Nested paths in 2D percolation

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Joint work with Youjin Deng, Jesper Jacobsen, Yu-Feng Song

An exercise in Google Translate

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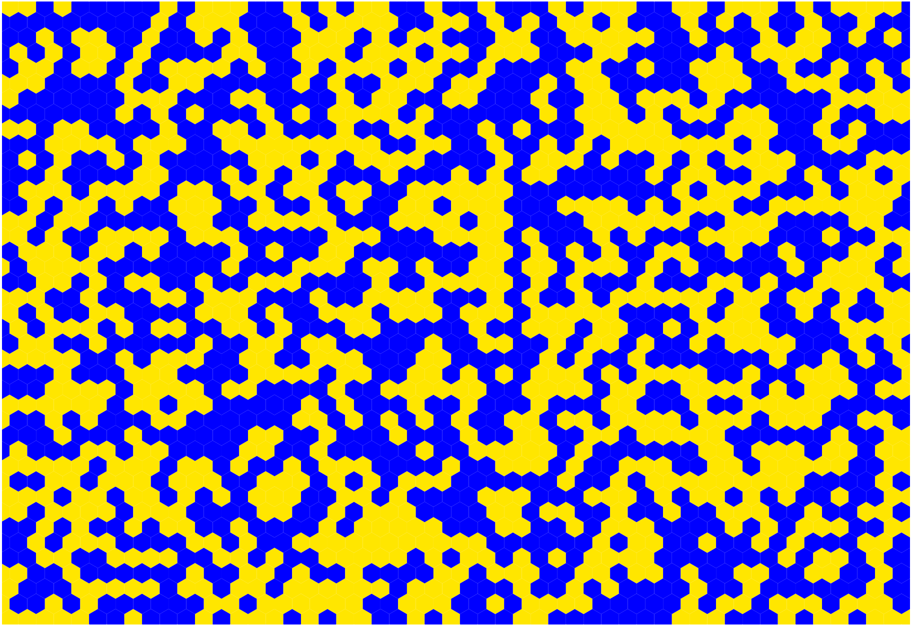
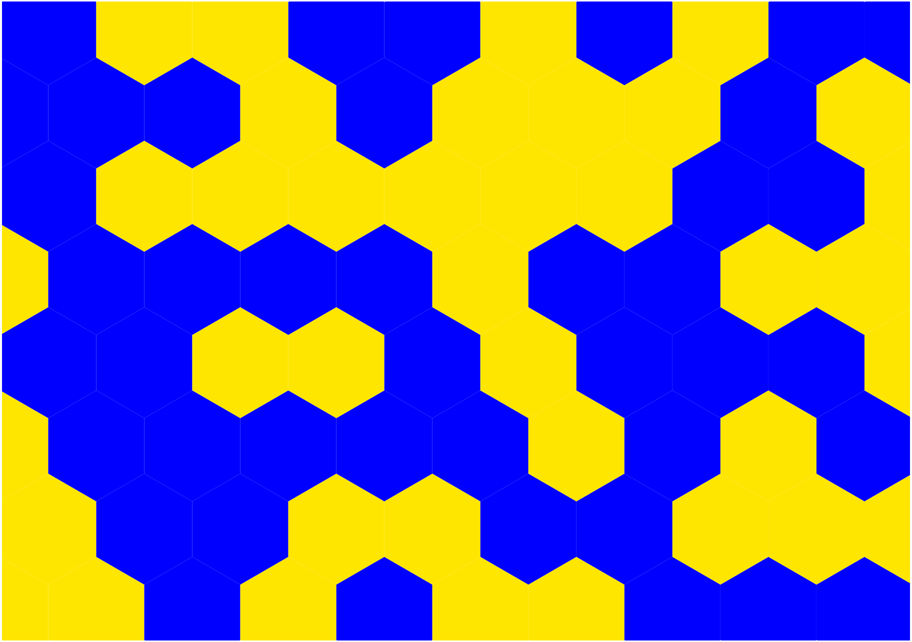
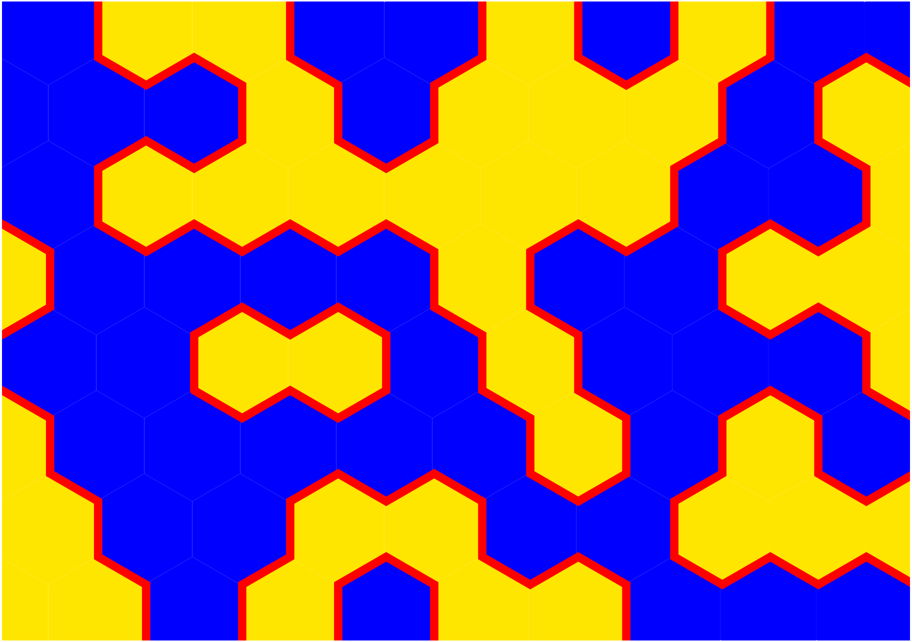
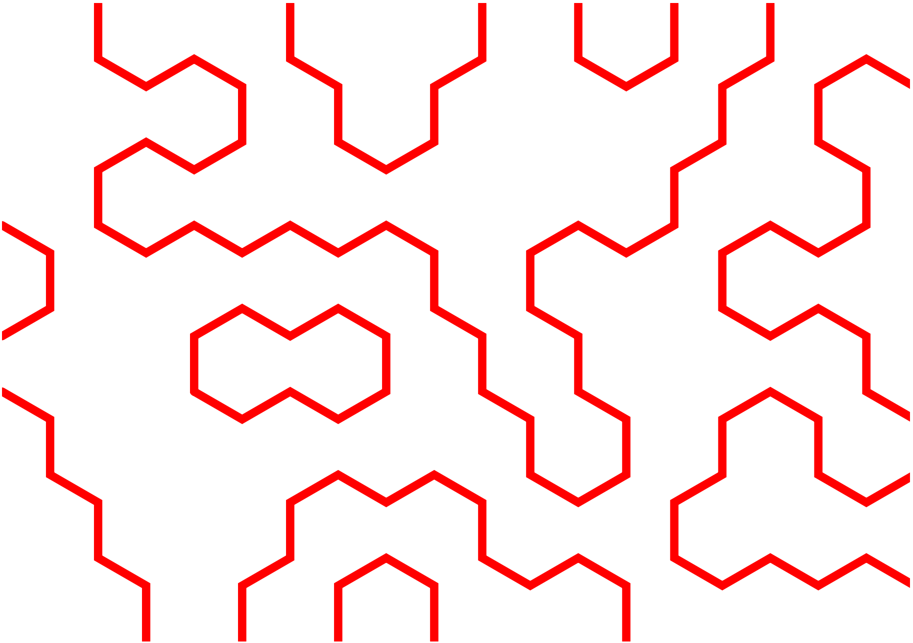
Everything is visual









