Integrable dynamics on polygons and the dimer integrable system

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- Several integrable dynamical systems on spaces of polygons have been studied in the last decades.

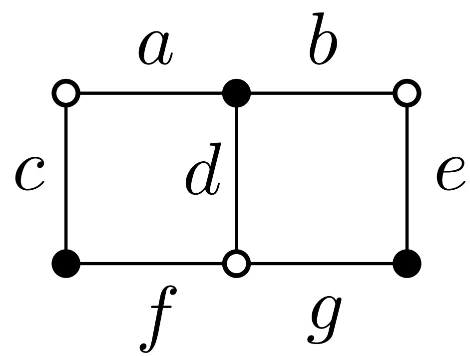
- Another class of integrable systems associated with bipartite dimer models on the torus was introduced by Goncharov and Kenyon in 2013.

- The setting of triple crossing diagram maps provides a common framework for both the geometric integrable systems and the dimer integrable system.

1 Integrable systems from bipartite dimer models on the torus

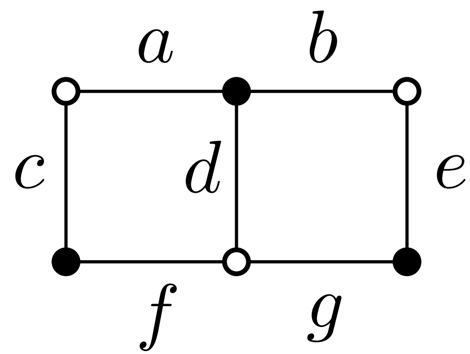
A model from statistical mechanics

- Setting: *planar bipartite graphs* (vertices can be colored black and white such that each edge has two endpoints of different colors) with *edge weights*.



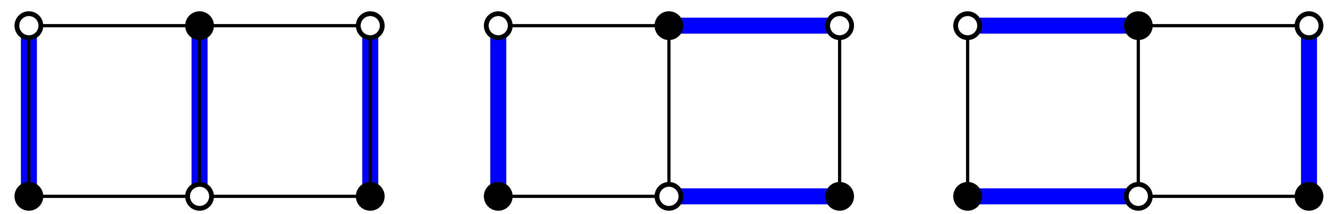
- In probability, edge weights are positive real numbers.

- For integrable systems and geometry purposes (this talk), edge weights are complex numbers.



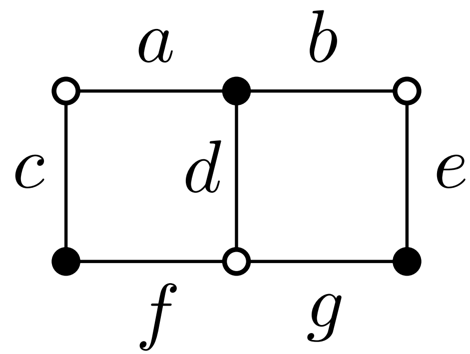
- *Dimer covering*: subset of edges such that each vertex is incident to exactly one edge.

Dimer coverings:



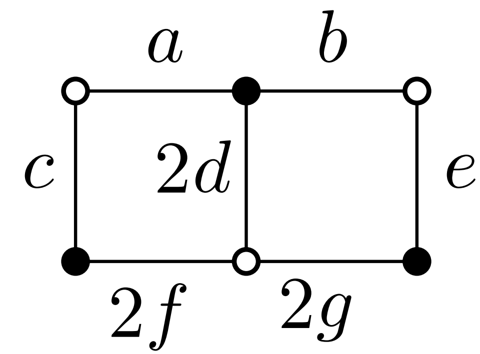
Weights: cde bcg aef

- *Boltzmann probability measure*: draw a dimer covering at random with probability proportional to its weight.



- Multiplying by $\lambda > 0$ the weight of every edge incident to a given vertex (*gauge transformation*) does not change the probability measure.

