Diagonal finite volume matrix elements in the sinh-Gordon model

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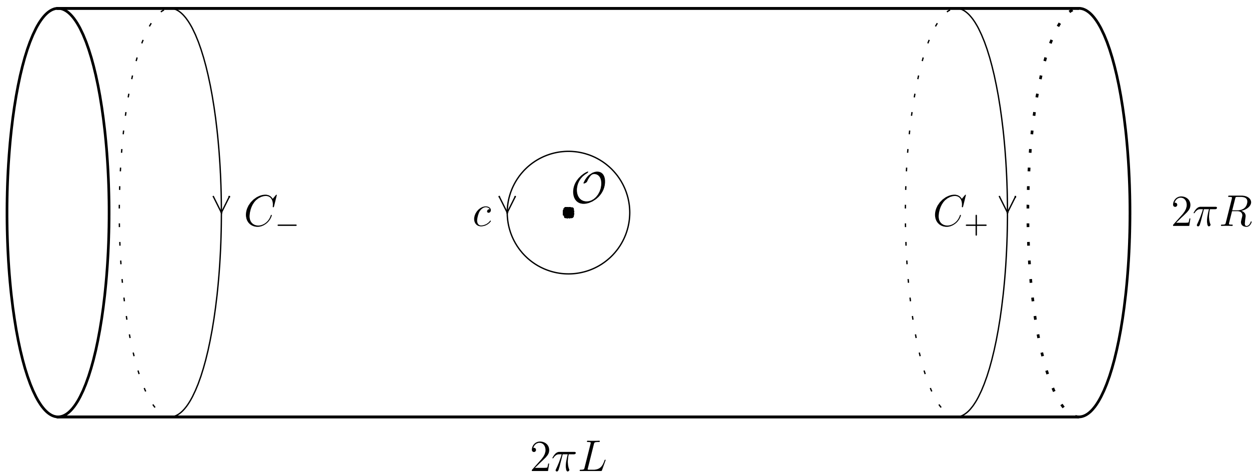
Based on joint work with Zoltan Bajnok

1. Introduction

Consider the Euclidean QFT on a cylinder



where we insert the local operator $\mathcal{O}$. Energy-momentum tensor

$$\partial_{\bar{z}} T = \partial_z \Theta \,, \quad \partial_z \bar{T} = \partial_{\bar{z}} \Theta \,,$$

produces

$$P_{\pm} = \int_{C_{\pm}} (T dz + \Theta d\bar{z}) \qquad \bar{P}_{\pm} = \int_{C_{\pm}} (\bar{T} d\bar{z} + \Theta dz) \,.$$

If the boundary conditions correspond to the same Matsubara eigenvector

($P_+ = P_-$, $\bar{P}_+ = \bar{P}_-$) the figure represents a diagonal matrix element.

There are several reasons to be interested in diagonal matrix elements.

- For $L \to \infty$ we are interested in the Matsubara ground state matrix elements (One-point functions, Negro, Smirnov, 2013). Using the OPE

$$O_i(x)O_j(0) = \sum_k c_{i,j}^k(x)O_k(0)\,,$$

which is a purely UV characteristics (a subject of PCFT in principle) we reduce all correlation functions to the one-point functions.

- The one-point function on the torus equals

$$\langle O(0)\rangle_{\mathrm{torus}} = \sum_K \frac{\langle K|O(0)|K\rangle}{\langle K|K\rangle} e^{-2\pi L e_K}\ .$$

This is an interesting object which, in particular must be invariant under $L \leftrightarrow R$.

- Diagonal matrix elements are used when the correction to the S-matrix for perturbed model are computed.