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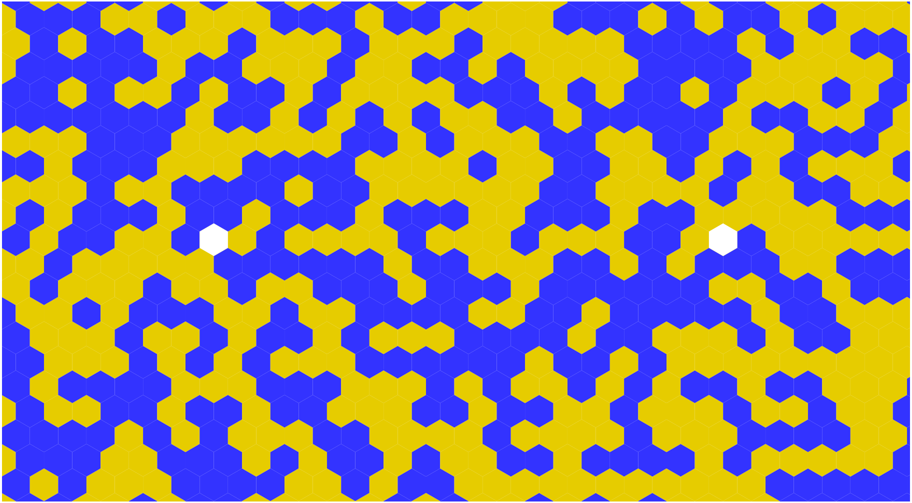
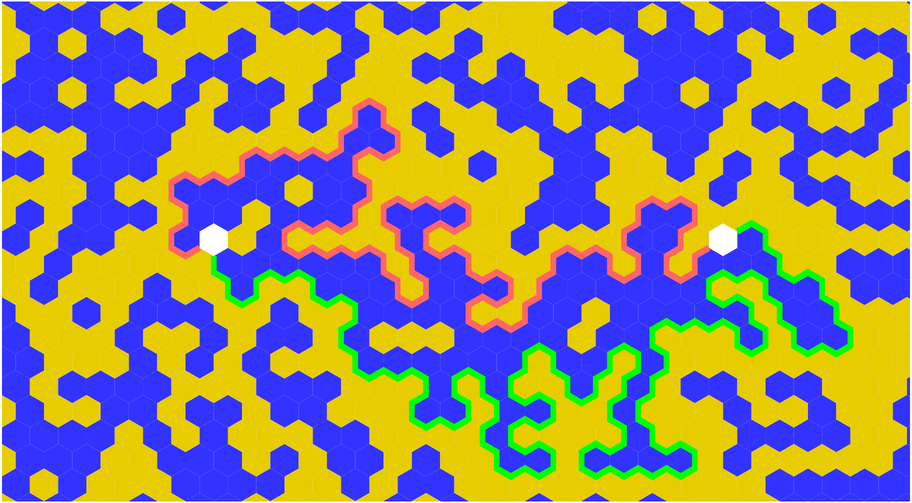
This talk

- Focus on point operators at the phase transition in 2D
- and on their critical exponents (conformal weights)
- Rehearse some known families of operators
- Introduce a new family & study its properties

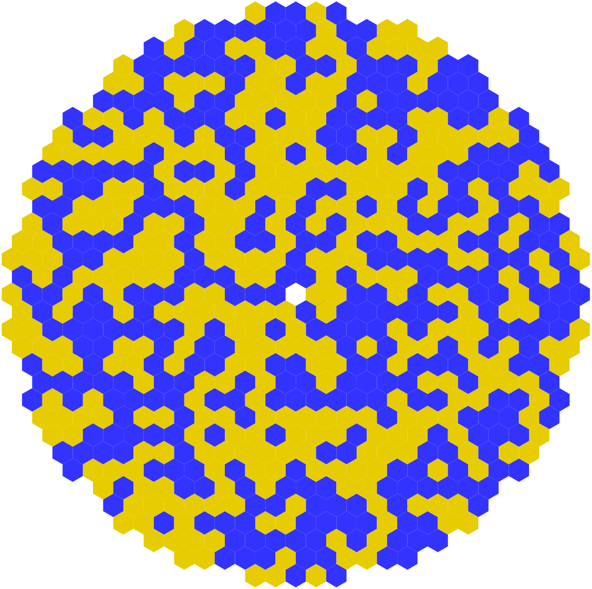


A two-point function: Insertion of two point-operators

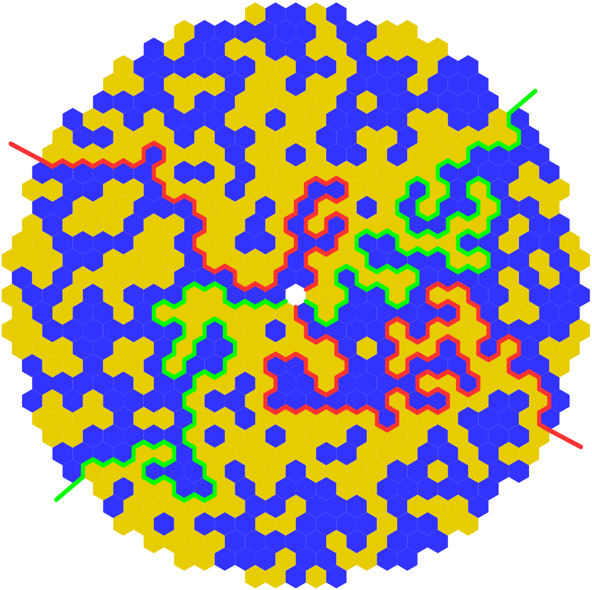


A two-point function:
The probability that, say N, domain wall connect both points.

2-point functions decay as a power of the distance d: $d^{-2X_{\mathrm{WM}}(N)}$



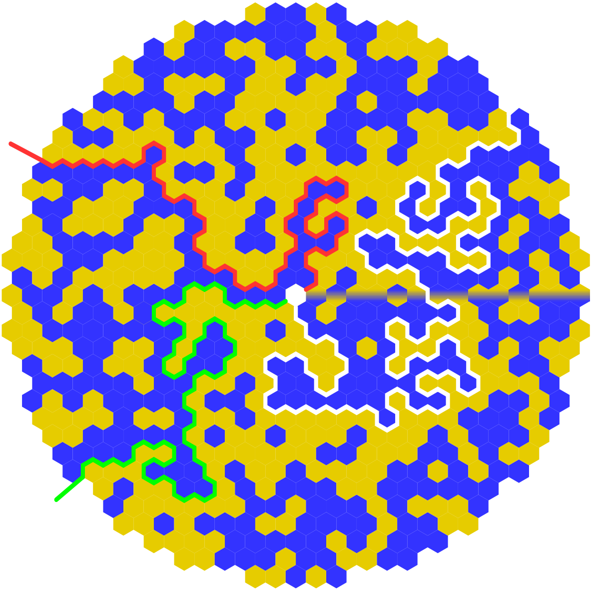
The one-point function of this operator measures the probability that N domain walls run from the center (operator insertion point) to the boundary.



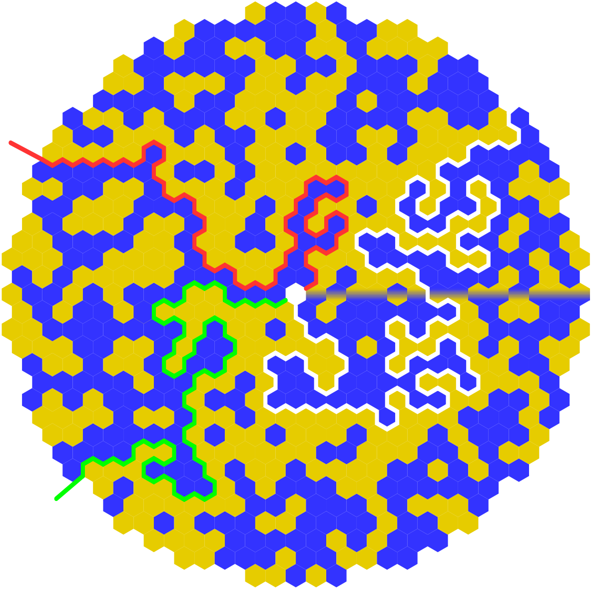
The one-point function of this operator measures the probability that N domain walls run from the center (operator insertion point) to the boundary.

