarrow$ inner product on equivalence classes $\{a(x, 0) : x \in \mathbb{Z}\}$ $\rightarrow$ Hilbert space $\mathscr{H}$ of **extensive observables**. Time evolution $\tau_t : \{a(x, 0)\} \mapsto \{a(x, t)\}$ is unitary on $\mathscr{H}$ (by stationarity of the state and Lieb-Robinson bound).

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Conserved quantities are all the extensive observables that are invariant under τ_t

$$\mathscr{Q} = \{A\in\mathscr{H}\ :\ \tau_t A = A\ \forall\ t\}$$

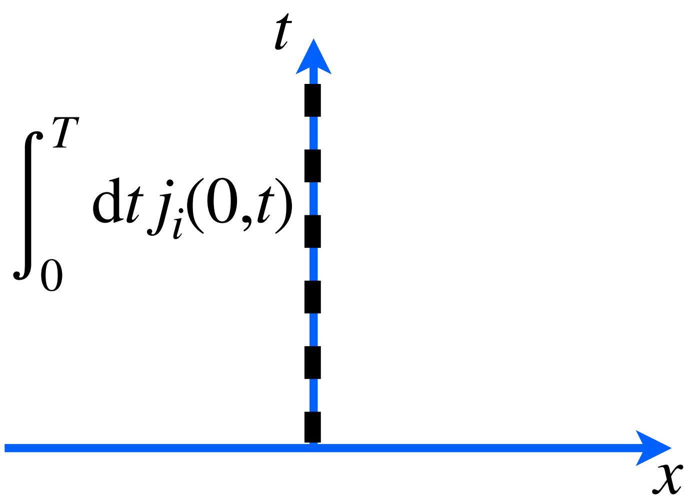
Then q_k are just a basis in the closed subspace $\mathscr{Q}$

$$\sum_k \mathsf{A}_i^{\ k} S_{q_k,q_j}(\kappa) = S_{\mathbb{P}j_i,q_j}(\kappa)\ :\ \text{projection } \mathscr{H}\rightarrow\mathscr{Q} \text{ and sum over a basis.}$$

Fluctuations

Fluctuations: consider the transport of conserved quantities from left to right

$$\langle e^{\lambda \int_0^T \mathrm{d}t\, j_i(0,t)} \rangle \asymp e^{T F(\lambda)} \quad (T \to \infty)$$



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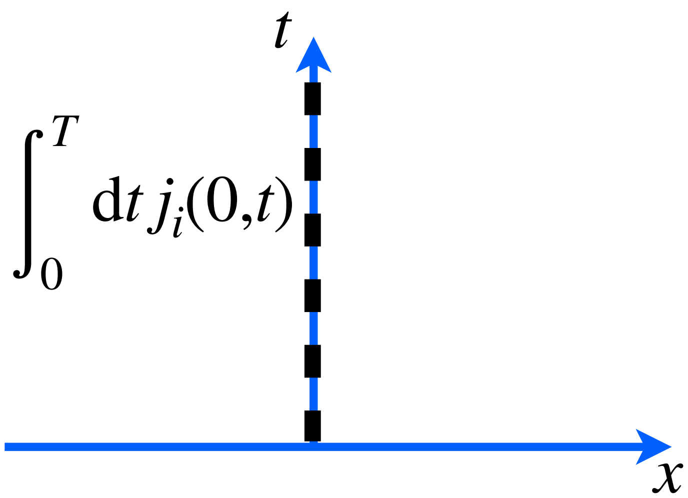
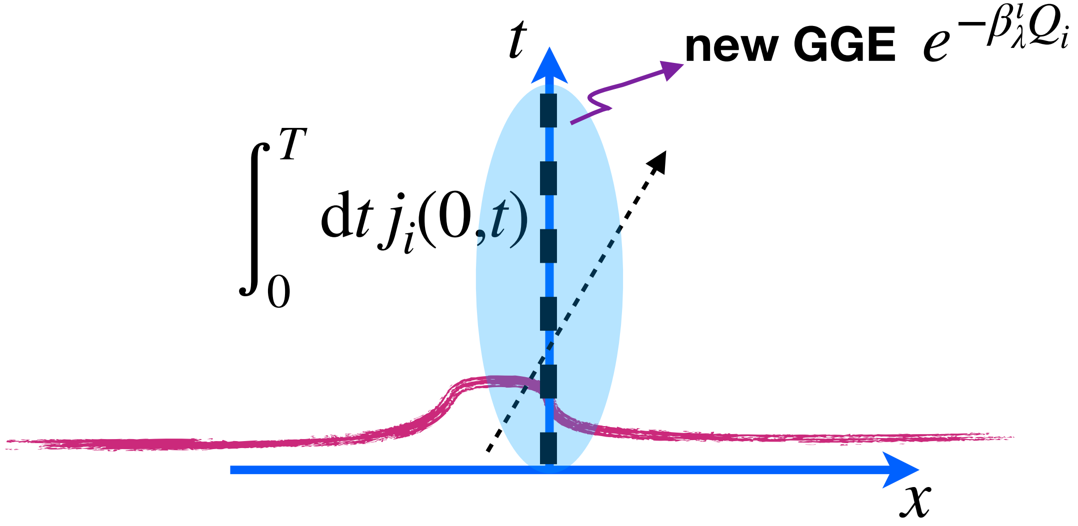
$F(\lambda)$ generates the scaled cumulants, which are time-integrated connected correlation functions (with $j_i(t) = j_i(0,t)$)