Sinh-Gordon model

We shall consider sinh-Gordon model The Lagrangian density of the

$$\mathcal{L} = \frac{1}{4\pi}(\partial\phi)^2 + \frac{2\mu^2}{\sin\pi b^2}\cosh(b\varphi)\,.$$

Duality:

$$b \to 1/b\,.$$

We use selfdual $Q = b + b^{-1}\,.$

The exact relation between the dimensional parameter and μ and the particle mass:

$$b^{-1}(\mu\Gamma(1+b^2))^{\frac{1}{1+b^2}} = Z(b)m\,,$$

where

$$Z(b) = \frac{1}{16Q\pi^{3/2}}\Gamma\big(\tfrac{b}{2Q}\big)\Gamma\big(\tfrac{b^{-1}}{2Q}\big).$$

The S-matrix

$$S(\theta) = \frac{\sinh\theta - i\sin(\frac{\pi}{1+b^2})}{\sinh\theta + i\sin(\frac{\pi}{1+b^2})}\,.$$

We shall use dimensionless $r = mR$.

Finite volume spectrum can be found using the Q-function: regular in θ it satisfies the quantum Wronskian relation

$$\mathcal{Q}(\theta+\tfrac{i\pi}{2})\mathcal{Q}(\theta-\tfrac{i\pi}{2})-\mathcal{Q}(\theta+\tfrac{i\pi}{2}\tfrac{1-b^2}{1+b^2})\mathcal{Q}(\theta-\tfrac{i\pi}{2}\tfrac{1-b^2}{1+b^2})=1\,,$$

and behaves asymptotically as

$$\log\mathcal{Q}(\theta)\simeq-\frac{r\cosh(\theta)}{2\sin(\frac{\pi}{1+b^2})}\,,\qquad \theta\to\pm\infty\,.$$

Let θ_k be zeros of $\mathcal{Q}(\theta)$ in the strip $|\mathrm{Im}(\theta)|<\pi$, then

$$\mathcal{Q}(\theta)=\prod_{k=1}^{N}\tanh\Big(\frac{\theta-\theta_k}{2}\Big)$$

$$\times\exp\Big(-\frac{r\cosh(\theta)}{2\sin(\frac{\pi}{1+b^2})}+\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{1}{\cosh(\theta-\theta')}\log(1+e^{-\epsilon(\theta')})d\theta'\Big)\,,$$

where

$$e^{-\epsilon(\theta)}=\mathcal{Q}(\theta+\tfrac{i\pi}{2}\tfrac{1-b^2}{1+b^2})\mathcal{Q}(\theta-\tfrac{i\pi}{2}\tfrac{1-b^2}{1+b^2})\,.$$

It is assumed that all θ_k are real.

From that we derive the TBA equation

$$\epsilon(\theta) = r\cosh\theta + \sum_{k=1}^{N} \log S(\theta - \theta_k - \tfrac{\pi i}{2}) - \int_{-\infty}^{\infty} K(\theta - \theta') \log(1 + e^{-\epsilon(\theta')}) d\theta' \,,$$

with the kernel

$$K(\theta) = \frac{1}{2\pi i} \frac{d}{d\theta} \log S(\theta) \,.$$

In addition we have for the discrete spectrum

$$f(\theta_k) = \pi N_k \,,$$

where

$$f(\theta) = r\sinh\theta + \sum_{k=1}^{N} \arg(-S(\theta - \theta_k)) - \int_{-\infty}^{\infty} K(\theta - \theta' + \tfrac{\pi i}{2}) \log(1 + e^{-\epsilon(\theta')}) d\theta' \,.$$

coincides at $\theta = \theta_k$ with the analytical continuation of $-i\epsilon(\theta + i\pi/2)$.

We use $\arg(-S(\theta - \theta_k))$ for convenience because $S(0) = -1$, Eigenvalues of the local integrals of motion:

$$I_s(r) = \frac{1}{C_s(b)}\Big(-\frac{1}{s}\sum_{j=1}^{m} e^{s\theta_k} + (-1)^{\frac{s-1}{2}}\frac{1}{2\pi}\int\limits_{-\infty}^{\infty} e^{s\theta}\log\Big(1+e^{-\epsilon(\theta)}\Big)d\theta\Big)\,,$$

$$\bar{I}_s(r) = \frac{1}{C_s(b)}\Big(-\frac{1}{s}\sum_{j=1}^{m} e^{-s\theta_k} + (-1)^{\frac{s-1}{2}}\frac{1}{2\pi}\int\limits_{-\infty}^{\infty} e^{-s\theta}\log\Big(1+e^{-\epsilon(\theta)}\Big)d\theta\Big)\,,$$

where

$$C_s(b) = -\frac{Z(b)^{-s}}{4\sqrt{\pi}Q\,\left(\frac{s+1}{2}\right)!}\,\Gamma\!\left(\tfrac{b}{2Q}s\right)\Gamma\!\left(\tfrac{b^{-1}}{2Q}s\right).$$

