Quantum exclusion processes

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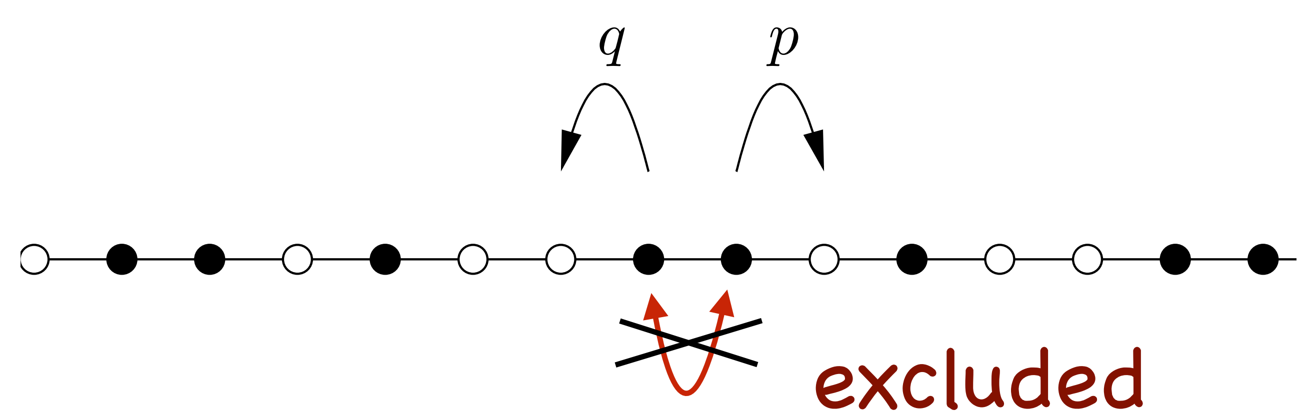
Classical exclusion processes and integrable spin chains

Lattice model of hard-core particles

"Configurations" $\boldsymbol{\tau}=(\tau_1,\ldots,\tau_L)$; $\tau_j=0,1$: site j empty/occupied

Prob. distr. of having config. $\boldsymbol{\tau}$ at time t: $P(\boldsymbol{\tau},t)$

Stochastic dynamics:



Master equation:

$$P(\boldsymbol{\tau},t+dt)=P(\boldsymbol{\tau},t)+\left[\sum_{\boldsymbol{\sigma}\neq\boldsymbol{\tau}}M(\boldsymbol{\sigma},\boldsymbol{\tau})P(\boldsymbol{\sigma},t)-M(\boldsymbol{\tau},\boldsymbol{\sigma})P(\boldsymbol{\tau},t)\right]dt$$

Mapping to a non-hermitian "spin-chain":

(Alcaraz, Droz, Henkel& Rittenberg '94)

- Configurations $\boldsymbol{\tau}=(\tau_1,\ldots,\tau_L)$ → basis states $|\tau_1,\ldots,\tau_L\rangle$
- Probability distr. $P(\boldsymbol{\tau},t)$ → State $|P(t)\rangle = \sum_{\boldsymbol{\tau}} P(\boldsymbol{\tau},t)\,|\,\tau_1,\ldots,\tau_N\rangle$
- Master eqn. → **imaginary time Schrödinger eqn**

$$\boxed{\frac{d}{dt}|P(t)\rangle = \mathscr{L}\,|P(t)\rangle}$$

$$\mathscr{L} = \sum_{j=1}^{L} q\sigma_j^+\sigma_{j+1}^- + p\sigma_j^-\sigma_{j+1}^+ - \frac{p+q}{4}\left[\sigma_j^z\sigma_{j+1}^z - 1\right]$$