This configurations contributes to the case $N = 4$.

Naturally N is always even.



The symmetry between open en closed elements *in this model*, permits the introduction of anti-cyclic closure, thus allowing odd N.



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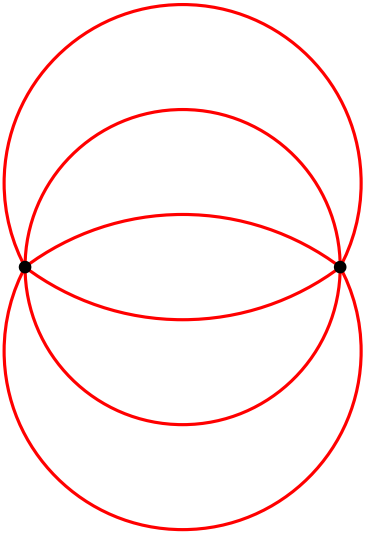
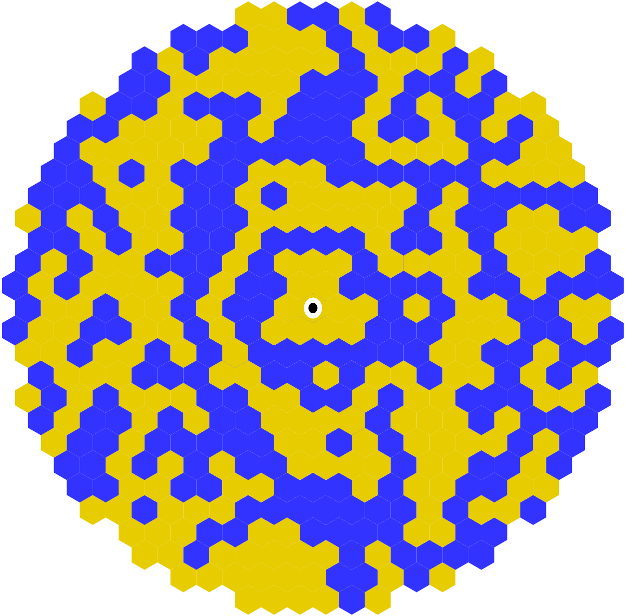
One-point functions decay with the disk radius r: $r^{-X_{\mathrm{WM}}(N)}$

The exponent $X_{\mathrm{WM}}(N)$ is known as the *watermelon* exponent suggested by the cartoon of the two-point diagrams.

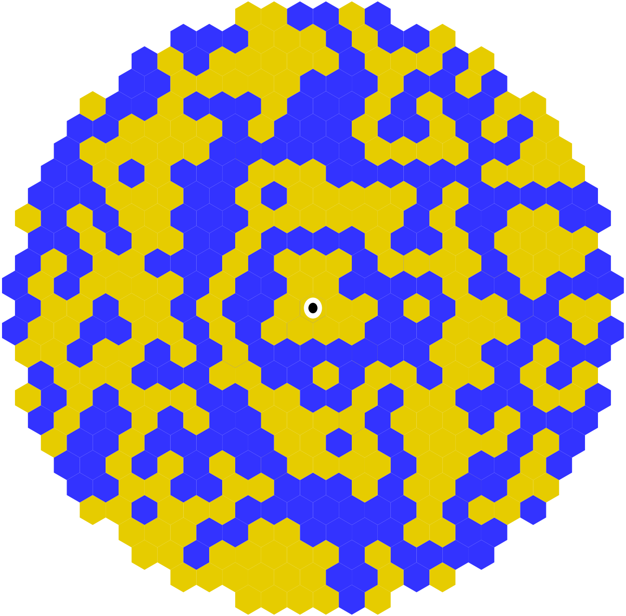


The value:

$$X_{\mathrm{WM}}(N) = \frac{N^2 - 1}{12}$$



Another operator relates to closed domain walls (loops), not terminating at the insertion point, but encircling it.



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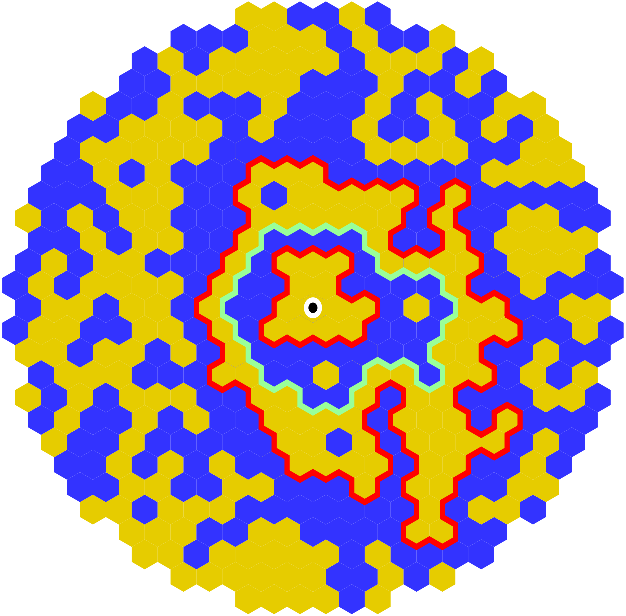
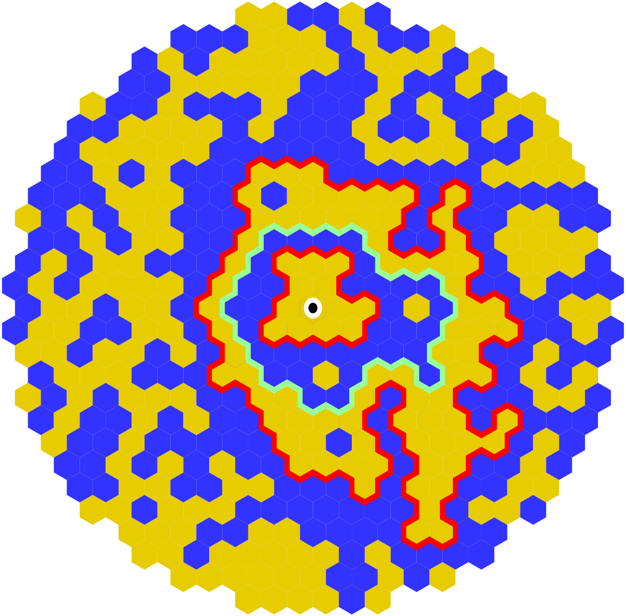
We again find power law behavior, not in the probability, P_N, of finding N such loops but in its generating function.



In this example three domain walls surround the center of the disk.



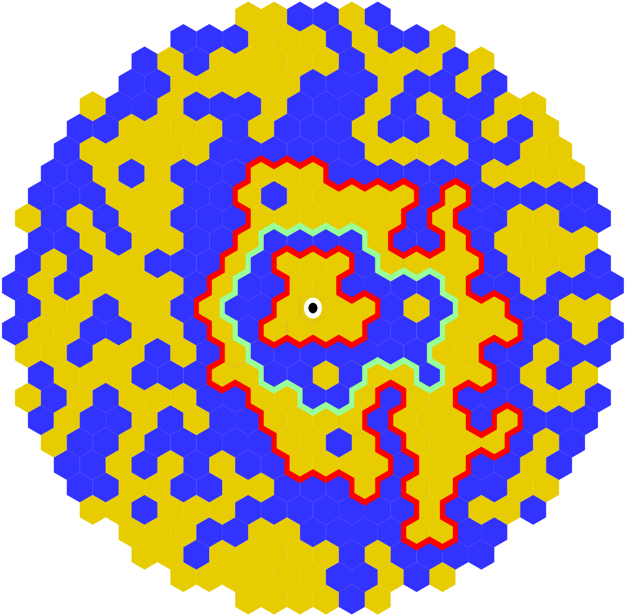
In this example three domain walls surround the center of the disk.