In particular,

$$E = I_1 + \bar{I}_1\,, \quad P = I_1 - \bar{I}_1\,.$$

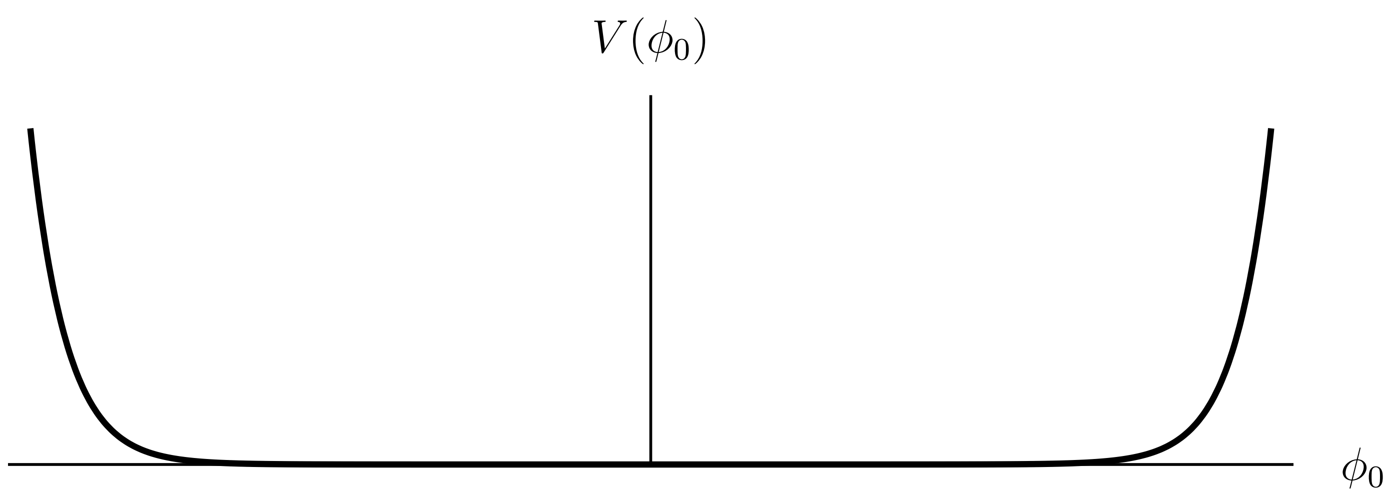
The coefficients $C_s(b)$ are introduced in order to fit with the UV CFT normalisation.

For large volume one reproduces the Luscher corrections. This is rather simple, I do not go into details.

Small R is more interesting. Everything depends on dimensionless

$$\tilde{\mu} = \mu R^{1+b^2} \; .$$

Replacing μ we obtain for the zero mode the potential of the form



Quantisation of the zero-mode:

$$4P_L(r)Q\log\Big(Z(b)rb^{\frac{b^2-1}{b^2+1}}\Big) = -\pi L + \frac{1}{i}\log\frac{\Gamma(1+2iP_L(r)b)\Gamma(1+2iP_L(r)/b)}{\Gamma(1-2iP_L(r)b)\Gamma(1-2iP_L(r)/b)}\,.$$

These equations we solve numerically.
It is easy to compare the first integrals with CFT

$$I_1^{\mathrm{CFT}} = P_L(r)^2 - \frac{1}{24} + M\,, \quad \bar{I}_1^{\mathrm{CFT}} = P_L(r)^2 - \frac{1}{24} + \overline{M}\,,$$