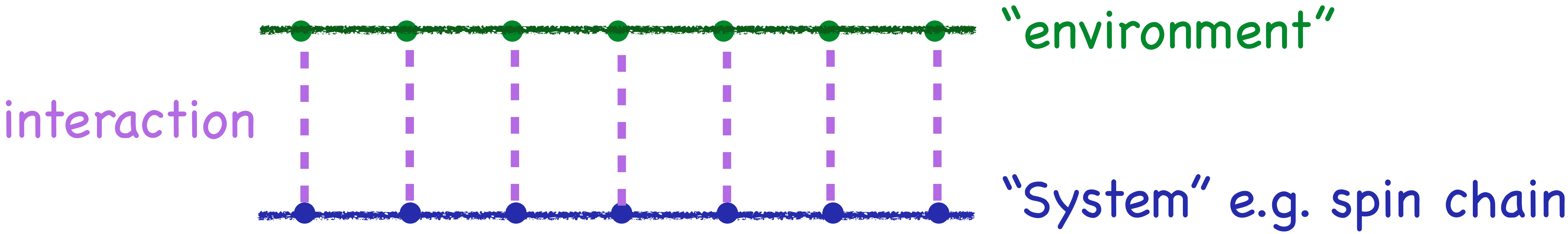
non-hermitian

$S\mathscr{L}S^{-1}$=XXZ spin-1/2 chain

Huge body of work since 1990ies

Gwa&Spohn, Derrida, …

Master equations for open quantum systems



Hamiltonian $$H = H_S + H_E + H_{\text{int}}$$

density matrix $$\rho(t) = e^{-iHt}\rho(0)e^{iHt}$$

reduced DM $$\rho_S(t) = \mathrm{Tr}_E\left[\rho(t)\right]$$

Goal: determine e.g. $\mathrm{Tr}[\rho_S(t)\, O_S(x)]$

$O_S(x)$ = local operator acting on system

Difficult in general...

Simplifying assumptions

1. Environment is Markovian

   $\rightarrow$ can neglect retardation when integrating out environment

2. Effect on system on environment is negligible

3. Initial density matrix is factorised $\rho(0) = \rho_E(0) \otimes \rho_S(0)$

Can average over environment degrees of freedom

Lindblad equation for reduced density matrix

Lindblad '76
Gorini et at '76

$$\frac{d}{dt}\rho_S(t) = i[H_S, \rho_S(t)] + \sum_{a=1}^{M} \gamma_a \left(L_a \rho_S(t) L_a^\dagger - \frac{1}{2}\{L_a^\dagger L_a, \rho_S(t)\} \right)$$

Linear!

rates

"jump operators"

describe coupling of system to environment

If correlation length in environment is short compared to lattice spacing & system/environment are homogeneous

$$\frac{d}{dt}\rho_S(t) = i[H_S, \rho_S(t)] + \gamma \sum_{j=1}^{M} \left(L_j \rho_S(t) L_j^\dagger - \frac{1}{2}\{L_j^\dagger L_j, \rho_S(t)\} \right)$$

Now follow the classical case and write this as an imaginary time Schrödinger equation

"Superoperator formalism"

$$\rho = \sum_{n,m=1}^{\dim\mathcal{H}} \langle n|\rho|m\rangle \; |m\rangle\langle n| \longrightarrow |\boldsymbol{\rho}\rangle = \sum_{n,m} \langle n|\rho|m\rangle \; |m\rangle|n\rangle\!\rangle$$

density matrix

state

Now define superoperators acting on these states.

Superoperators arising from operators acting from the left:

$$\mathcal{O}\rho \to \mathcal{O}\,|\boldsymbol{\rho}\rangle \equiv \sum_{n,m} \langle n|\rho|m\rangle \; \big(\mathcal{O}\,|m\rangle\big)\,|n\rangle\!\rangle$$

Superoperators arising from operators acting from the right:

$$\rho\mathcal{O} \to \widetilde{\mathcal{O}}\,|\boldsymbol{\rho}\rangle \equiv \sum_{n,m} \langle n|\rho|m\rangle \; |m\rangle\big(\widetilde{\mathcal{O}}\,|n\rangle\!\rangle\big)$$

$$\langle n'|\,\widetilde{\mathcal{O}}\,|n\rangle = \langle n|\,\mathcal{O}\,|n'\rangle$$

for bosonic ops

Lindblad eqn becomes

$$\boxed{\frac{\partial}{\partial t}|\rho\rangle = \mathscr{L}\,|\rho\rangle}$$

$$\boxed{\mathscr{L} = -iH + i\widetilde{H} + \sum_a \gamma_a \left[L_a \widetilde{L}_a^\dagger - \frac{1}{2}\left(L_a^\dagger L_a + \widetilde{L}_a \widetilde{L}_a^\dagger\right)\right]}$$

There are Lindblad equations for which $\mathscr{L}$ is the (non-hermitian) Hamiltonian of a quantum integrable model!