$$F(\lambda) = \langle j_i \rangle + \lambda \int_{-\infty}^{\infty} \mathrm{d}t\, \langle j_i(0) j_i(t) \rangle^{\mathrm{c}} + \frac{\lambda^2}{2} \int_{-\infty}^{\infty} \mathrm{d}t_1 \int_{-\infty}^{\infty} \mathrm{d}t_2\, \langle j_i(0) j_i(t_1) j_i(t_2) \rangle^{\mathrm{c}} + \ldots$$



Fluctuations

Using hydrodynamic linear response, one finds that large-scale fluctuation of total currents is controlled by linear waves passing through the point $x = 0$: each mode contributes positively or negatively according to its velocity, in order to form a "new" GGE that knows about the insertion of the time-integrated current in the exponential



$$F(\lambda) = \int_0^\lambda \mathrm{d}\lambda' \mathtt{j}_i[\boldsymbol{\beta}_{\lambda'}], \quad \mathtt{j}_i[\boldsymbol{\beta}_\lambda] = \frac{1}{Z}\mathrm{Tr}\, e^{-\beta_\lambda^k Q_k} j_i, \quad \frac{\mathrm{d}}{\mathrm{d}\lambda}\beta_\lambda^j = -\mathrm{sgn}\big(\mathsf{A}[\boldsymbol{\beta}_\lambda]\big)_i^{\ j}$$

[BD, Myers - AHP 2020 (AHP-Birkhauser prize 2020)]

Fluctuations

Example: TASEP $j[\rho] = \rho(1-\rho), \quad \mathsf{A}(\rho) = 1 - 2\rho, \quad -\frac{\partial \rho}{\partial \beta} = \rho(1-\rho)$

we reproduce the known results ($\rho < 1/2$) [de Gier, Essler - PRL 2011; Lazarescu, Mallick - JPA 2011]