Later we use

$$\Delta = P^2 + \frac{Q^2}{4}\,, \qquad c = 1 + 6Q^2\,.$$

For example $\mathcal{N} = \{-2, 0, 2\}$ corresponds to the primary field with $L = 4$.
We compute for $r = .001$ the ratios:

$$\frac{I_1}{I_1^{\mathrm{CFT}}} = 1.00003\,, \quad \frac{\bar{I}_1}{\bar{I}_1^{\mathrm{CFT}}} = 0.999989\,.$$

Counting the CFT operators

$\mathcal{N}$	L	M	$\bar{M}$
{ }	1	0	0
{0}	2	0	0
{-1,1}	3	0	0
{-2,0,2}	4	0	0
{2}	1	1	0
{-1,3}	2	1	0
{-2,0,4}	3	1	0
{-3,3}	1	1	1
{4}	1	2	0
{1,3}	1	2	0
{-1,5}	2	2	0
{-2,2,4}	2	2	0
{-3,5}	1	2	1
{-5,5}	1	2	2
{1,5}	1	3	0
{0,2,4}	1	3	0
{-3,-1,1,5}	4	1	0

2. Expectation values

Since a is generic we have one-to-one correspondence between local fields in the Liouville and sinh-Gordon models.

Our goal is to compute the diagonal matrix elements

$$\langle\theta_1,\dots,\theta_m|\mathcal{O}|\theta_m,\dots,\theta_1\rangle_R\,.$$

Obviously $\mathcal{O}$ are defined modulo $[I_s,\cdot]$. The primary field

$$\Phi_a=\frac{1}{\mathcal{F}(a,b)}e^{a\varphi}\,,$$

is normalised by Lukyanov-Zamolodchikov one-point function.

We have fermionic operators acting on the space of local operators.

Define

$$\beta^*_M\gamma^*_N\bar\beta^*_{\bar M}\bar\gamma^*_{\bar N}\Phi_a=\beta^*_{m_1}\dots\beta^*_{m_k}\gamma^*_{n_1}\dots\gamma^*_{n_k}\bar\beta^*_{\bar m_1}\dots\bar\beta^*_{\bar m_{\bar k}}\bar\gamma^*_{\bar n_1}\dots\bar\gamma^*_{\bar n_{\bar k}}\Phi_a\,.$$

with the requirement

$$\#(M)+\#(\overline{M})=\#(N)+\#(\overline{N})\,.$$

General rule: In UV limit the Virasoro descendants descendants of Φ_a are sent to the descentants of Φ_{a-mb},

$$m = \#(N) - \#(M)\,.$$

Explicit relation to Virasoro descendants will be discussed later.

To understand $m \neq 0$ we need $\beta^*_{-j} = \gamma_j$, $\gamma^*_{-j} = \beta_j$ satisfying

$$\{\beta_k, \beta^*_n\} = \{\bar\gamma_k, \bar\gamma^*_n\} = -t_k(a,b)\delta_{k,n}; \qquad t_n(a,b) = \frac{1}{2\sin\left(\frac{\pi}{Q}(2a - nb)\right)}\,.$$

The femionmic descendant with $m \neq 0$ describe Virasoro descendants in shifted modules:

$$\beta^*_M \gamma^*_N \bar\beta^*_{\bar M} \bar\gamma^*_{\bar N} \Phi_{a+mb}$$
$$= \frac{1}{\prod_{j=1}^m t_{2j-1}(a,b)} \beta^*_{M+2m} \gamma^*_{N-2m} \bar\beta^*_{\bar M-2m} \bar\gamma^*_{\bar N+2m} \beta^*_{I_{\mathrm{odd}}(m)} \bar\gamma^*_{I_{\mathrm{odd}}(m)} \Phi_a\,.$$

where $I_{\mathrm{odd}}(m) = \{1, 3, \ldots, 2m-1\}$.

