Correlation functions in many-body systems:
Euler hydrodynamics, macroscopic fluctuation theory,
and long-range correlations.

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Plan

Correlation functions **at large scales of space and time**

\[ \langle a_1(x_1, t_1)a_2(x_2, t_2) \cdots \rangle, \quad x_i, t_i \to \infty \]
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in **statistical ensembles, in or out of equilibrium**

e.g. \( \langle \cdots \rangle = \frac{1}{Z} \text{Tr} e^{-\beta H} \cdots \) or \( \langle \cdots \rangle = \frac{1}{Z} \text{Tr} e^{-\int_{\mathbb{R}} dx \beta(x) h(x)} \cdots \) etc.
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in one-dimensional many-body systems with short-range interactions such as spin chains, one-dimensional gases of quantum or classical particles, field theories, etc.

\[ \text{e.g. } H = \sum_{x \in \mathbb{Z}} \left( \sigma_x^1 \sigma_{x+1}^1 + \sigma_x^2 \sigma_{x+1}^2 \right) \text{ or } H = \sum_i \frac{p_i^2}{2} + \sum_{ij} V(x_i - x_j) \text{ etc.} \]

of local observables

\[ \text{e.g. } a(x, t) = \sigma_x^3(t) := e^{iHt} \sigma_x^3 e^{-iHt} \text{ or } a(x, t) = \sum_i \delta(x - x_i(t)) \]
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Using ideas of hydrodynamics (understood in a general sense)
Plan

- Hydrodynamic linear response for two-point functions in stationary states.
  
  [Spohn, BD - SciPost 2017; BD - SciPost 2018; Del Vecchio Del Vecchio, BD - 2021; BD - CMP 2021; Ampelogiannis, BD - 2022]

- Fluctuations: full counting statistics and twist field correlation functions in stationary states.
  
  [Myers, Bhaeeen, Harris, BD - SciPost 2019; BD, Myers - AHP 2020; Del Vecchio Del Vecchio, BD - 2021]

- A general macroscopic fluctuation theory for the Euler scale: generic long-range correlations in non-stationary states of interacting models.
  
  [BD, Perfetto, Sasamoto, Yoshimura - in prep]
Basic correlation results

By Araki 1969 and Lieb & Robinson 1972:

\[ \langle a(x, t)b(0, 0) \rangle^c = \langle a(x, t)b(0, 0) \rangle - \langle a(x, t) \rangle \langle b(0, 0) \rangle \]

In state \( \langle \cdot \rangle = Z^{-1} \text{Tr} \, e^{-W} \cdot \) with \( W, H \) short range interaction

\[ \leq 2 \| a \| \| b \| \]

\[ \leq C \| a \| \| b \| \, e^{-\mu(x-v_{LR}t)} \]
Basic correlation results

**Almost-everywhere ergodicity**: take stationary state and translation invariance,

\[ [W, H] = 0 \]

**Theorem**: then inside any correlation function, for any \( \omega \in \mathbb{R} \),

\[
\lim_{T \to \infty} \frac{1}{T} \int_0^T dt \ e^{i\omega t} a(vt, t) = \langle a \rangle \, 1_{\delta_{\omega,0}} \quad \text{for almost all} \ v \in \mathbb{R}
\]

[BD - CMP 2021; Ampelogiannis, BD - 2022]
Hydrodynamic linear response

This is enough to give rise to the hydrodynamic structure inside the LR cone: the true relevant velocities for correlations are the hydrodynamic velocities $v^{\text{eff}}_i$.
Hydrodynamic linear response:

leading large-scale correlations are due to travelling waves
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Hydrodynamic linear response:

**leading large-scale correlations are due to travelling waves**

Conserved densities ($\partial_x$ is “discrete derivative” in the case of quantum chains)

\[ \partial_t q_i + \partial_x j_i = 0 \]

e.g. in XX model: 
\[ q_0(x) = \sigma_x^3, \quad q_1(x) = \sigma_x^1 \sigma_{x+1}^1 + \sigma_x^2 \sigma_{x+1}^2 \text{ etc.} \]

and their currents: 
\[ j_0(x) = 2(\sigma_{x-1}^1 \sigma_x^2 - \sigma_{x-1}^2 \sigma_x^1), \text{ etc.} \]
Hydrodynamic linear response