$$F(\lambda) = \frac{(1-\rho)(e^{\lambda}-1)}{\rho(e^{\lambda}-1)+1}$$

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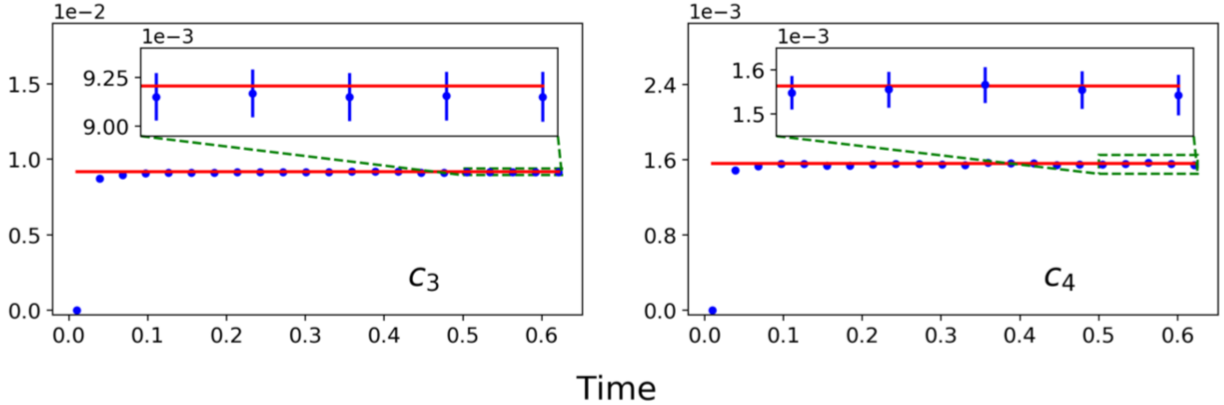
Example: energy transport in hard rods: GHD [Myers, Bhaseen, Harris, BD - SciPost 2019]



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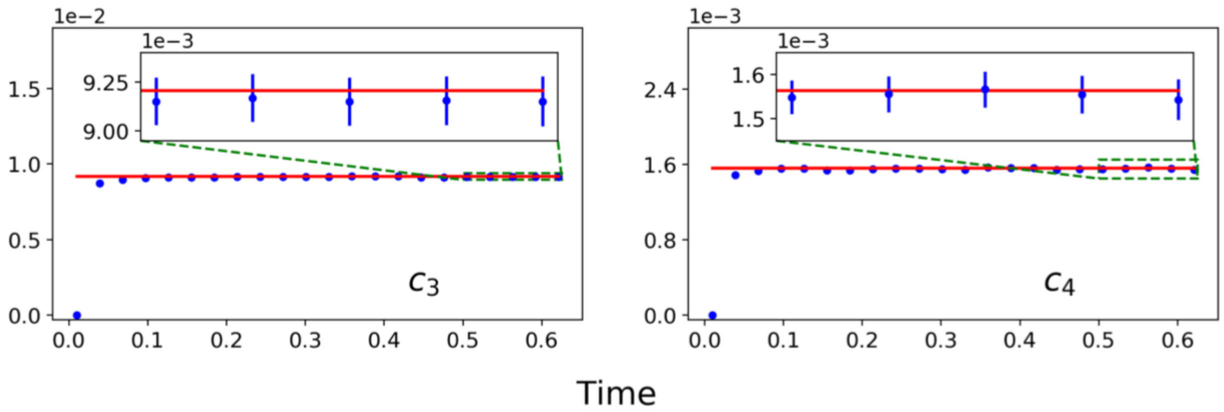

Example: Free fermion: reproduces Levitov-Lesovik formula

Example: CFT energy transfer [Bernard, BD - JPA 2012] proven in [Gawedzki, Kozlowski - CMP 2020]

Example: box-ball system, shown analytically [Kuniba, Misguich, Pasquier - 2020]

Fluctuations

More generally, one can obtain asymptotics of **twist field correlation functions**.

XX model: written in terms of free fermions $a_x(t), a_x^\dagger(t)$, the spin variables $\sigma^\pm$ are expressed using Jordan-Wigner strings. It is argued in [Del Vecchio Del Vecchio, BD - 2021] that this can be recast into **space-time Jordan Wigner strings**

$$\langle\sigma_x^+(t)\sigma_0^-(0)\rangle \asymp \langle a_x^\dagger(t)a_0(0)\rangle_{\boldsymbol{\beta}_\pi}\langle\exp\mathrm{i}\pi\int_{0,0}^{x,t}\mathrm{d}s_\mu j_0^\mu(\vec{s})\rangle,\quad j_0^\mu(x,t):\ \text{spin 2-current}$$