Medvedyeva, Essler & Prosen '16
Rowlands & Lamacraft '18, Shibata & Katsura '19
Essler& Ziolkowska '20, Essler& Piroli '21, Robertson & Essler '21
Buca et al '20, Nakagawa, Kawakami & Ueda `20
de Leeuw, Paletta & Pozsgay '21 …

Example: XX model with dephasing noise

Medvedyeva et al '16

$$H = \sum_j \sigma_j^+ \sigma_{j+1}^- + \sigma_j^- \sigma_{j+1}^+ \qquad L_j = \sqrt{\frac{u}{2}} \sigma_j^z$$

After a Jordan-Wigner transformation Lindbladian becomes

$$\boxed{\mathscr{L} = -i \sum_{j,\sigma} c_{j,\sigma}^\dagger c_{j+1,\sigma} + c_{j+1,\sigma}^\dagger c_{j,\sigma} + 4u \sum_j \left(n_{j,\uparrow} - \frac{1}{2}\right)\left(n_{j,\downarrow} - \frac{1}{2}\right)}$$

imaginary t Hubbard model

N.B. The i does not affect integrability but dramatically changes the BAE.

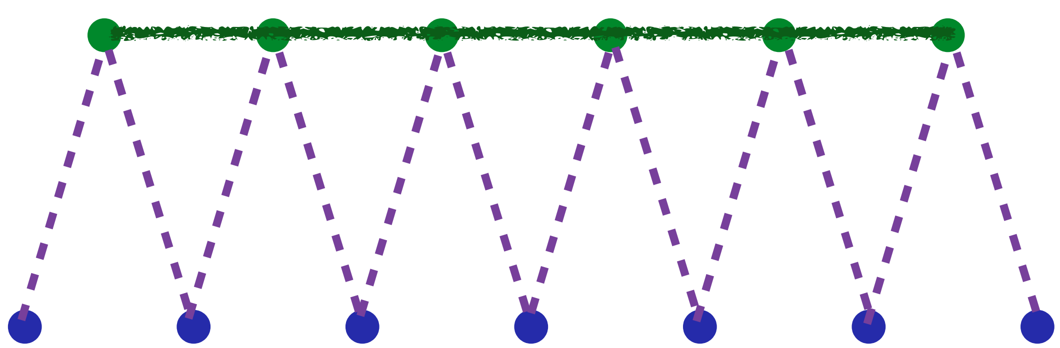
Are there quantum versions of exclusion processes?

And if so, are they integrable?

"Quantum ASEP"

Jin, Krajenbrink & Bernard '20
Bauer, Bernard & Jin '19, '20

spin-1/2 chain coupled to "quantum noise"



Time-dep. Hamiltonian $$H(t) = \sum_{j=1}^{L} \kappa_j(t)\sigma_j^+\sigma_{j+1}^- + \bar{\kappa}_j(t)\sigma_j^-\sigma_{j+1}^+$$

spin-flips induced by Markovian (quantum) environment

$$\mathbb{E}\left[\kappa_j(t)\right] = 0 \qquad \mathbb{E}\left[\kappa_j(t)\bar{\kappa}_k(t')\right] = J_1\delta_{j,k}\delta(t-t')$$

$$\mathbb{E}\left[\bar{\kappa}_j(t)\right] = 0 \qquad \mathbb{E}\left[\bar{\kappa}_j(t)\kappa_k(t')\right] = J_2\delta_{j,k}\delta(t-t')$$

Time evolution of the full density matrix

$$\rho(t) = U(t,t_0)\rho(t_0)U^\dagger(t,t_0) \qquad U(t,t_0) = T\, e^{-i\int_{t_0}^{t} ds H(s)}$$

Bernard, Essler, Hruza& Medenjak `21

Time evolution of the reduced density matrix of the system

$$\boxed{\frac{d}{dt}\rho_S(t) = \frac{d}{dt}\mathbb{E}\Big[\rho(t)\Big] = \mathscr{L}\left[\rho_S(t)\right]}$$

$$\mathscr{L}[\rho_S] = \sum_{j=1}^{L} J_2 \ell_j^{(2)}\rho_S \ell_j^{(2)\dagger} + J_1 \ell_j^{(1)}\rho_S \ell_j^{(1)\dagger} - \frac{1}{2}\left\{J_1 \ell_j^{(1)\dagger}\ell_j^{(1)} + J_2 \ell_j^{(2)\dagger}\ell_j^{(2)}, \rho_S\right\}$$

$$\ell_j^{(1)} = \sigma_j^- \sigma_{j+1}^+ \,, \qquad \ell_j^{(2)} = \sigma_j^+ \sigma_{j+1}^-$$