The generating function

$$W_z = \sum_n P_n \, z^n$$

depends on the disk radius as

$$W_z(r) \propto r^{-X_{\rm NL}(z)}$$



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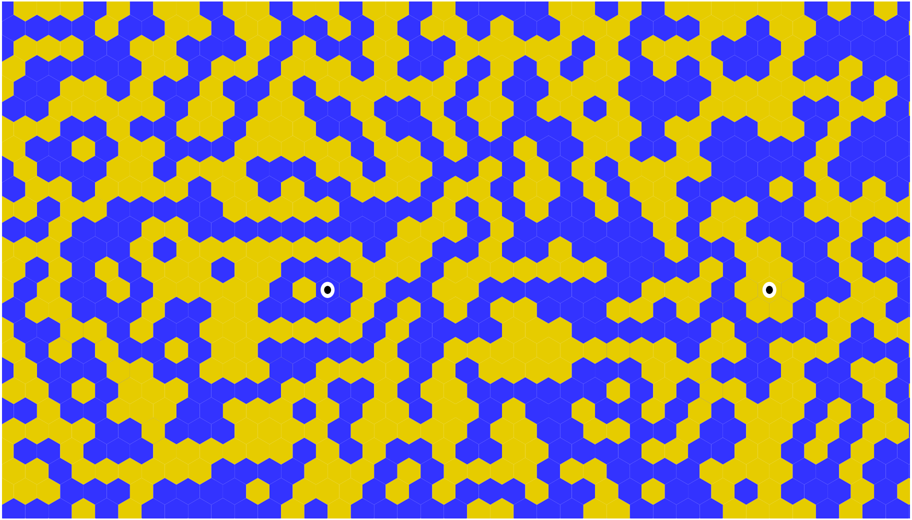
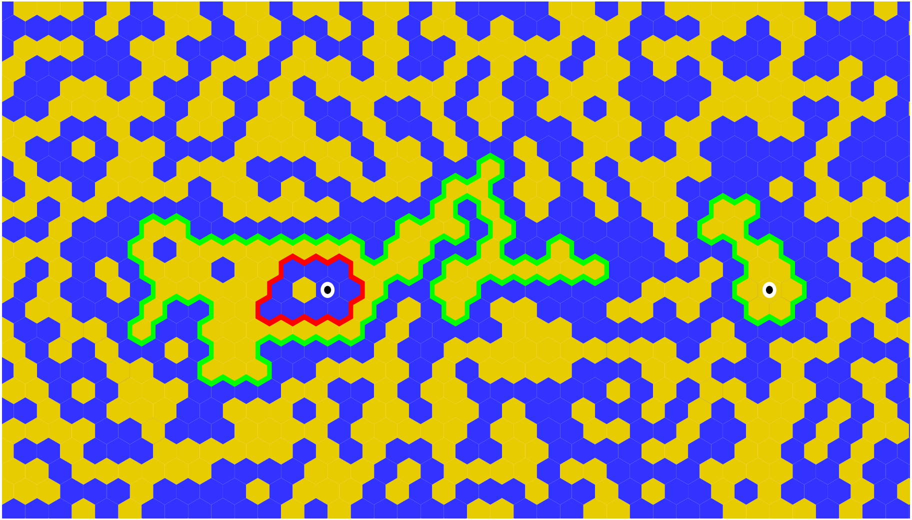


In the two-point function the relevant loops separate the two points.



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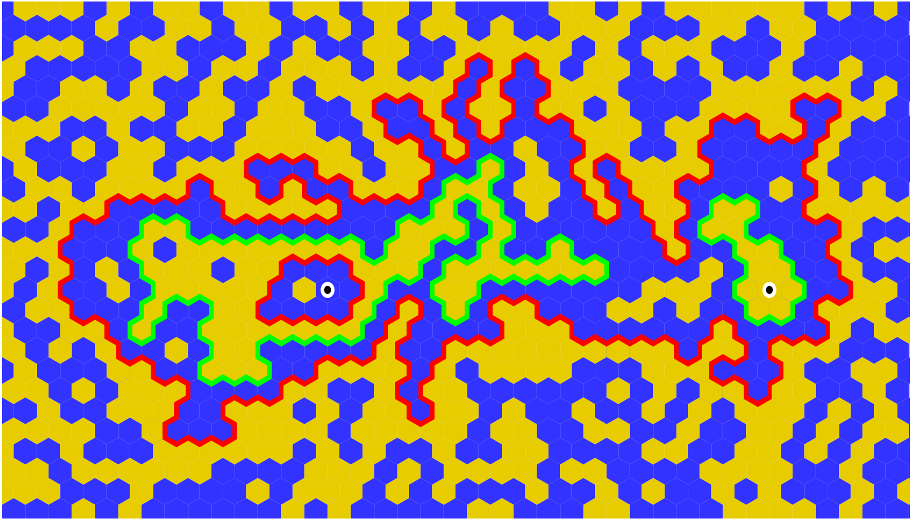

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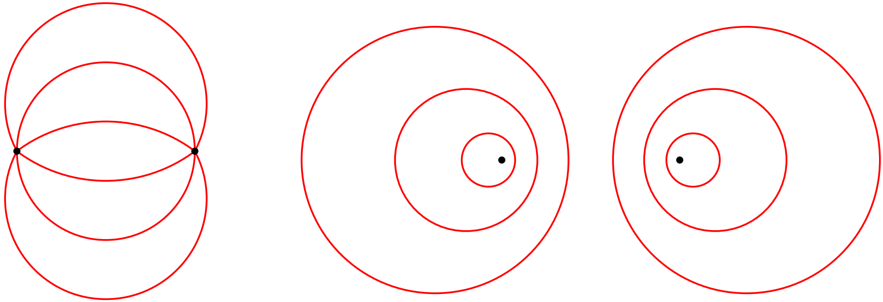
They surround only one of the two.

But the loops surrounding both, are not counted (i.e. given weight z).



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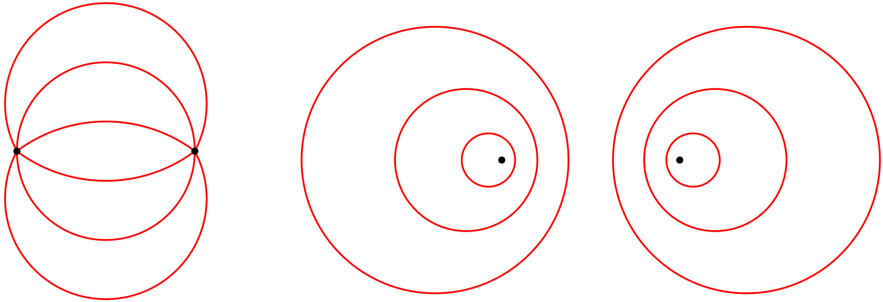

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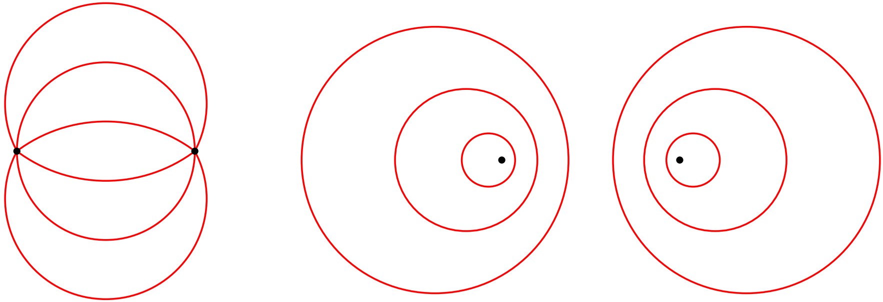
$$X_{\mathrm{NL}}(z) = \frac{3}{4}\phi^2 - \frac{1}{12} \qquad \text{where} \qquad z = 2\cos\phi\pi$$