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The scaled cumulant generating function on an arbitrary ray is known, here:

$$\langle \exp \mathrm{i}\pi \int_{0,0}^{x,t} \mathrm{d}s_\mu j_0^\mu(\vec{s})\rangle \asymp e^{F(x,t)}$$

where

$$F(x,t) = \int_0^{\mathrm{i}\pi} \mathrm{d}\lambda' \left(t\mathtt{j}_0[\boldsymbol{\beta}_{\lambda'}] - x\mathsf{q}_0[\boldsymbol{\beta}_{\lambda'}]\right), \quad \frac{\mathrm{d}}{\mathrm{d}\lambda}\beta_\lambda^j = -\mathrm{sgn}\big(t\mathsf{A}[\boldsymbol{\beta}_\lambda] - x\mathbf{1}\big)_0^j$$

Fluctuations

This reproduces the old results [Its, Izergin, Korepin, Slavnov - PRL 1993; Jie. - Ph.D. Thesis 1998] and gives new results (last line)

$$F(x,t) = \begin{cases} f_{x,t} & (|\xi| \leq 4) \\ |x| f_{1,0} & (|\xi| > 4,\ |h| \leq 2) \\ -\,|x| \min\bigl(\mathrm{arccosh}(h/2), M_\xi\bigr) + |x| f_{1,0} & (|\xi| > 4,\ |h| > 2) \end{cases}$$

with $M_\xi = \mathrm{arccosh}(\xi/4) - \sqrt{1 - \frac{16}{|\xi|^2}}$ and

$$f_{x,t} = \int_{-\pi}^{\pi} \frac{\mathrm{d}k}{2\pi}\ |x - v(k)t| \log \left| \tanh \frac{\beta E(k)}{2} \right|.$$

Ballistic macroscopic fluctuation theory

[BD, Perfetto, Sasamoto, Yoshimura - in prep]

Take again long-wavelength initial state

$$\langle \cdots \rangle_\ell \;:\; \rho \propto e^{-\int \mathrm{d}x\, \beta^i(x/\ell) q_i(x)}$$

We can reproduces **all Euler-scale correlations**:

$$\lim_{\ell\to\infty} \ell^{n-1} \langle \overline{a}_1(\ell x_1, \ell t_1) \cdots \overline{a}_n(\ell x_n, \ell t_n) \rangle_\ell^{\mathrm{c}}$$

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[BD, Perfetto, Sasamoto, Yoshimura - in prep]

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$$\langle \cdots \rangle_\ell \;:\; \rho \propto e^{-\int \mathrm{d}x\, \beta^i(x/\ell) q_i(x)}$$

We can reproduces **all Euler-scale correlations**:

$$\lim_{\ell\to\infty} \ell^{n-1} \langle \overline{a}_1(\ell x_1, \ell t_1) \cdots \overline{a}_n(\ell x_n, \ell t_n) \rangle_\ell^{\rm c}$$

replacing fluid-cell-averaged observables **by random variables** $\check{q}_i$:

$$\overline{q}_i(\ell x, \ell t) \to \check{q}_i(x,t)$$

with every observable functions of these given by their GGE values

$$\overline{a}(\ell x, \ell t) \to \check{a}(x,t) = \frac{1}{Z} \mathrm{Tr}\, e^{-\check{\beta}^i(x,t) Q_i} a \;(\text{at } \check{q}_i = \mathsf{q}_i[\check{\boldsymbol{\beta}}])$$

Ballistic macroscopic fluctuation theory

[BD, Perfetto, Sasamoto, Yoshimura - in prep]