"jump operators"

Quantum-ASEP in the super-operator formalism

Vectorization

$|n\rangle\langle m| \Rightarrow |n\rangle|m\rangle\rangle$

$$|\uparrow\rangle_j\ {}_j\langle\uparrow| \Rightarrow |\uparrow\rangle_j|\uparrow\rangle\rangle_j \equiv |1\rangle_j$$
$$|\downarrow\rangle_j\ {}_j\langle\uparrow| \Rightarrow |\downarrow\rangle_j|\uparrow\rangle\rangle_j \equiv |2\rangle_j$$
$$|\uparrow\rangle_j\ {}_j\langle\downarrow| \Rightarrow |\uparrow\rangle_j|\downarrow\rangle\rangle_j \equiv |3\rangle_j$$
$$|\downarrow\rangle_j\ {}_j\langle\downarrow| \Rightarrow |\downarrow\rangle_j|\downarrow\rangle\rangle_j \equiv |4\rangle_j$$

Basis of super-operators: $E_j^{ab} \equiv |a\rangle_j\ {}_j\langle b|\ , \quad a,b \in \{1,2,3,4\}$

Lindblad equation:

$$\boxed{\frac{d|\rho_S(t)\rangle}{dt} = \mathscr{L}|\rho_S(t)\rangle}$$

where the Lindbladian is

$$\mathscr{L} = \sum_j J_1 E_j^{14} E_{j+1}^{41} + J_2 E_j^{41} E_{j+1}^{14} - J_1 E_j^{44} E_{j+1}^{11} - J_2 E_j^{11} E_{j+1}^{44}$$

$$-\frac{1}{2} \sum_j (E_j^{22} + E_j^{33})(J_1 E_{j+1}^{11} + J_2 E_{j+1}^{44}) + (E_{j+1}^{22} + E_{j+1}^{33})(J_2 E_j^{11} + J_1 E_j^{44})$$

$$+\frac{J_1 + J_2}{4} \sum_j (E_j^{22} E_{j+1}^{33} + E_j^{33} E_{j+1}^{22}) \, .$$

Integrable?

"Operator space fragmentation"

Essler&Piroli '20

$\mathscr{L}$ has an extensive number of strictly local conservation laws

$$\boxed{[\mathscr{L}, E_j^{22}] = 0 = [\mathscr{L}, E_j^{33}]}$$

j=1,..,L

→ $\mathscr{L}$ is **block-diagonal**

sites in states $|2\rangle, |3\rangle$ are frozen under dynamics→ static "defects"