For $m=0$ the fermionic basis $\beta^*_M\gamma^*_N\bar\beta^*_{\bar M}\bar\gamma^*_{\bar N}\Phi_a$ has a remarkable property of solving the reflection relations of FFLZZ, namely, under both reflections

$$a\to -a,\qquad a\to Q-a$$

we have

$$\boldsymbol{\beta}^*\longleftrightarrow\boldsymbol{\gamma}^*\,.$$

This can be taken for definition, contrary to sine-Gordon case we do not have scaling limit from the lattice.

Our main conjecture is that like in the sine-Gordon case

$$\frac{\langle\theta_1,\ldots,\theta_m|\beta^*_M\gamma^*_N\bar\beta^*_{\bar M}\bar\gamma^*_{\bar N}\Phi_a|\theta_m,\ldots,\theta_1\rangle_R}{\langle\theta_1,\ldots,\theta_m|\Phi_a|\theta_m,\ldots,\theta_1\rangle_R}=\mathcal{D}(\{M\cup(-\bar M)\}|\{N\cup(-\bar N)\}|a)\,.$$

where for the index sets $M=\{m_1,\ldots,m_k\}$ and $N=\{n_1,\ldots,n_k\}$ the determinant is

$$\mathcal{D}(M|N|a)=\prod_{j=1}^{k}\frac{\mathrm{sgn}(m_j)\mathrm{sgn}(n_j)}{\pi}\mathrm{Det}\left(\Omega_{m_i,n_j}\right)_{i,j=1,\cdots,k}\,,$$

$$\Omega_{m,n}=\omega_{m,n}-\pi\mathrm{sgn}(n)\delta_{m,-n}t_n(a)\,.$$

The matrix $||\omega_{m,n}||$ is defined by analogy with other cases.

3. Definition of ω

Using

$$\partial_r f(\theta_k) + \partial_\theta f(\theta_k)\frac{d\theta_k}{dr} = 0\,,$$

compute the variation

$\partial_r \epsilon(\theta)$

$$= \cosh\theta - 2\pi i \sum_{k=1}^{N} K(\theta - \theta_k + \tfrac{\pi i}{2})\frac{d\theta_k}{dr} + \int_{-\infty}^{\infty} K(\theta-\theta')\partial_r\epsilon(\theta')\frac{1}{1+e^{\epsilon(\theta')}}d\theta'$$

$$= \cosh\theta + 2\pi i \sum_{k=1}^{N} K(\theta - \theta_k + \tfrac{\pi i}{2})\frac{\partial_r f(\theta_k)}{\partial_\theta f(\theta_k)} + \int_{-\infty}^{\infty} K(\theta-\theta')\partial_r\epsilon(\theta')\frac{1}{1+e^{\epsilon(\theta')}}d\theta'\,,$$

Motivated by this we introduce the convolution

$$G * H = 2\pi i \sum \frac{1}{\partial_\theta f(\theta_k)} g_k h_k + \int_{-\infty}^{\infty} g(\theta)h(\theta)\frac{d\theta}{1+e^{\epsilon(\theta)}}\,.$$

Now we are ready to define

$$\omega_{n,m} = e_n * (1 + \mathcal{K}_a + \mathcal{K}_a * \mathcal{K}_a + \dots) * e_m \equiv e_n * (1 + \mathcal{R}_{\mathrm{dress},a}) * e_m \,,$$

where $e_n = \{e^{n(\theta_1 + \frac{\pi i}{2})}, \dots, e^{n(\theta_m + \frac{\pi i}{2})}, e^{n\theta}\}$, and $\mathcal{K}_a$ has a matrix structure

$$\mathcal{K}_a = \begin{pmatrix} K_a(\theta_k - \theta_l) & K_a(\theta_k - \theta' + \frac{\pi i}{2}) \\ K_a(\theta - \theta_l - \frac{\pi i}{2}) & K_a(\theta - \theta') \end{pmatrix} \,,$$

and K_a is the deformation of the TBA kernel

$$K_a(\theta) = \frac{1}{2\pi i}\Big(\frac{A^{-1}}{\sinh(\theta - \frac{\pi i}{1+b^2})} - \frac{A}{\sinh(\theta + \frac{\pi i}{1+b^2})}\Big) \,, \quad A = \exp\Big\{2\pi i \frac{a}{Q}\Big\} \,.$$