- Alternating products of edge weights around faces are coordinates on the space of edge weights modulo gauge. They are called *face weights*.



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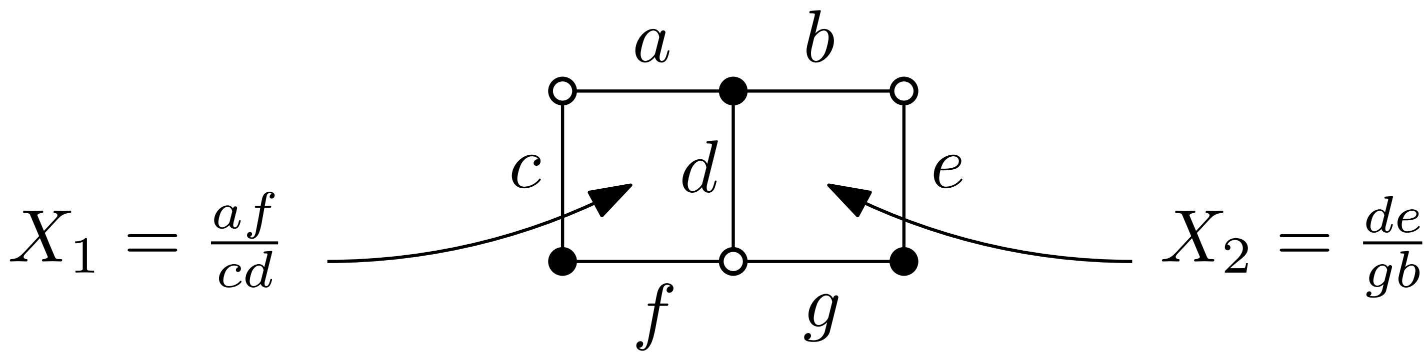

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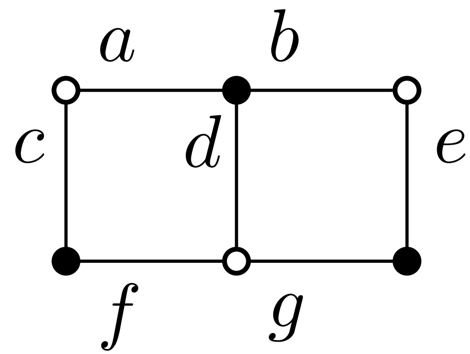
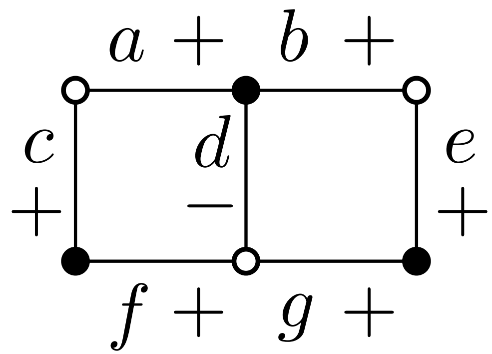
Kasteleyn edge weights



- To each edge, one can associate a *Kasteleyn sign*, such that the product of these signs around a face of degree $2k$ is $(-1)^{k+1}$.

- Multiplying the edge weights by the Kasteleyn signs, one gets *Kasteleyn edge weights*.

Kasteleyn edge weights



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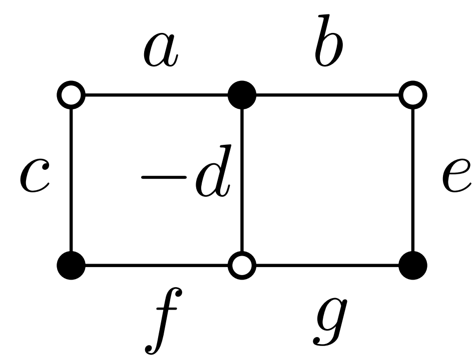