Simplest block: no defects

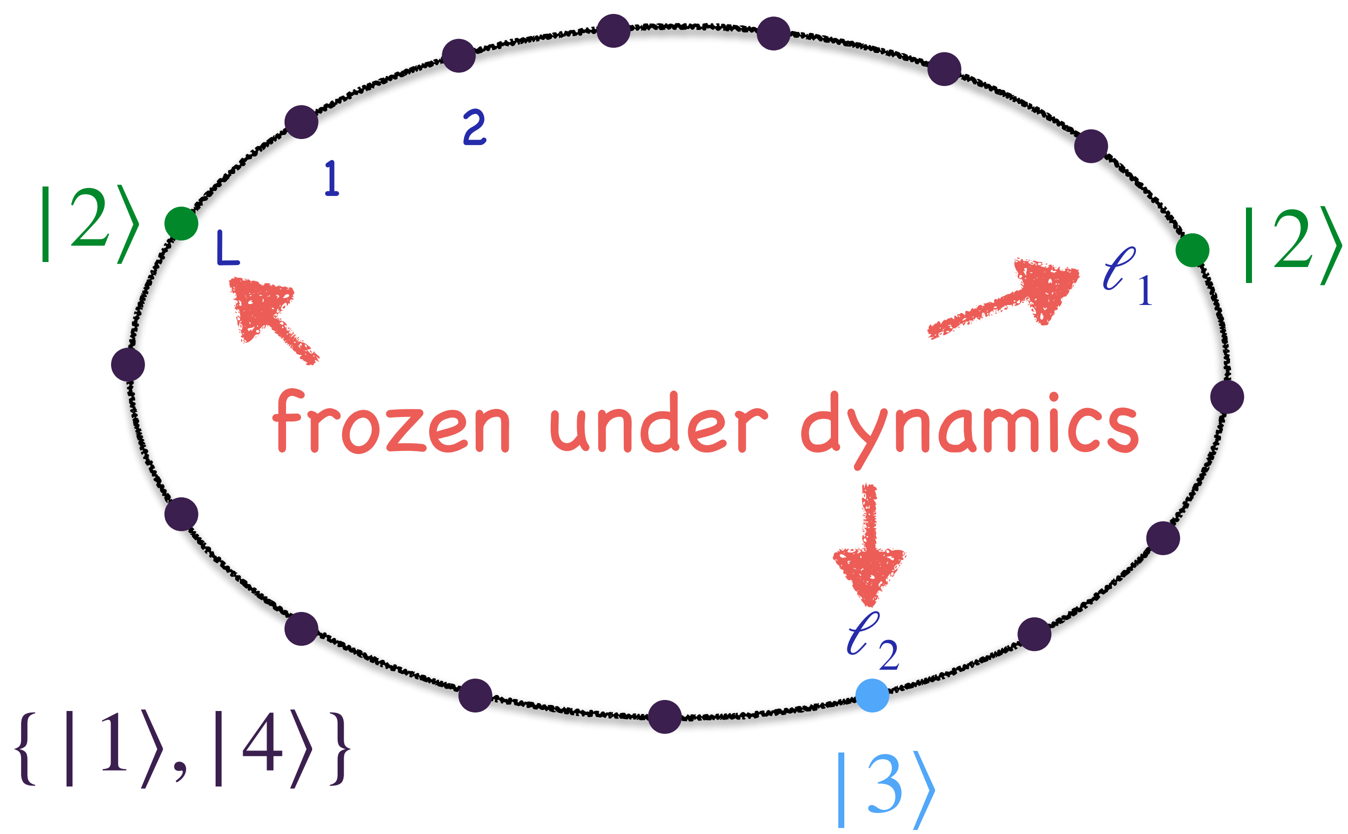
$$\boxed{\mathscr{L}_{\mathrm{ASEP}} = \sum_{j=1}^{L} \left[J_1 \sigma_j^+ \sigma_{j+1}^- + J_2 \sigma_j^- \sigma_{j+1}^+ + \frac{J_1 + J_2}{4} \left(\sigma_j^z \sigma_{j+1}^z - 1 \right) \right].}$$

Classical ASEP!

Sectors with defects:

consider $|\Psi(t)\rangle = e^{\mathscr{L}t}|\Psi(0)\rangle$ where

$$|\Psi(0)\rangle = |\psi_{[1,\ell_1-1]}\rangle \otimes |2\rangle_{\ell_1} \otimes |\psi_{[\ell_1+1,\ell_2-1]}\rangle \otimes |3\rangle_{\ell_2} \otimes \psi_{[\ell_2+1,L-1]}\rangle \otimes |2\rangle_L$$



Lindbladian is block-diagonal: $\mathscr{L} \to \mathscr{L}_{[1,\ell_1-1]} + \mathscr{L}_{[\ell_1+1,\ell_2-1]} + \mathscr{L}_{[\ell_2+1,L-1]}$

$$S^{-1}\mathscr{L}_{[m,n]}S = -\sqrt{\frac{J_1 J_2}{2}}\left(2\Delta + \sum_{j=m}^{n-1}\left[\sigma_j^x\sigma_{j+1}^x + \sigma_j^y\sigma_{j+1}^y + \Delta\left(\sigma_j^z\sigma_{j+1}^z - 1\right)\right]\right)$$

$$2\Delta = \sqrt{\frac{J_1}{J_2}} + \sqrt{\frac{J_2}{J_1}}$$

open Heisenberg spin-1/2 chain!