4. UV limit

On the cylinder we have two Virasoro algebras

$$T(z) = \sum_{n=-\infty}^{\infty} \mathbf{l}_n z^{-n-2}, \quad T(z) = \sum_{n=-\infty}^{\infty} L_n e^{\frac{nz}{R}} - \frac{c}{24},$$

here we can set $R = 1$, it is easy to reconstruct the dependance on R due to the scale invariance.
We are interested in

$$\langle \Psi | \mathcal{O}_a | \Psi \rangle \, ,$$

where

$$|\Psi\rangle = L_{-n_1} \cdots L_{-n_p} |\Delta\rangle \, , \quad \mathcal{O}_a = \mathbf{l}_{-2m_1} \cdots \mathbf{l}_{-2m_r} \Phi_a \, ,$$

Φ_a is the primary field with dimension

$$\Delta_a = a(Q - a) \, .$$

Below we give examples relating the fermionic basis to the Virasoro one.

Recall $m = \#(\gamma^*) - \#(\beta^*)$.

Consider first examples with $m = 0$

$$\beta_1^*\gamma_1^*\Phi_a \equiv D_1(a,b)D_1(Q-a,b)\mathbf{l}_{-2}\Phi_a \,,$$

$$\beta_1^*\gamma_3^*\Phi_a \equiv D_1(a,b)D_3(Q-a,b)\left(\mathbf{l}_{-2}^2 + \Big(\frac{2c-32}{9} + \frac{2}{3}d(a,b)\Big)\mathbf{l}_{-4}\right)\Phi_a \,,$$

$$\beta_3^*\gamma_1^*\Phi_a \equiv D_3(a,b)D_1(Q-a,b)\left(\mathbf{l}_{-2}^2 + \Big(\frac{2c-32}{9} - \frac{2}{3}d(a,b)\Big)\mathbf{l}_{-4}\right)\Phi_a \,,$$

where

$$d(a,b) = (b^2 - b^{-2})(Q/2 - a)\,,$$

$$D_n(a,b) = \frac{1}{2i\sqrt{\pi}}Z(b)^{-n}\Gamma\Big(\frac{(2a+nb)}{2Q}\Big)\Gamma\Big(\frac{(2a+nb^{-1})}{2Q}\Big)\,.$$

Using these formulae we find the UV asymptotics

$$\Omega_{1,1} \simeq r^{-2} D_1(a,b) D_1(Q-a,b) \frac{\langle \mathbf{l}_{-2}\Phi_a\rangle}{\langle\Phi_a\rangle}\,,$$

$$\Omega_{3,1} \simeq r^{-4}\frac{1}{2} D_3(a,b) D_1(Q-a,b)\Big\{\frac{\langle \mathbf{l}_{-2}^2\Phi_a\rangle}{\langle\Phi_a\rangle} + \Big(\frac{2c-32}{9} + \frac{2}{3}d(a,b)\Big)\frac{\langle \mathbf{l}_{-4}\Phi_a\rangle}{\langle\Phi_a\rangle}\Big\}\,,$$

$$\Omega_{1,3} \simeq r^{-4}\frac{1}{2} D_1(a,b) D_3(Q-a,b)\Big\{\frac{\langle \mathbf{l}_{-2}^2\Phi_a\rangle}{\langle\Phi_a\rangle} + \Big(\frac{2c-32}{9} - \frac{2}{3}d(a,b)\Big)\frac{\langle \mathbf{l}_{-4}\Phi_a\rangle}{\langle\Phi_a\rangle}\Big\}\,.$$