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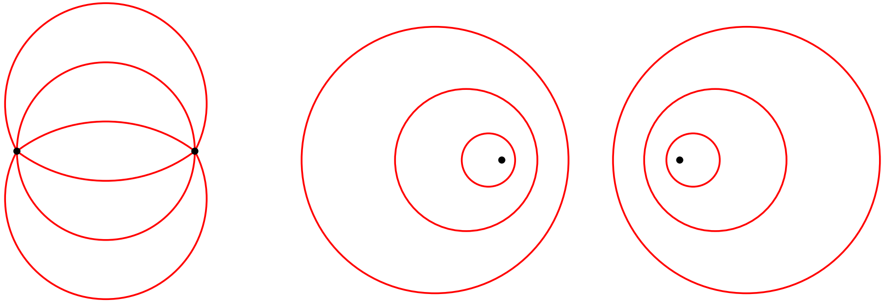
For $\phi = \frac{\pi}{3}$, the weight $z = 1$, and the exponent $X_{\mathrm{NL}}(1) = 0$ as expected.



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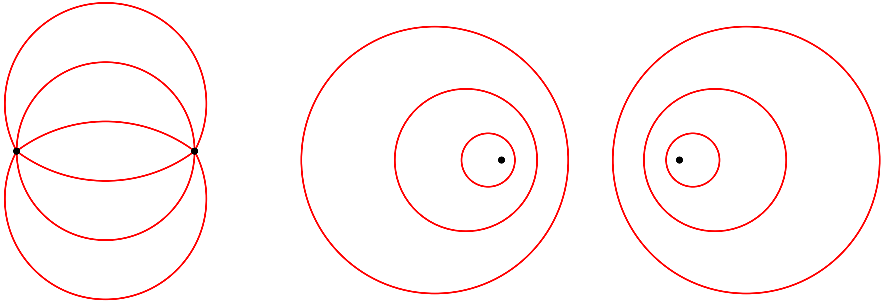

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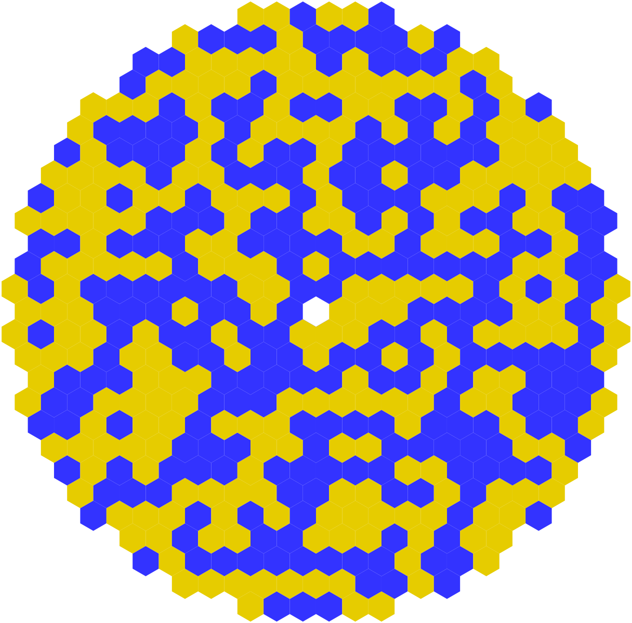
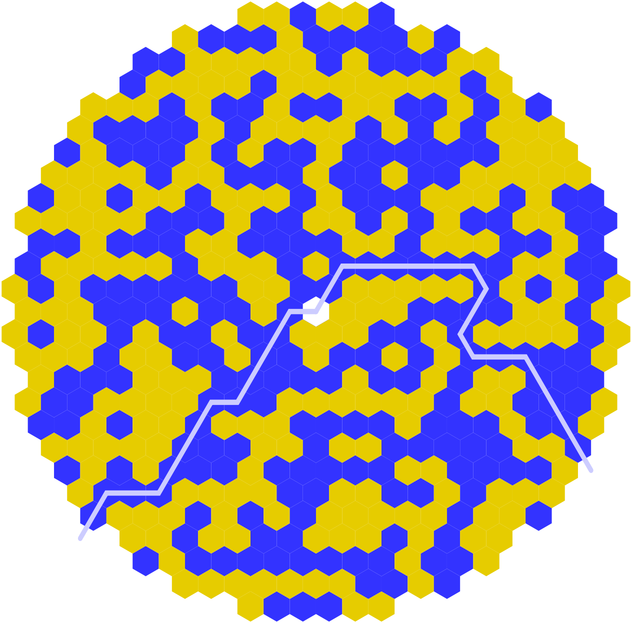
$z = 0$ selects configurations with at least one path (between insertion points) over hexagons of the same color.



In this example there is indeed a path from the center to the boundary over blue hexagons.

Many different paths are possible.



In this example there is indeed a path from the center to the boundary over blue hexagons.

Many different paths are possible.

But, in this case, only two non-overlapping paths at the same time

This exponent has been named Monochromatic Arm exponent, $X_{\mathrm{MA}}(N)$
Its value is not known analytically, except for $N = 1$.

But especially $X_{\mathrm{MA}}(2)$ is well studied, usually called the backbone exponent, from its relation to the percolating cluster without its singly connected elements.

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Some values:

N	$X_{\mathrm{MA}}(N)$	$(1+4N^2)/48$
1	5/48	0.10417
2	0.35435	0.35417
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but disagrees with the best estimates of
$X_{\mathrm{MA}}(2) = 0.3569 \pm 0.0006$ (Jacobsen, Zinn-Justin, 2002)
$X_{\mathrm{MA}}(2) = 0.3566 \pm 0.0001$ (Xu, Wang, Zhou, Garoni, Deng, 2014)

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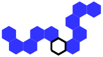
Somewhat counter-intuitively: $\mathbb{P}($  $) > \mathbb{P}($  $)$

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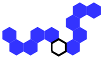
Somewhat counter-intuitively: $\mathbb{P}($  $) > \mathbb{P}($  $)$

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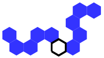
Somewhat counter-intuitively: $\mathbb{P}($  $) > \mathbb{P}($  $)$

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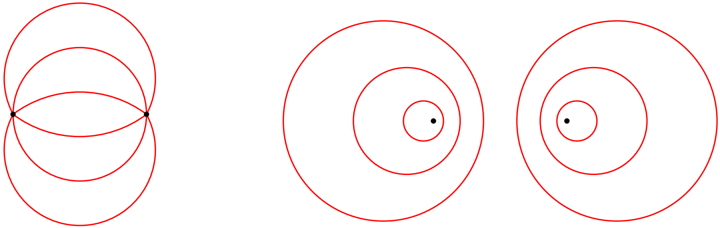
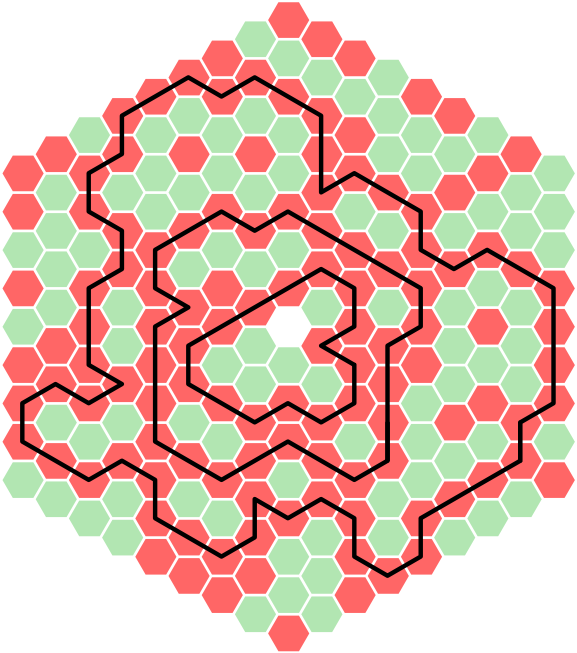
Proof that $X_{\mathrm{BA}}(N) = X_{\mathrm{WM}}(N)$ claimed by Aizenman, Duplantier & Aharony PRL 1999.