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replacing fluid-cell-averaged observables **by random variables** $\check{q}_i$, using the **BMFT measure**

$$\begin{aligned} \mathrm{d}\mathbb{P} &= [\mathrm{d}\check{\boldsymbol{q}}(\cdot,\cdot)] \\ &\quad \times \exp\Big[-\ell \int \mathrm{d}x\, \Big(\beta^i(x)(\check{q}_i(x,0) - \mathsf{q}_i(x)) + s[\mathsf{q}(x)] - s[\check{\boldsymbol{q}}(x,0)] \Big) \Big] \\ &\quad \times \delta\big[\partial_t \check{\boldsymbol{q}} + \partial_x \mathsf{j}[\check{\boldsymbol{q}}]\big] \end{aligned}$$

Ballistic macroscopic fluctuation theory

That is:

$$\lim_{\ell\to\infty} \ell^{n-1}\langle \overline{a}_1(\ell x_1,\ell t_1)\cdots \overline{a}_n(\ell x_n,\ell t_n)\rangle_\ell^{\rm c} = \lim_{\ell\to\infty} \ell^{n-1}\int {\rm d}\mathbb{P}\,\check a_1(x_1,t_1)\cdots\check a_n(x_n,t_n)$$

with

$$\begin{aligned}{\rm d}\mathbb{P} &= [{\rm d}\check{\boldsymbol{q}}(\cdot,\cdot)]\\ &\quad\times \exp\Big[-\ell\int {\rm d}x\,\Big(\beta^i(x)(\check q_i(x,0)-\mathsf{q}_i(x)) + s[\mathsf{q}(x)] - s[\check{\boldsymbol{q}}(x,0)]\Big)\Big]\\ &\quad\times\delta\big[\partial_t\check{\boldsymbol{q}} + \partial_x \mathtt{j}[\check{\boldsymbol{q}}]\big]\end{aligned}$$

Ballistic macroscopic fluctuation theory

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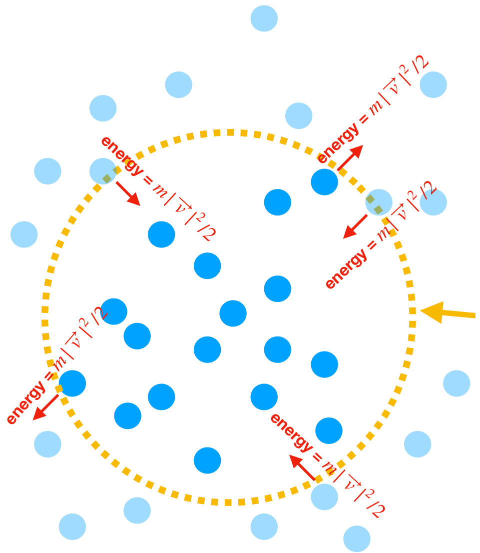
Main principle: **separation of scales on fluctuations**:

● average out “quick” fluctuations within the microcanonical shell due to all kinds of non-conserving processes that happen in the bulk of the fluid cell

● keep “slow” fluctuations amongst different microcanonical shells due to fluctuations of conserved quantities that happens at the surface of the fluid cell.



Ballistic macroscopic fluctuation theory

$\ell \to \infty$: saddle point equations in terms of an auxiliary "Lagrange parameter" $H^i(x,t)$ for the delta function $\delta[\partial_t q + \partial_x j] = \int [\mathrm{d}H] e^{\ell \int_{\mathbb{R}} \mathrm{d}x \int_0^T \mathrm{d}t \, H(\partial_t q + \partial_x j)}$.

With source terms, here for conserved densities $a_r = q_{k_r}$:

$$
\begin{aligned}
H^i(x,0) &= \beta^i - \check{\beta}^i \\
H^i(x,T) &= 0 \\
\partial_t \check{\beta}^i + \check{\mathsf{A}}_j^{\,i} \partial_x \check{\beta}^j &= 0 \\
\partial_t H^i + \check{\mathsf{A}}_j^{\,i} \partial_x H^j &= \sum_{r=1}^{n} \lambda_r \delta^i_{k_r} \delta(x - x_r) \delta(t - t_r)
\end{aligned}
$$