Hydrodynamics: from initial **inhomogeneous states** of large wavelength $\ell$, e.g.

$$\rho \propto e^{-\int dx \beta^i(x/\ell)q_i(x)}$$

in “fluid cells” of mesoscopic sizes $L$ much greater than microscopic lengths $\ell_{\text{micro}}$.

$$\ell_{\text{micro}} \ll L \ll \ell$$
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in “fluid cells” of mesoscopic sizes \( L \) much greater than microscopic lengths \( \ell_{\text{micro}} \),

\[
\ell_{\text{micro}} \ll L \ll \ell
\]

we have maximisation of entropy with respect to all available conservation laws

\[
\text{Tr}_{\mathbb{R}^\ell \backslash [x-L/2,x+L/2]} \rho(\ell t) \approx e^{-\sum_i \beta^i(x,t)Q_i^{(L)}} : \rho_{\beta(x,t)}, \quad Q_i^{(L)} = \int_0^L dx q_i(x)
\]
Hydrodynamic linear response

Then averages of densities and currents

\[ q_i(x, t) = \text{Tr} \rho \beta(x, t) q_i, \quad j_i(x, t) = \text{Tr} \rho \beta(x, t) j_i \]

satisfy continuity equation

\[ \partial_t q_i(x, t) + \partial_x j_i(x, t) = 0 \]

This is an equation for \( q_i \) using the bijection \( \{q_i\} \leftrightarrow \{\beta^i\} \).
Hydrodynamic linear response

Linear response theory: take a state that is nearly stationary, \( \rho \propto e^{-W - \int dx \delta \beta^i(x) q_i(x)} \), then small disturbance propagates according to linearised hydrodynamics

\[
\partial_t \delta q_i(x, t) + A_{ij} \partial_x \delta q_j(x, t) = 0, \quad A_{ij} = \left. \frac{\delta j_i}{\delta q_j} \right|_{\text{stationary}} e^{-W}
\]
**Hydrodynamic linear response**

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which you can diagonalise to normal modes \( \delta n_i(x, t) \) with velocities \( v^\text{eff}_i \in \text{spec}(A) \subset \mathbb{R} \), i.e. \( \partial_t \delta n_i + v^\text{eff}_i \partial_x \delta n_i = 0 \)

![Diagram of hydrodynamic linear response](image-url)
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This translates into a linear equation for correlation functions: reduction of degrees of freedom for calculating correlation functions!

\[
\partial_t \langle q_i(x, t) q_j(0, 0) \rangle^c + A^k_i \partial_x \langle q_k(x, t) q_j(0, 0) \rangle^c = 0
\]
Hydrodynamic linear response

In integrable systems of fermionic type: use GHD to get [BD, Spohn - SciPost 2017; BD - SciPost 2018]

\[
\langle q_i(\xi t, t) q_j(0, 0) \rangle^c \sim \frac{1}{t} \frac{\rho_p(p)(1 - n(p)) h_i^{\text{dr}}(p) h_j^{\text{dr}}(p)}{|v_{\text{eff}}'(p)|} \bigg|_{v_{\text{eff}}(p) = \xi}
\]

where \( \rho_p(p) \) is the Bethe root density, \( n(p) \) is the occupation function, \( \text{dr} \) is the TBA dressing operation, and \( h_i(p) \) is the one-particle eigenvalue of the charge \( Q_i \).
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- the GHD effective velocities \( v_{\text{eff}}(p) \) are identified with the hydrodynamic mode velocities
- the formula has been shown using / reproduces calculations from finite-density form factors in integrable models [De Nardis, Panfil - JSTAT 2018; Cortés Cubero, Panfil 2019]
- in the XX model, free fermions, \( v_{\text{eff}}(p) = 4 \sin p \) is the group velocity from the dispersion relation, and the dressing is trivial.