We studied domain walls connecting distant points, as well as separating them.
Why not do the same with percolation paths?

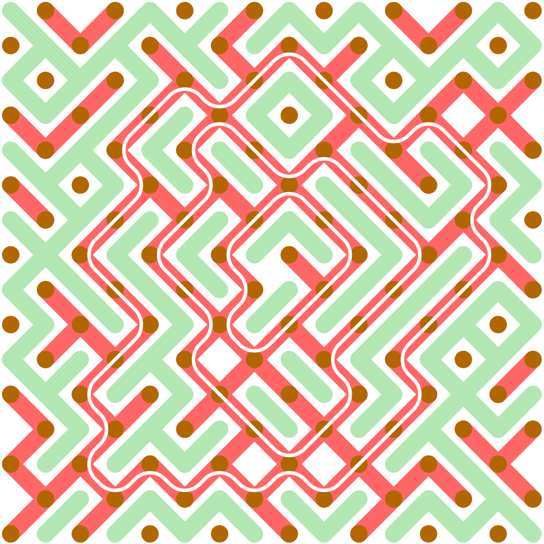
It is natural to expect that this gives another family of universal percolation exponents.



Conventions:
(for 1-point fn.)

Count possible paths surrounding the center

All in one cluster connecting the center to the boundary.

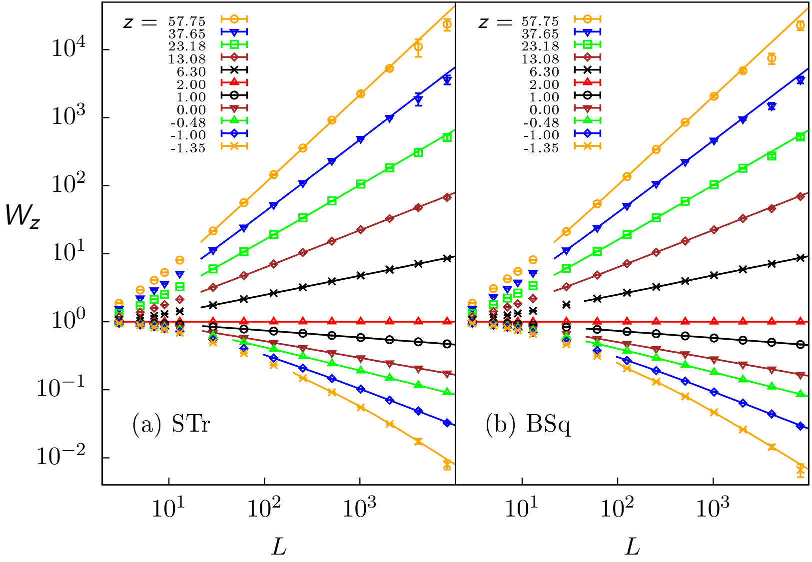


To test universality we do the same with bond percolation.

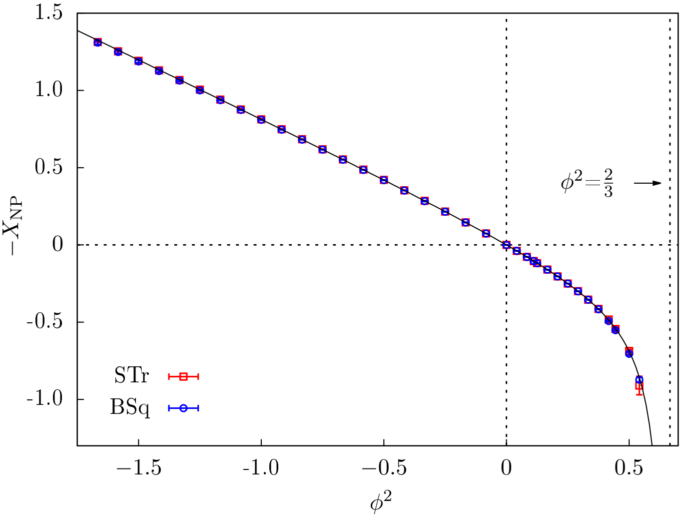
The opposite clusters are now on dual lattice.

non-overlapping now means no edge in common

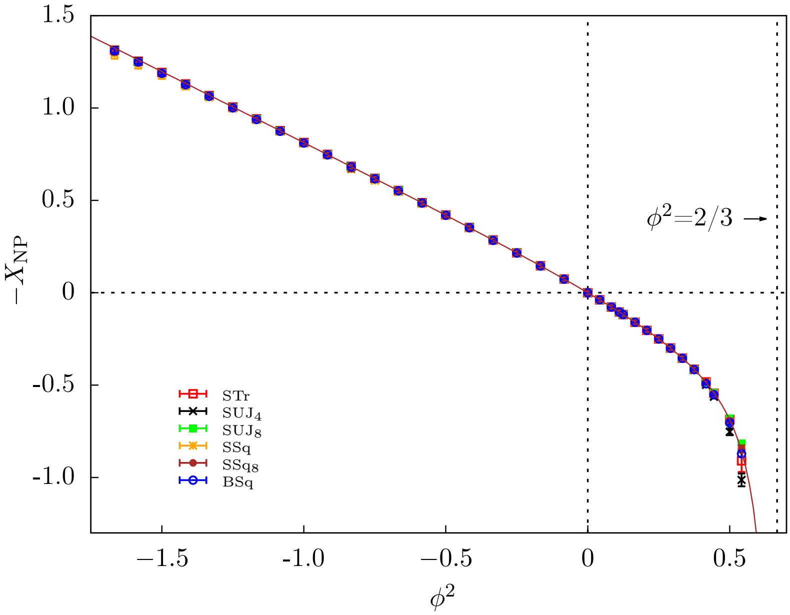
different paths may pass the same site



labels: STr for site percolation on the triangular lattice
BSq for bond percolation on the square lattice.



In analogy with the exonent $X_{\mathrm{NL}}(z) = \frac{3}{4}\phi^2 - \frac{1}{12}$ for $z = 2\cos(\phi\pi)$, we also plot $X_{\mathrm{NP}}(z)$ versus ϕ^2.



To test universality we did the computation for a few more lattices.

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$z = 0$ Forbidding paths around the center, while demanding a path from the center to the boundary, effectively enforces two bichromatic paths from the center to the boundary.
$X_{\mathrm{NP}}(0) = X_{\mathrm{BA}}(2) = X_{\mathrm{WM}}(2) = 1/4$

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$z = 1$ Ignoring paths around the center, while demanding a path from the center to the boundary: $X_{NP}(1) = X_{MA}(1) = X_{NL}(0) = 5/48$

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And what do we see?

$z \to \infty$ An asymptotic slope of $3/4$, equal to that of $X_{\mathrm{NL}}(z)$.

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$z = 1$ Ignoring paths around the center, while demanding a path from the center to the boundary:
$X_{\mathrm{NP}}(1) = X_{\mathrm{MA}}(1) = X_{\mathrm{NL}}(0) = 5/48$

And what do we see?

$z \to \infty$ An asymptotic slope of $3/4$, equal to that of $X_{\mathrm{NL}}(z)$.

$z = 2$ Or $\phi = 0$. Strong suggestion that $X_{\mathrm{NP}}(2) = 0$. (proof later)

In an attempt to guess $X_{\mathrm{NP}}(z)$, we assumed it is a simple function of ϕ^2.