Now we consider $m = -1$. Simplest case is

$$\beta_1^* \bar{\gamma}_1^* \Phi_a \equiv t_1(a,b) \Phi_{a-b} \,.$$

For primary fields using the Liouville three-point function we get

$$\frac{\langle \Phi_{a-b} \rangle_\Delta}{\langle \Phi_a \rangle_\Delta}$$
$$= F(a,b) \frac{\gamma^2(ab-b^2)}{\gamma(2ab-2b^2)\gamma(2ab-b^2)} \gamma(ab-b^2-2ibP)\gamma(ab-b^2+2ibP) \,.$$

where $F(a,b)$ is a ratio of two LZ one-point functions:

$$F(a,b) = 2(1+b^2) Z(b)^{2(\Delta_a - \Delta_{a-b})}$$
$$\times \gamma\Big(\frac{2a+b^{-1}}{2Q}\Big) \gamma\Big(\frac{2(Q-a)+b}{2Q}\Big) \gamma\big(2ab-b^2\big) \,.$$

We find

$$\Omega_{1,-1} \simeq r^{2(\Delta_a - \Delta_{a-b})} t_1(a,b) \frac{\langle \Phi_{a-b} \rangle}{\langle \Phi_a \rangle} \,.$$

Numerics.

Below we give numerical results for

$$r = .0001, \quad a = \frac{87}{80}, \quad b = \frac{2}{5}\,.$$

We take the first really non-trivial case of level 2. This is the simplest one with degenerate $I_1 = L_0 - \frac{c}{24}$. For simplicity we consider states with $\overline{M} = 0$.

Use the basis:

$$L_{-2}|\Delta\rangle, \quad L_{-1}^2|\Delta\rangle\,.$$

The degeneration is lifted by

$$I_3 = 2\sum_{n=1}^{\infty} L_{-n}L_n + L_0^2 - \frac{c+2}{12}L_0 + \frac{c(5c+22)}{2880}\,,$$

with eigenvalues

$$\lambda_\pm(\Delta) = \frac{17}{3} + \frac{c(5c+982)}{2880} - \frac{c-142}{12}\Delta + \Delta^2 \pm \frac{1}{2}\sqrt{288\Delta + (c-4)^2}\,,$$

and eigenvectors

$$\psi_\pm = \begin{pmatrix} \frac{1}{12}c - 4 \pm \sqrt{288\Delta + (c-4)^2} \\ 1 \end{pmatrix}$$

Numerical computation gives for $b = 2/5, r = 10^{-3}$

$$\mathcal{N}_- = \{1,3\}\,, \quad I_3(r) = 21.3719\,, \quad \lambda_-(\Delta(r)) = 21.3767\,,$$
$$\mathcal{N}_+ = \{4\}\,, \qquad I_3(r) = 74.8342\,, \quad \lambda_+(\Delta(r)) = 74.8399\,.$$

We compute (using natural notations)

$$\frac{\langle \Phi_a \rangle_{\Delta+2}}{\langle \Phi_a \rangle_{\Delta}} =$$
$$\begin{pmatrix} 4\Delta - 4\Delta_a + 4\Delta_a^2 + c/2 & 2(3\Delta - \Delta_a + \Delta_a^3) \\ 2(3\Delta - \Delta_a + \Delta_a^3) & 8\Delta^2 + \Delta(4 - 8\Delta_a + 8\Delta_a^2) - 2\Delta_a + 3\Delta_a^2 - 2\Delta_a^3 + \Delta_a^4 \end{pmatrix}$$

and

$$\frac{\langle \mathcal{O}_a \rangle_{\Delta+2}}{\langle \Phi_a \rangle_{\Delta}} = \begin{pmatrix} M_{1,1}(\mathcal{O}_a) & M_{1,2}(\mathcal{O}_a) \\ M_{1,2}(\mathcal{O}_a) & M_{2,2}(\mathcal{O}_a) \end{pmatrix} \,,$$

the matrix is symmetric.