For instance for \( q_0(x) = \sigma_3^x \), we take \( h_{0}^{dr}(p) = h_{0}(p) = 1 \).
Hydrodynamic linear response

The need for fluid-cell averaging:

But wait, in the XX model, by Wick’s theorem and saddle point analysis ($\xi \in (-4, 4)$)

$$\langle \sigma^3_{\xi t}(t)\sigma^3_0(0) \rangle^c \sim \frac{2}{\pi |t| \sqrt{16 - \xi^2}} \sum_{a=\pm} \times$$

$$\times n_a \left(1 - n_a + ai (1 - n_{-a})(-1)^x e^{-2ai(x \arcsin(\xi/4)+t\sqrt{16-\xi^2})} \right)$$

where

$$n_\pm = \frac{1}{1 + \exp \left[ \pm \beta \sqrt{16 - \xi^2} \right]}$$

There is an oscillatory term!
Hydrodynamic linear response

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There is an oscillatory term!

Hydrodynamic results for correlation functions are valid under fluid cell averaging:

$$\langle \bar{a}(\ell x, \ell t) \cdots \rangle^c \text{ for } \bar{a}(\ell x, \ell t) = \frac{1}{LT} \int_{-L/2}^{L/2} dy \int_{-T/2}^{T/2} ds a(\ell x + y, \ell t + s)$$
Hydrodynamic linear response

Rigorous result: the general form of linearised Euler equation holds for every translation invariant quantum chains with short range interactions.

Fluid-cell mean: time average, space average

\[ S_{a,b}(\kappa) = \lim_{T \to \infty} \frac{1}{T - T_0} \int_{T_0}^{T} dt \sum_{x \in \mathbb{Z}} e^{i\kappa x/t} \langle a(x, t)b(0, 0) \rangle^c \]

**Theorem:** then linearised Euler equation holds

\[ \frac{d}{d\kappa} S_{q_i, q_j}(\kappa) = iA_{k}^i S_{q_k, q_j}(\kappa) \]

[BD - CMP 2021]
Hydrodynamic linear response

Equal-time total connected correlator:

\[(a, b) := \sum_{x \in \mathbb{Z}} \langle a(x) b(0) \rangle^c\]

Positive semidefinite \((a, a) \geq 0 \rightarrow\) inner product on equivalence classes \(\{a(x, 0) : x \in \mathbb{Z}\}\) \(\rightarrow\) Hilbert space \(\mathcal{H}\) of extensive observables. Time evolution \(\tau_t : \{a(x, 0)\} \mapsto \{a(x, t)\}\) is unitary on \(\mathcal{H}\) (by stationarity of the state and Lieb-Robinson bound).
Hydrodynamic linear response

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→ Hilbert space \(\mathcal{H}\) of **extensive observables**. Time evolution \(\tau_t : \{a(x, 0)\} \mapsto \{a(x, t)\}\)
is unitary on \(\mathcal{H}\) (by stationarity of the state and Lieb-Robinson bound).

**Conserved quantities** are all the extensive observables that are invariant under \(\tau_t\)

\[
\mathcal{Q} = \{ A \in \mathcal{H} : \tau_t A = A \forall t \}
\]

Then \(q_k\) are just a basis in the closed subspace \(\mathcal{Q}\)

\[
\sum_k A^k_i S_{q_k,q_j}(\kappa) = S_{\mathbb{P}j_i,q_j}(\kappa) : \text{projection } \mathcal{H} \rightarrow \mathcal{Q} \text{ and sum over a basis.}
\]
Fluctuations

Fluctuations: consider the transport of conserved quantities from left to right

$$\langle e^{\lambda \int_0^T dt j_i(0,t)} \rangle \approx e^{TF(\lambda)} \quad (T \to \infty)$$
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\[ \langle e^{\lambda \int_0^T dt j_i(0,t)} \rangle \approx e^{TF(\lambda)} \quad (T \to \infty) \]