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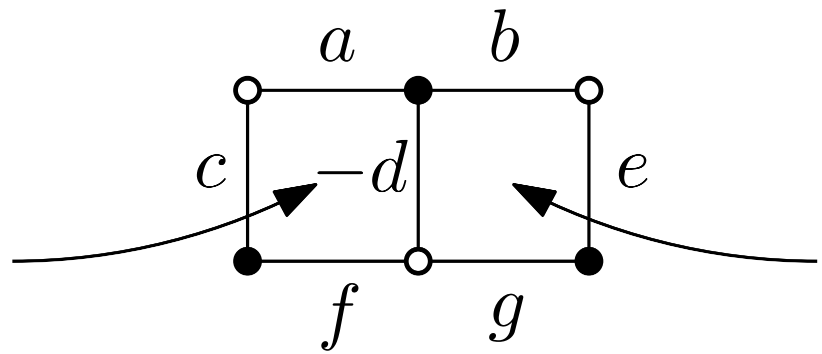
Kasteleyn edge weights

$$X_1 = (-1)^3 \frac{af}{c(-d)}$$



$$X_2 = (-1)^3 \frac{(-d)e}{gb}$$

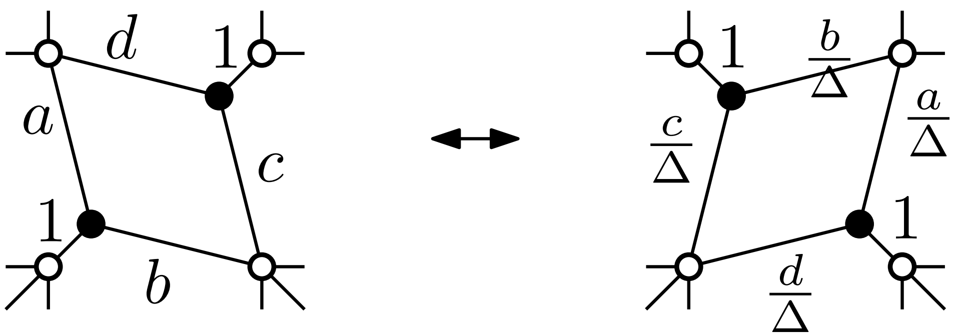
- If a face of degree $2k$ has face weight X, then the alternating product of Kasteleyn edge weights around the face equals $(-1)^{k+1}X$.

- The Kasteleyn matrix is the adjacency matrix of the graph with Kasteleyn edge weights. It is used to compute the partition function and correlations for the dimer model (Kasteleyn, Temperley-Fisher, 60s).

Two local moves

- These moves preserve the Boltzmann measure.

1. Spider move:



$$\Delta := ac + bd$$

Two local moves

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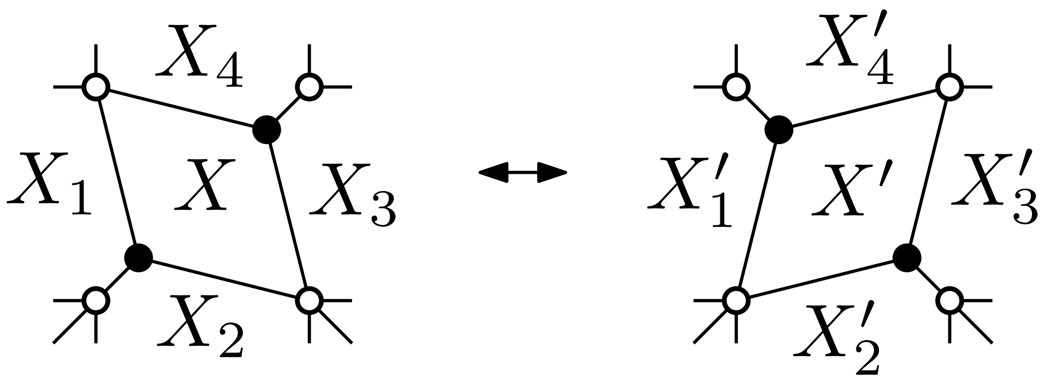


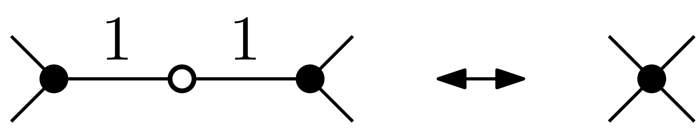
$$X' = X^{-1}$$
$$X_1' = X_1(1+X)$$
$$X_2' = \frac{X_2}{1+X^{-1}}$$
$$X_3' = X_3(1+X)$$
$$X_4' = \frac{X_4}{1+X^{-1}}$$

- The change in the face weights is a special instance of mutation of coefficient variables in cluster algebras.