What do we know?

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$z < -1$ Some singularity, perhaps a pole?

z	ϕ	$X_{\rm NP}$
1	1/3	1/4
0	1/2	5/48
∞	$i\infty$	$-3/4\ \phi^2$
2	0	0
< -1	$> .7$	pole

Proposal:

$$X_{\rm NP}(z) = \frac{3}{4}\phi^2 - \frac{a\ \phi^2}{\phi^2 - b}$$

The rational function is chosen to agree with the numerical observations (lines 3-5 of table).

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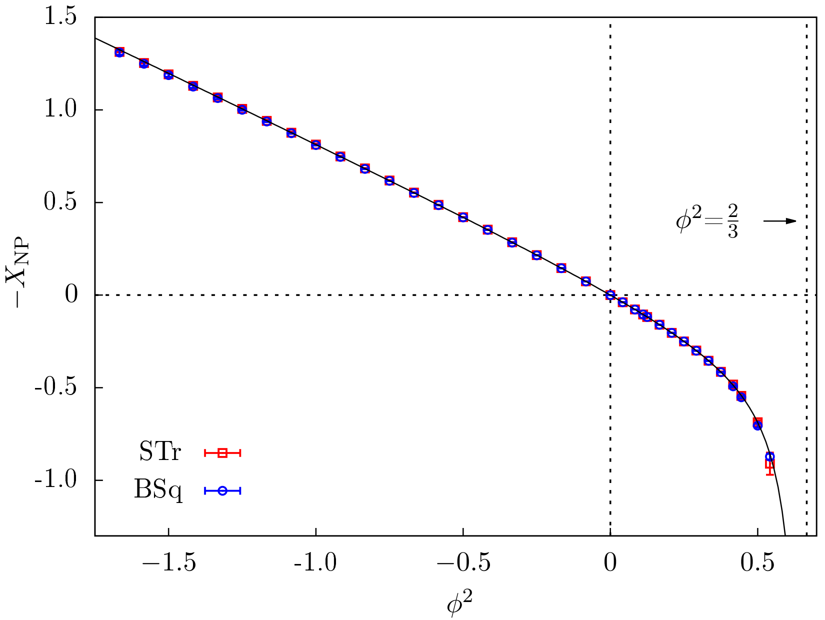
Formula looks credible, ageement with numerics is excellent, but I offer not even a trace of understanding.
The pole, and its position ($\phi = \sqrt{2/3}$) are a challenge to our faith.


1.5
1.0
0.5
0
-0.5
-1.0
$-X_{\rm NP}$
-1.5
-1.0
-0.5
0
0.5
ϕ^2
$\phi^2 = \frac{2}{3}$
STr
BSq

What remains is the status of $X_{\mathrm{NP}}(2) = 0$.

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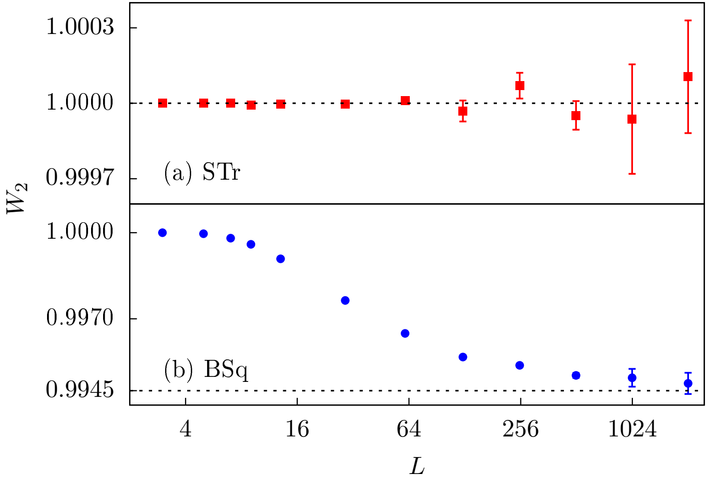

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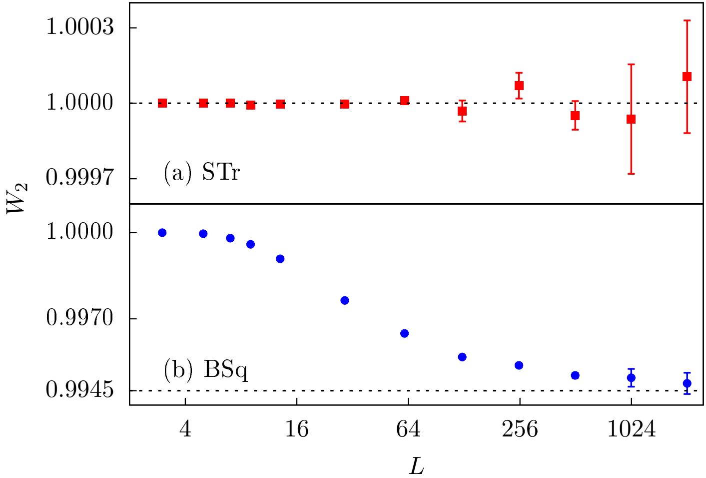

For STr up to diagonal $L = 7$, $W_2(L) = 1$ exactly, for larger L, data are consistent with $W_2(L) = 1$.

Now the proof

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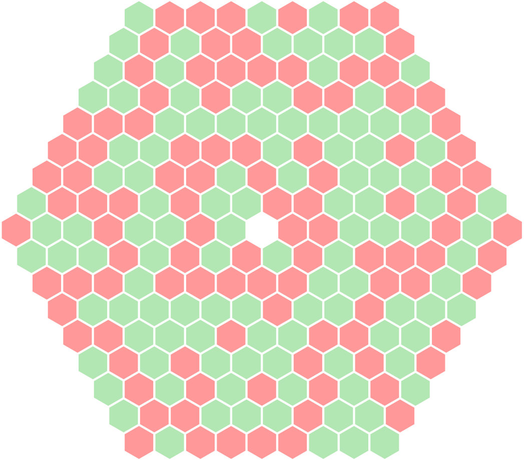
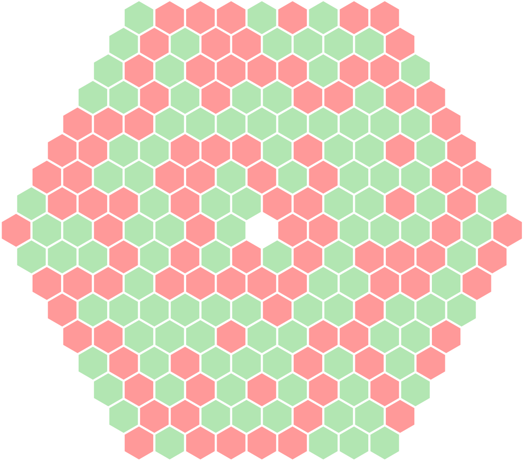


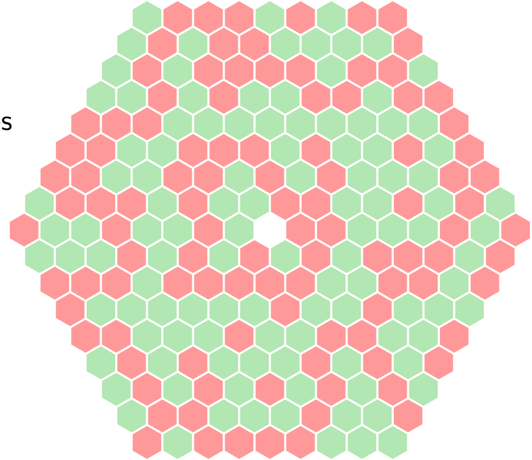
Now the proof

- Consider any configuration in STr percolation
- Consider the maximal set of bichromatic nested paths.