\( F(\lambda) \) generates the scaled cumulants, which are time-integrated connected correlation functions (with \( j_i(t) = j_i(0, t) \))

\[
F(\lambda) = \langle j_i \rangle + \lambda \int_{-\infty}^{\infty} dt \langle j_i(0) j_i(t) \rangle^c + \frac{\lambda^2}{2} \int_{-\infty}^{\infty} dt_1 \int_{-\infty}^{\infty} dt_2 \langle j_i(0) j_i(t_1) j_i(t_2) \rangle^c + \ldots
\]
Fluctuations

Using hydrodynamic linear response, one finds that large-scale fluctuation of total currents is controlled by linear waves passing through the point $x = 0$: each mode contributes positively or negatively according to its velocity, in order to form a “new” GGE that knows about the insertion of the time-integrated current in the exponential

$$F(\lambda) = \int_0^\lambda d\lambda' j_i[\beta_{\lambda'}], \quad j_i[\beta_{\lambda}] = \frac{1}{Z} \text{Tr} e^{-\beta_{\lambda}^k Q_k j_i}, \quad \frac{d}{d\lambda} \beta_{\lambda}^j = -\text{sgn}(A[\beta_{\lambda}])^j_i$$

[BD, Myers - AHP 2020 (AHP-Birkhauser prize 2020)]
**Fluctuations**

**Example:** TASEP $j[\rho] = \rho(1 - \rho), \quad A(\rho) = 1 - 2\rho, \quad -\frac{\partial \rho}{\partial \beta} = \rho(1 - \rho)$

we reproduce the known results ($\rho < 1/2$) [de Gier, Essler - PRL 2011; Lazarescu, Mallick - JPA 2011]

$$F(\lambda) = \frac{(1 - \rho)(e^\lambda - 1)}{\rho(e^\lambda - 1) + 1}$$
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Example: energy transport in hard rods: GHD [Myers, Bhaseen, Harris, BD - SciPost 2019]
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Example: Free fermion: reproduces Levitov-Lesovik formula


Example: box-ball system, shown analytically [Kuniba, Misguich, Pasquier - 2020]
Fluctuations

More generally, one can obtain asymptotics of twist field correlation functions.

XX model: written in terms of free fermions $a_x(t), a_x^\dagger(t)$, the spin variables $\sigma^\pm$ are expressed using Jordan-Wigner strings. It is argued in [Del Vecchio Del Vecchio, BD - 2021] that this can be recast into space-time Jordan Wigner strings

$$\langle \sigma_x^+(t) \sigma_0^- (0) \rangle \simeq \langle a_x^\dagger(t) a_0(0) \rangle_{\beta \pi} \langle \exp i \pi \int_{0,0}^{x,t} ds \mu j_0^\mu (\vec{s}) \rangle, \quad j_0^\mu (x, t) : \text{spin 2-current}$$
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$$\langle \sigma_x^+(t)\sigma_0^-(0) \rangle \asymp \langle a_x^\dagger(t)a_0(0) \rangle_{\beta, \pi} \langle \exp i\pi \int_{0,0}^{x,t} ds_{\mu} j_0^\mu(\vec{s}) \rangle, \quad j_0^\mu(x, t) : \text{spin 2-current}$$

The scaled cumulant generating function on an arbitrary ray is known, here:

$$\langle \exp i\pi \int_{0,0}^{x,t} ds_{\mu} j_0^\mu(\vec{s}) \rangle \asymp e^{F(x,t)}$$

where

$$F(x, t) = \int_0^{i\pi} d\lambda' (t j_0[\beta_{\lambda'}] - x q_0[\beta_{\lambda'}]), \quad \frac{d}{d\lambda} \beta^j_\lambda = -\text{sgn}(t A[\beta_\lambda] - x 1)_j^j$$
Fluctuations