For example, for $\mathcal{O}_a = \mathbf{1}_{-2}\Phi_a$,

$$M_{1,1} = \frac{1}{48}\big(48c - c^2 - 672\Delta_a + 102c\Delta_a + 976\Delta_a^2 - 8c\Delta_a^2 - 16\Delta_a^3 + \Delta(384 + 16c + 560\Delta_a + 192\Delta_a^2) + 192\Delta^2\big)\,,$$

$$M_{1,2} = \frac{1}{12}\big(-72\Delta_a + 7c\Delta_a + 14\Delta_a^2 + 6c\Delta_a^2 + 84\Delta_a^3 - c\Delta_a^3 - 2\Delta_a^4 + \Delta(144 - 3c + 258\Delta_a + 144\Delta_a^2 + 24\Delta_a^3) + 72\Delta^2\big)\,,$$

$$M_{2,2} = \frac{1}{24}\big(-96\Delta_a + 2c\Delta_a + 52\Delta_a^2 - 3c\Delta_a^2 - 6\Delta_a^3 + 2c\Delta_a^3 + 52\Delta_a^4 - c\Delta_a^4 - 2\Delta_a^5 + \Delta(192 - 4c + 232\Delta_a + 8c\Delta_a + 568\Delta_a^2 - 8c\Delta_a^2 + 128\Delta_a^3 + 24\Delta_a^4) + \Delta^2(480 - 8c + 560\Delta_a + 192\Delta_a^2) + 192\Delta^3\big)\,.$$

The general formulas the CFT limit take the form

$$\Omega^{\pm}_{1,1} \simeq r^{-2} D_1(a,b) D_1(Q-a,b) \frac{\psi^t_{\pm} \cdot \langle \mathbf{l}_{-2} \Phi_a \rangle_{\Delta+2} \cdot \psi_{\pm}}{\psi^t_{\pm} \cdot \langle \Phi_a \rangle_{\Delta+2} \cdot \psi_{\pm}} \,,$$

$$\Omega^{\pm}_{1,3} \simeq r^{-2} D_1(a,b) D_3(Q-a,b) \\ \times \frac{\psi^t_{\pm} \cdot \langle \mathbf{l}^2_{-2} + \frac{2}{9}(c-16-3d(a,b))\mathbf{l}_{-4} \Phi_a \rangle_{\Delta+2} \cdot \psi_{\pm}}{\psi^t_{\pm} \cdot \langle \Phi_a \rangle_{\Delta+2} \cdot \psi_{\pm}} \,,$$

$$\Omega^{\pm}_{3,1} \simeq r^{-2} D_3(a,b) D_1(Q-a,b) \\ \times \frac{\psi^t_{\pm} \cdot \langle \mathbf{l}^2_{-2} + \frac{2}{9}(c-16+3d(a,b))\mathbf{l}_{-4} \Phi_a \rangle_{\Delta+2} \cdot \psi_{\pm}}{\psi^t_{\pm} \cdot \langle \Phi_a \rangle_{\Delta+2} \cdot \psi_{\pm}} \,,$$

$$\Omega^{\pm}_{1,-1} \simeq r^{2(\Delta_a - \Delta_{a-b})} t_1(a,b) \frac{\psi^t_{\pm} \cdot \langle \Phi_{a-b} \rangle_{\Delta+2} \cdot \psi_{\pm}}{\psi^t_{\pm} \cdot \langle \Phi_a \rangle_{\Delta+2} \cdot \psi_{\pm}} \,.$$

Numerical results

	Ω^-	CFT	Ω^+	CFT
$1,1$	$-1.08264 \cdot 10^{10}$	$-1.08263 \cdot 10^{10}$	$-1.88711 \cdot 10^{10}$	$-1.88706 \cdot 10^{10}$
$1,3$	$2.18196 \cdot 10^{20}$	$2.18194 \cdot 10^{20}$	$1.20262 \cdot 10^{21}$	$1.20259 \cdot 10^{21}$
$3,1$	$2.04976 \cdot 10^{20}$	$2.04975 \cdot 10^{20}$	$1.20746 \cdot 10^{21}$	$1.20742 \cdot 10^{21}$
$1,-1$	-0.000266622	-0.000266688	-0.000309866	-0.000309887

This supports our conjecture.
Another evidence is given by comparing with the analogue of LeClair-Mussardo formula which was found by Pozsgay.