Now the proof

- Consider any configuration in STr percolation
- Consider the maximal set of bichromatic nested paths.
- Draw their unique innermost version given the interior ones



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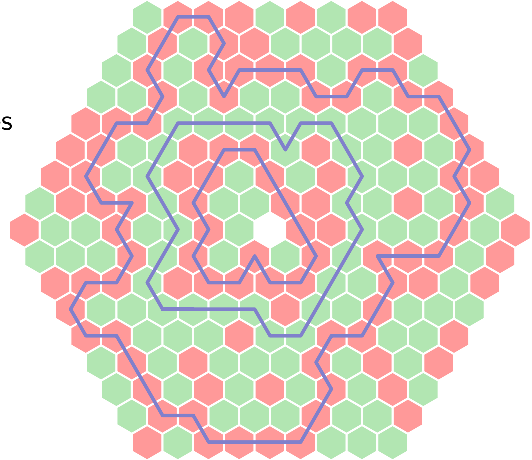
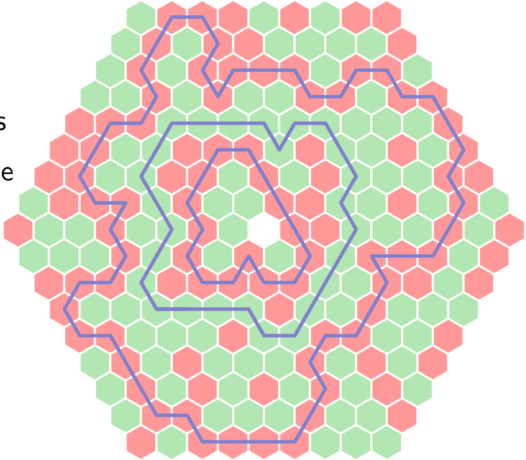


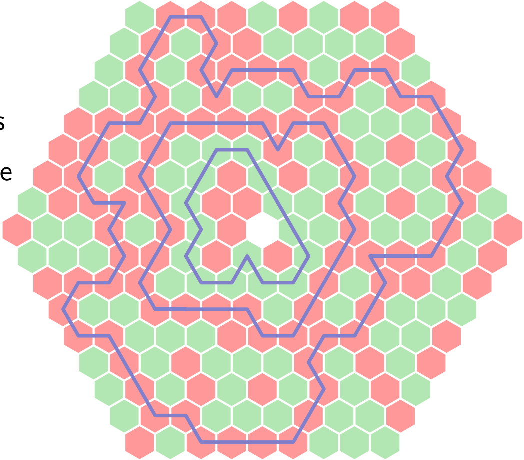
Now the proof

- Consider any configuration in STr percolation
- Consider the maximal set of bichromatic nested paths.
- Draw their unique innermost version given the interior ones
- Define P_n: the color-flip of the n-th path and its interior



Now the proof

- Consider any configuration in STr percolation
- Consider the maximal set of ℓ bichromatic nested paths.
- Draw their unique innermost version given the interior ones
- Define P_n: the color-flip of the n-th path and its interior



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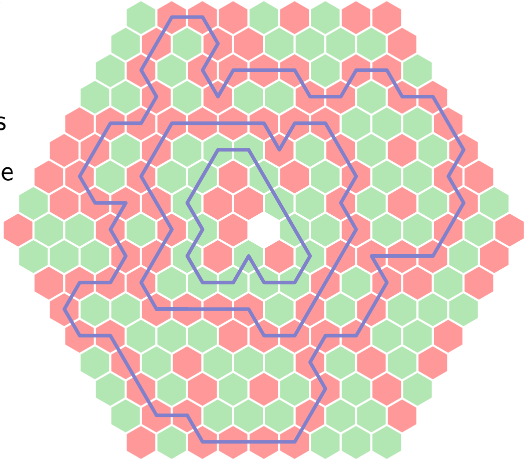

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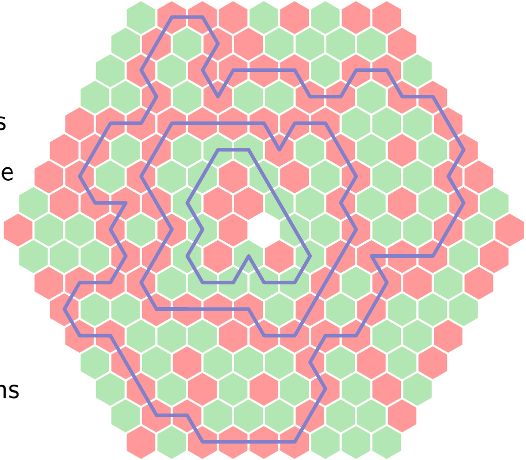

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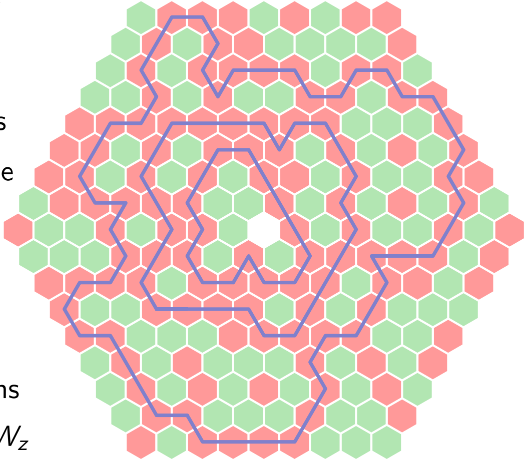
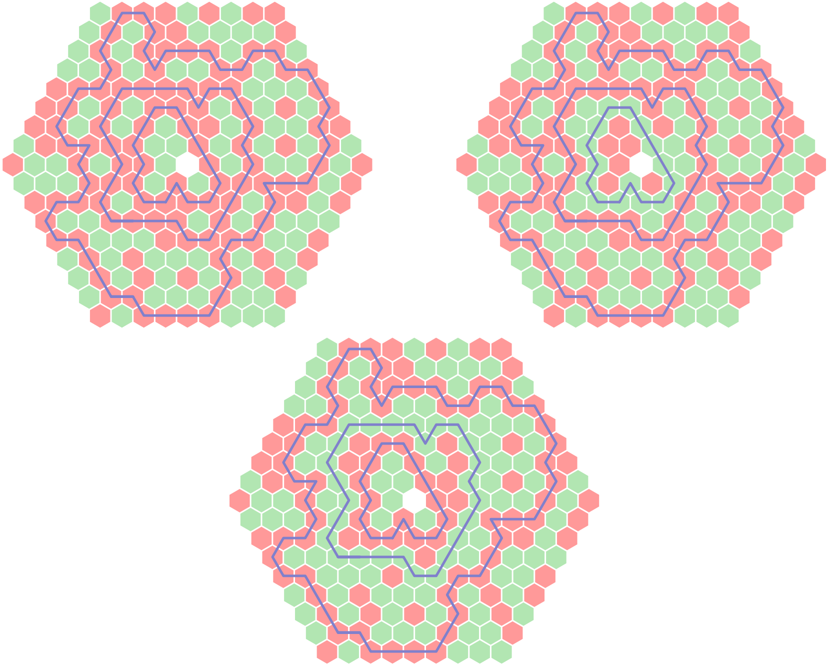
- One of the 2^ℓ has ℓ open paths
- Only this one contributes to W_z but with a multiplier z^ℓ.
- Therefore $W_2 = 1$ □





Summary & outlook

- WM, NL, MA operators complemented with NP.
- $X_{\mathrm{NP}}(z) = \frac{3}{4}\phi^2 - \frac{5}{48}\frac{\phi^2}{\phi^2-2/3}$
- proof that $X_{\mathrm{NP}}(2) = 0$, or even that $W_2(L) = 1$
- Beffara & Nolins proposal for $X_{\mathrm{MA}}(N) = \frac{4N^2+1}{48}$.
- statistics on # nested paths can be derived and is tested.
- Generalization to Potts models, Kasteleyn Fortuin clusters is well underway.