This reproduces the old results [Iits, Izergin, Korepin, Slavnov - PRL 1993; Jie. - Ph.D. Thesis 1998] and gives new results (last line)

\[
F(x, t) = \begin{cases} 
  f_{x,t} & (|\xi| \leq 4) \\
  |x| f_{1,0} & (|\xi| > 4, |h| \leq 2) \\
  -|x| \min\left(\arccosh(h/2), M_\xi\right) + |x| f_{1,0} & (|\xi| > 4, |h| > 2)
\end{cases}
\]

with \( M_\xi = \arccosh(\xi/4) - \sqrt{1 - \frac{16}{|\xi|^2}} \) and

\[
f_{x,t} = \int_{-\pi}^{\pi} \frac{dk}{2\pi} |x - v(k)t| \log \left| \tanh \frac{\beta E(k)}{2} \right|.
\]
Ballistic macroscopic fluctuation theory

[BD, Perfetto, Sasamoto, Yoshimura - in prep]

Take again long-wavelength initial state

\[ \langle \cdots \rangle_\ell : \rho \propto e^{-\int \text{d}x \beta^i(x/\ell)q_i(x)} \]

We can reproduces all Euler-scale correlations:

\[ \lim_{\ell \to \infty} \ell^{n-1} \langle \bar{a}_1(\ell x_1, \ell t_1) \cdots \bar{a}_n(\ell x_n, \ell t_n) \rangle^c_\ell \]
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\]

replacing fluid-cell-averaged observables by random variables \( \tilde{q}_i \):

\[
\overline{q}_i(\ell x, \ell t) \to \tilde{q}_i(x, t)
\]

with every observable functions of these given by their GGE values

\[
\overline{a}(\ell x, \ell t) \to \tilde{a}(x, t) = \frac{1}{Z} \text{Tr} e^{-\tilde{\beta}^i(x,t)Q_i} a \quad \text{(at } \tilde{q}_i = q_i[\tilde{\beta}]\text{)}
\]
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We can reproduce all Euler-scale correlations:

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\]

replacing fluid-cell-averaged observables by random variables \( \tilde{q}_i \), using the BMFT measure

\[
d\mathbb{P} = [d\tilde{q}(\cdot, \cdot)] \\
\quad \times \exp \left[ -\ell \int dx \left( \beta^i(x)(\tilde{q}_i(x, 0) - q_i(x)) + s[q(x)] - s[\tilde{q}(x, 0)] \right) \right] \\
\quad \times \delta[\partial_t \tilde{q} + \partial_x j[\tilde{q}]]
\]
Ballistic macroscopic fluctuation theory

That is:

$$\lim_{\ell \to \infty} \ell^{n-1} \langle \bar{a}_1(\ell x_1, \ell t_1) \cdots \bar{a}_n(\ell x_n, \ell t_n) \rangle^{\mathcal{C}}_\ell = \lim_{\ell \to \infty} \ell^{n-1} \int d\mathcal{P} \, \bar{a}_1(x_1, t_1) \cdots \bar{a}_n(x_n, t_n)$$

with

$$d\mathcal{P} = [d\mathcal{q}(\cdot, \cdot)]$$

$$\times \exp \left[ -\ell \int dx \left( \beta^i(x)(\ddot{q}_i(x, 0) - q_i(x)) + s[q(x)] - s[\ddot{q}(x, 0)] \right) \right]$$

$$\times \delta[\partial_t \ddot{q} + \partial_x j[\ddot{q}]]$$
Ballistic macroscopic fluctuation theory

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\lim_{\ell \to \infty} \ell^{n-1} \langle \bar{a}_1(\ell x_1, \ell t_1) \cdots \bar{a}_n(\ell x_n, \ell t_n) \rangle^\ell = \lim_{\ell \to \infty} \ell^{n-1} \int d\mathbb{P} \bar{a}_1(x_1, t_1) \cdots \bar{a}_n(x_n, t_n)
\]