Ballistic macroscopic fluctuation theory

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\end{aligned}
$$

- Reproduces linear response $\partial_t \langle q_i(x,t) q_j(0,0)\rangle + \mathsf{A}_i^{\ k} \partial_x \langle q_k(x,t) q_j(0,0)\rangle = 0$
- Reproduces c_2, c_3 and can be argued to reproduce the full $F(\lambda)$
- Gives the Cohen-Gallavotti fluctuation relations

(see Takato's talk)

Ballistic macroscopic fluctuation theory

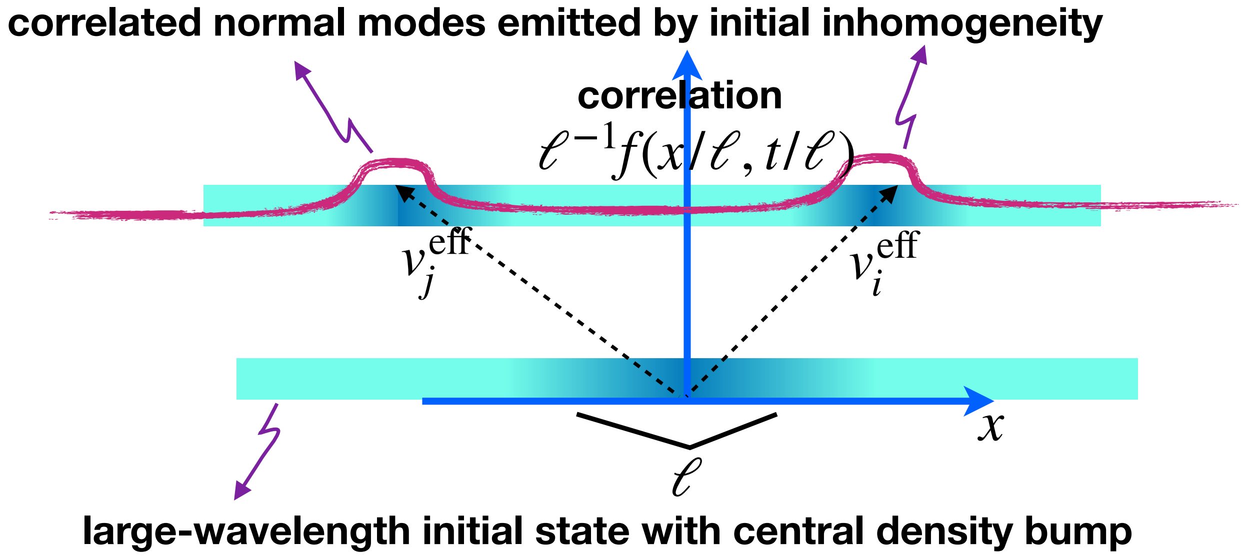
In particular, the theory implies that in interacting models with at least two different hydrodynamic velocities, from non-stationary state, there are **long range correlations:**

$$\lim_{\ell\to\infty} \ell\langle \overline{q}_i(\ell x, \ell t)\overline{q}_j(0,\ell t)\rangle_\ell^{\rm c} = f(x,t) \neq 0$$

The density matrix **does not take the exponential form** at later times even at the Euler approximation scale,

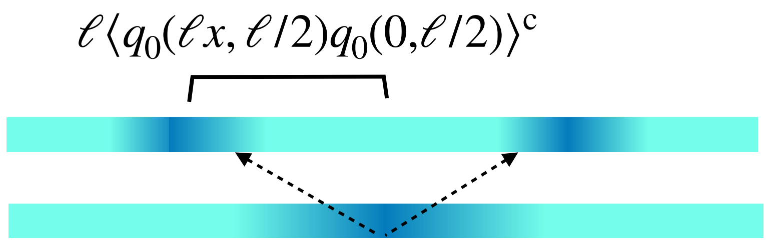
$$\rho(t) \not\propto e^{-\int {\rm d}x\, \beta^i(x/\ell, t/\ell) q_i(x)}$$