$\rightarrow$ can use integrability methods to determine spectrum of $\mathscr{L}$

At late times QASP reduces to ASEP!

$$|\rho(t)\rangle = \sum_{\boldsymbol{\alpha}} \rho_{\boldsymbol{\alpha}\boldsymbol{\alpha}} e^{\mathscr{L}_{\mathrm{ASEP}}t}\,|\boldsymbol{\alpha}\rangle\,|\boldsymbol{\alpha}\rangle\rangle + \mathcal{O}(e^{-\max(J_1,J_2)t/2})$$

"Dynamics becomes classical".

$$\mathrm{Tr}\left[\rho(t)\sigma^z_{j_1}\ldots\sigma^z_{j_n}\right]$$

dynamics as in classical ASEP (approach stationary values)

$$\mathrm{Tr}\left[\rho(t)\sigma^+_{j_1}\sigma^-_{j_2}\sigma^z_{j_3}\ldots\sigma^z_{j_n}\right] = \mathcal{O}(e^{-\max(J_1,J_2)t/2})$$

"quantum" correlations decay exponentially to zero (more $\sigma^{\pm}_m$ lead to faster decay)

Can we calculate "quantum" correlation functions?

$$\mathrm{Tr}\left[\rho(t)\sigma_1^+\sigma_\ell^-\right] = \langle \boldsymbol{\phi} \,|\, e^{\mathscr{L}_{[2,\ell-1]}t} e^{\mathscr{L}_{[\ell+1,L]}t} \,|\, \rho(0)\rangle$$

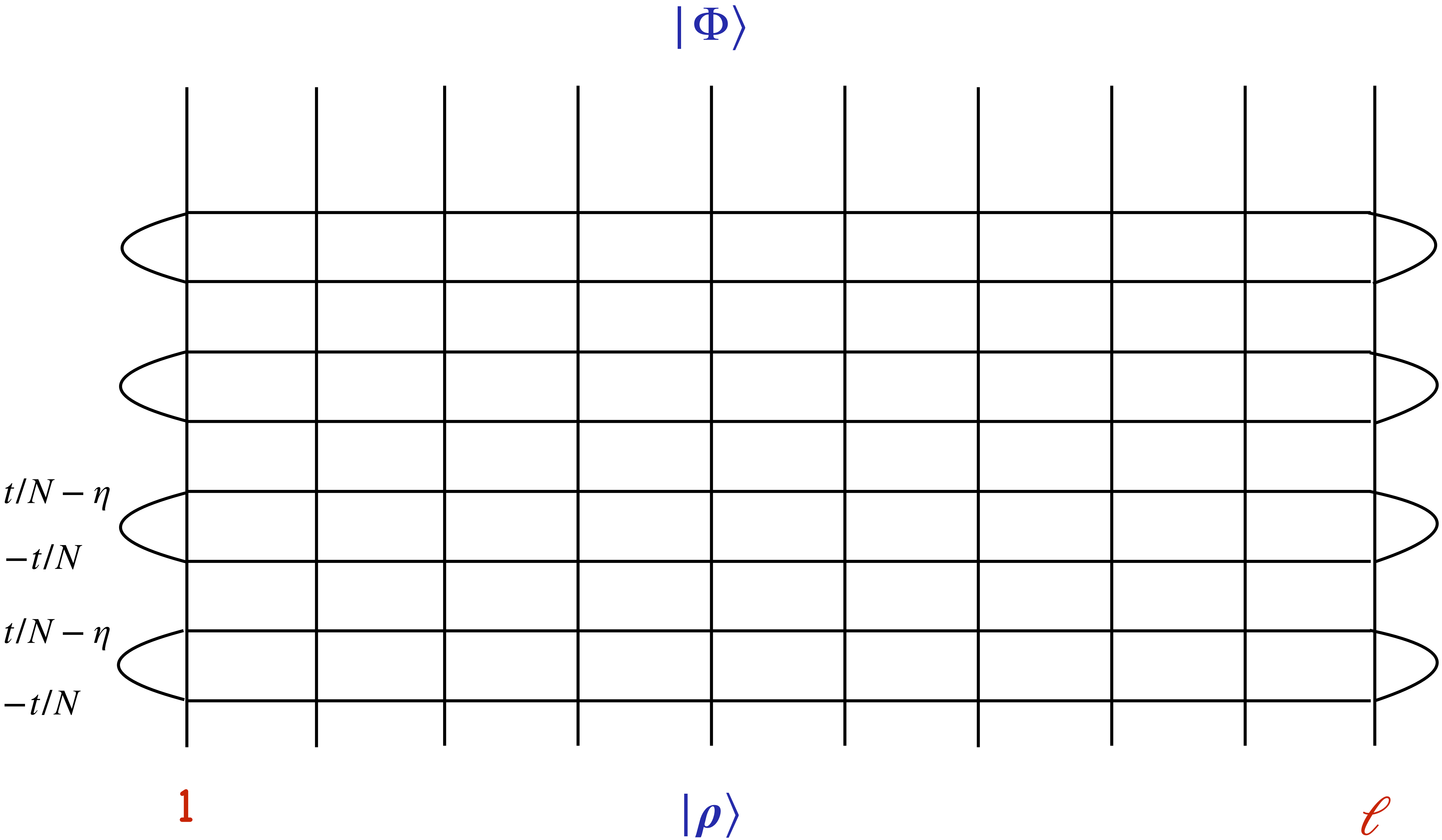
where $\langle \boldsymbol{\phi} \,| = {}_1\langle 2\,| \otimes_{j=2}^{\ell-1} \left[{}_j\langle 1\,| + {}_j\langle 4\,|\right] \otimes {}_\ell\langle 3\,| \otimes_{j=\ell+1}^{L} \left[{}_j\langle 1\,| + {}_j\langle 4\,|\right]$