Main principle: **separation of scales on fluctuations:**

- average out “quick” fluctuations within the microcanonical shell due to all kinds of non-conserving processes that happen in the bulk of the fluid cell
- keep “slow” fluctuations amongst different microcanonical shells due to fluctuations of conserved quantities that happens at the surface of the fluid cell.
Ballistic macroscopic fluctuation theory

\( \ell \to \infty \): saddle point equations in terms of an auxiliary “Lagrange parameter” \( H^i(x, t) \) for the delta function \( \delta[\partial_t q + \partial_x j] = \int [dH]e^{\ell \int_x dx \int_0^T dt H(\partial_t q + \partial_x j)}. \)

With source terms, here for conserved densities \( a_r = q_{k_r} \):

\[
\begin{align*}
H^i(x, 0) &= \beta^i - \tilde{\beta}^i \\
H^i(x, T) &= 0 \\
\partial_t \tilde{\beta}^i + \tilde{A}_j^i \partial_x \tilde{\beta}^j &= 0 \\
\partial_t H^i + \tilde{A}_j^i \partial_x H^j &= \sum_{r=1}^{n} \lambda_r \delta_{k_r}^i \delta(x - x_r) \delta(t - t_r)
\end{align*}
\]
Ballistic macroscopic fluctuation theory

ℓ → ∞: saddle point equations in terms of an auxiliary “Lagrange parameter” $H^i(x, t)$ for the delta function $\delta[\partial_t q + \partial_x j] = \int [dH] e^{\ell} \int_{R} dx \int_0^T dt H(\partial_t q + \partial_x j)$.

With source terms, here for conserved densities $a_r = q_{k_r}$:

\[ H^i(x, 0) = \beta^i - \tilde{\beta}^i \]
\[ H^i(x, T) = 0 \]
\[ \partial_t \tilde{\beta}^i + \tilde{A}_j^i \partial_x \tilde{\beta}^j = 0 \]
\[ \partial_t H^i + \tilde{A}_j^i \partial_x H^j = \sum_{r=1}^n \lambda_r \delta^i_{k_r} \delta(x - x_r) \delta(t - t_r) \]

- Reproduces linear response $\partial_t \langle q_i(x, t) q_j(0, 0) \rangle + A^k_i \partial_x \langle q_k(x, t) q_j(0, 0) \rangle = 0$
- Reproduces $c_2, c_3$ and can be argued to reproduce the full $F(\lambda)$
- Gives the Cohen-Gallavotti fluctuation relations (see Takato’s talk)
Ballistic macroscopic fluctuation theory

In particular, the theory implies that in interacting models with at least two different hydrodynamic velocities, from non-stationary state, there are long range correlations:

$$\lim_{\ell \to \infty} \ell \langle \bar{q}_i(\ell x, \ell t) \bar{q}_j(0, \ell t) \rangle^c_{\ell} = f(x, t) \neq 0$$

The density matrix does not take the exponential form at later times even at the Euler approximation scale,

$$\rho(t) \propto e^{-\int dx \beta(x/\ell, t/\ell) q_i(x)}$$

as instead there are correlations between fluid cells.
Ballistic macroscopic fluctuation theory

Example: evolution of two-velocity $p = \pm 1$ hard rod gas, from bump initial condition

$$\ell \langle q_0(\ell x, \ell/2)q_0(0,\ell/2) \rangle^c$$
Conclusion

Hydrodynamics gives a lot of general principles that can predict / reproduce asymptotic of many types of correlation functions at large space-time separations.

To do (cf grant applications):

- revisit recent nonlinear response results in light of BMFT and the new type of long-range correlations we uncovered
- apply fluctuation formalism to twist fields for entanglement entropy (work in progress with V. Alba, G. Del Vecchio Del Vecchio, P. Ruggiero)
- generalise fluctuation formalism / BMFT of integrable systems to diffusive scale (and beyond?)
- apply BMFT to non-integrable systems and reproduce KPZ scaling
- prove rigorously the large-scale correlation function formula in integrable models (prove that space of conserved charges $Q$ is the spanned by Bethe quasiparticles)