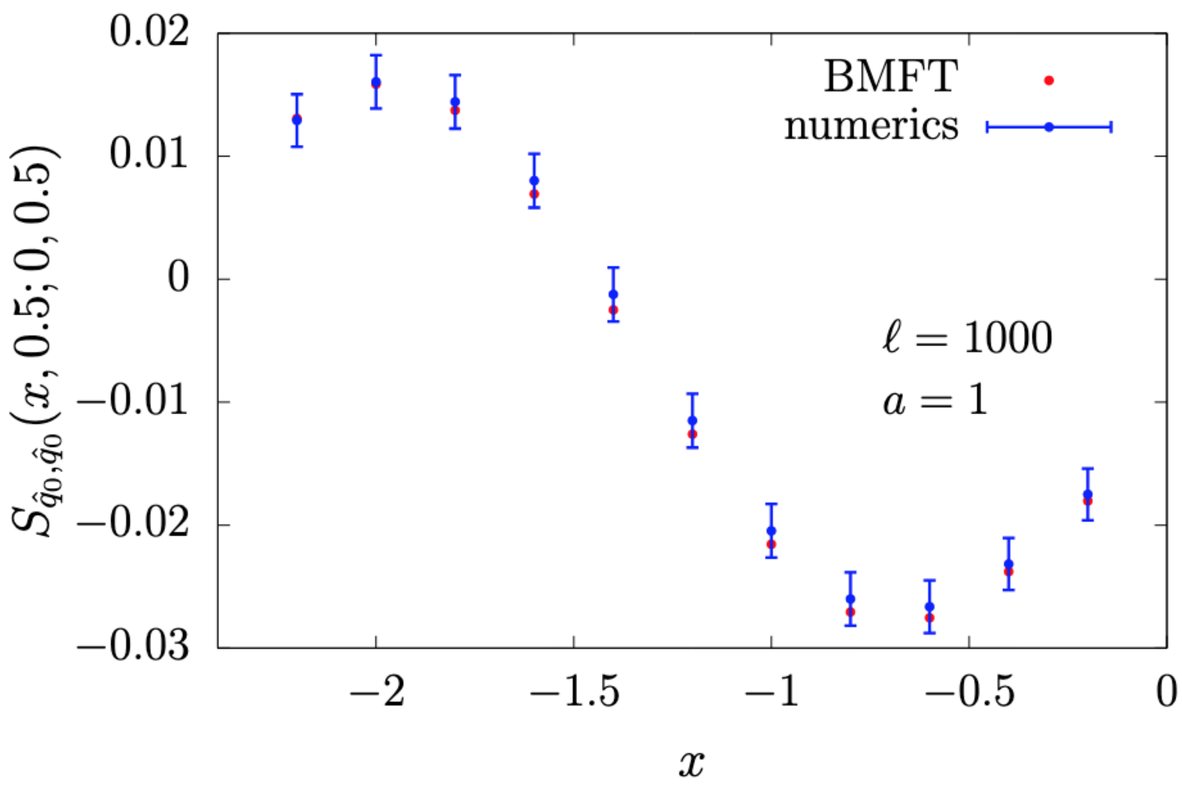
as instead there are **correlations between fluid cells**.



Ballistic macroscopic fluctuation theory

Example: evolution of two-velocity $p = \pm 1$ hard rod gas, from bump initial condition





Conclusion

Hydrodynamics gives a lot of general principles that can predict / reproduce asymptotic of many types of correlation functions at large space-time separations.

To do (cf grant applications):

- revisit recent nonlinear response results in light of BMFT and the new type of long-range correlations we uncovered
- apply fluctuation formalism to twist fields for entanglement entropy (work in progress with V. Alba, G. Del Vecchio Del Vecchio, P. Ruggiero)
- generalise fluctuation formalism / BMFT of integrable systems to diffusive scale (and beyond?)
- apply BMFT to non-integrable systems and reproduce KPZ scaling
- prove rigorously the large-scale correlation function formula in integrable models (prove that space of conserved charges $\mathfrak{Q}$ is the spanned by Bethe quasiparticles)