Not known how to calculate this even for unentangled $|\,\rho(0)\rangle$.

cf. Piroli, Pozsgay& Vernier '18

"Loschmidt amplitude" $\langle \rho(0)\,|\, e^{H_{\mathrm{XXZ}}\,t} \,|\, \rho(0)\rangle$

Building block: $\langle \mathbf{\Phi} | e^{\mathscr{L}_{[1,\ell]} t} | \boldsymbol{\rho} \rangle$ $\rightarrow$ 2-boundary QTM



A more complicated simpler problem

Robertson & Essler '21

Consider Lindblad equation for spin-1/2 with **four** jump operators

$$L_j^{(1)} = \left(L_j^{(2)}\right)^\dagger = \sigma_j^+ \sigma_{j+1}^- \qquad L_j^{(3)} = \left(L_j^{(4)}\right)^\dagger = \sigma_j^+ \sigma_{j+1}^+$$

Quantum-ASEP

additional couplings to environment

Lindbladian:

$$\mathscr{L} = \mathscr{L}_{QASEP} + \sum_j J_3 E_j^{14} E_{j+1}^{14} + J_4 E_j^{41} E_{j+1}^{41} - \sum_j J_3 E_j^{44} E_{j+1}^{44} + J_4 E_j^{11} E_{j+1}^{11}$$

$$-\frac{1}{2} \sum_j (E_j^{22} + E_j^{33})(J_4 E_{j+1}^{11} + J_3 E_{j+1}^{44}) + (E_{j+1}^{22} + E_{j+1}^{33})(J_4 E_j^{11} + J_3 E_j^{44})$$

$$-\frac{J_3 + J_4}{2} (E_j^{22} E_{j+1}^{33} + E_j^{33} E_{j+1}^{22})$$

Exhibits operator-space fragmentation $[\mathscr{L}, E_j^{22}] = 0 = [\mathscr{L}, E_j^{33}]$

j=1,..,L

Observation: if $J_1 + J_2 = J_3 + J_4$ then $\mathscr{L}_{[1,\ell]}$ can be mapped to a non-Hermitian open free-fermion chain

$$\boxed{\begin{aligned}\mathscr{L}_{[1,\ell]} = &-\frac{J_1+J_2}{2}(\ell+1) - \mu \sum_{j=1}^{\ell} (2c_j^\dagger c_j - 1) \\ &+ \sum_{j=1}^{\ell-1} \left\{ J_1 c_{j+1}^\dagger c_j + J_2 c_j^\dagger c_{j+1} - J_3 c_j c_{j+1} - J_4 c_{j+1}^\dagger c_j^\dagger) \right\}\end{aligned}}$$

$$c_j = \prod_{n=1}^{j-1} \left(E_j^{11} - E_j^{44}\right) E_j^{41} \ , \quad \{c_j, c_k^\dagger\} = \delta_{j,k}$$

Can use (non-standard) free-fermion techniques to determine correlation functions.

Summary

1. Certain quantum master equations can be related to Yang-Baxter integrable models in interesting ways.
2. Spectral properties can be analysed using integrability.
3. Calculation of observables requires new methods.