Two local moves

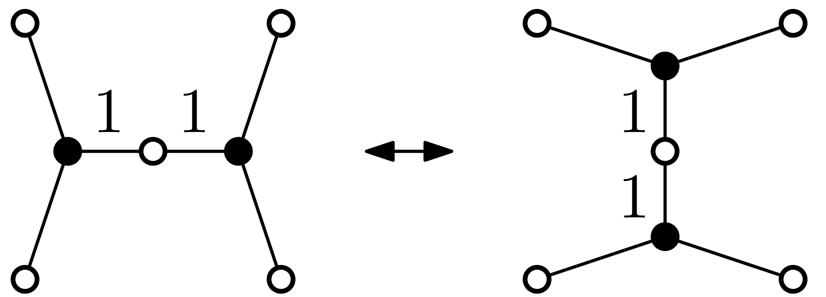
- These moves preserve the Boltzmann measure.

2. Contraction/expansion of degree-two vertex:



- Face weights don't change.

- May be recombined into a *resplit* move:

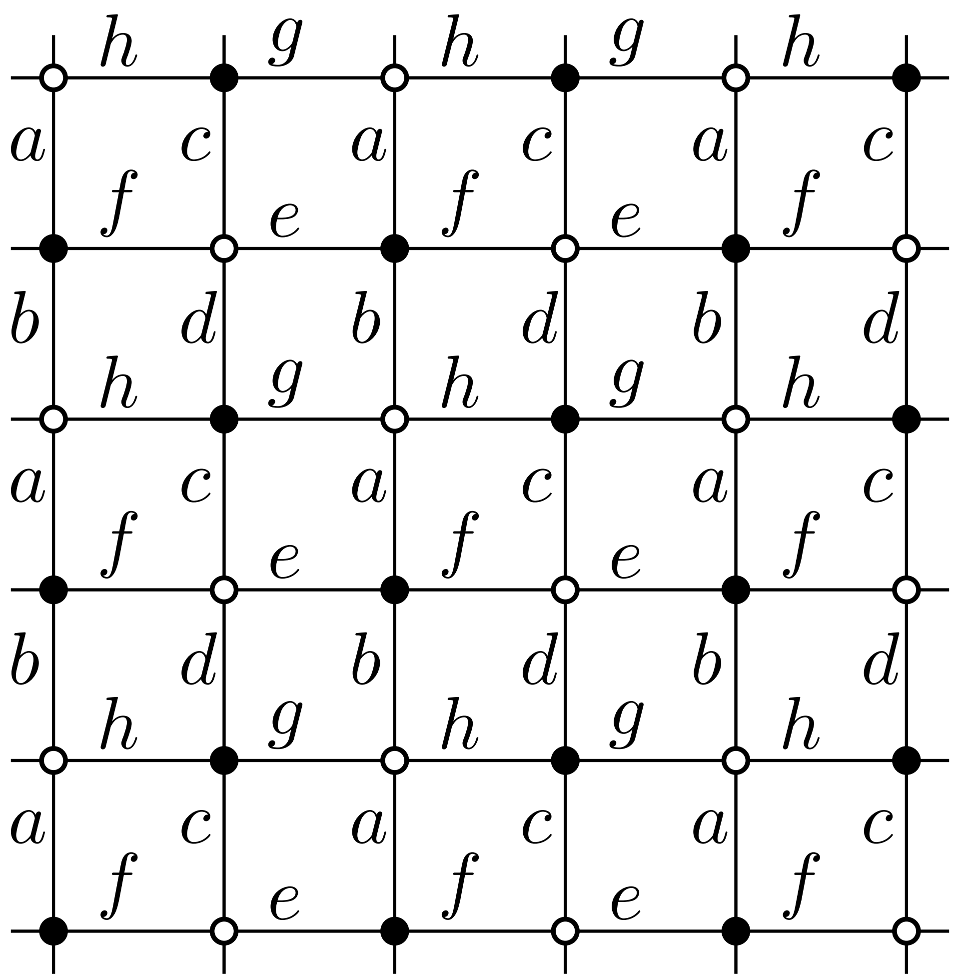


Discrete-time integrable dynamics

- We use these moves to define discrete-time dynamics on face-weighted bipartite graphs on the torus.

- One step of the dynamics will bring us back to the same combinatorial graph, but the face weights will potentially have changed.

- An equivalent way of working with torus graphs is to consider infinite planar graphs that are periodic in two directions, with weights also periodic in two directions.



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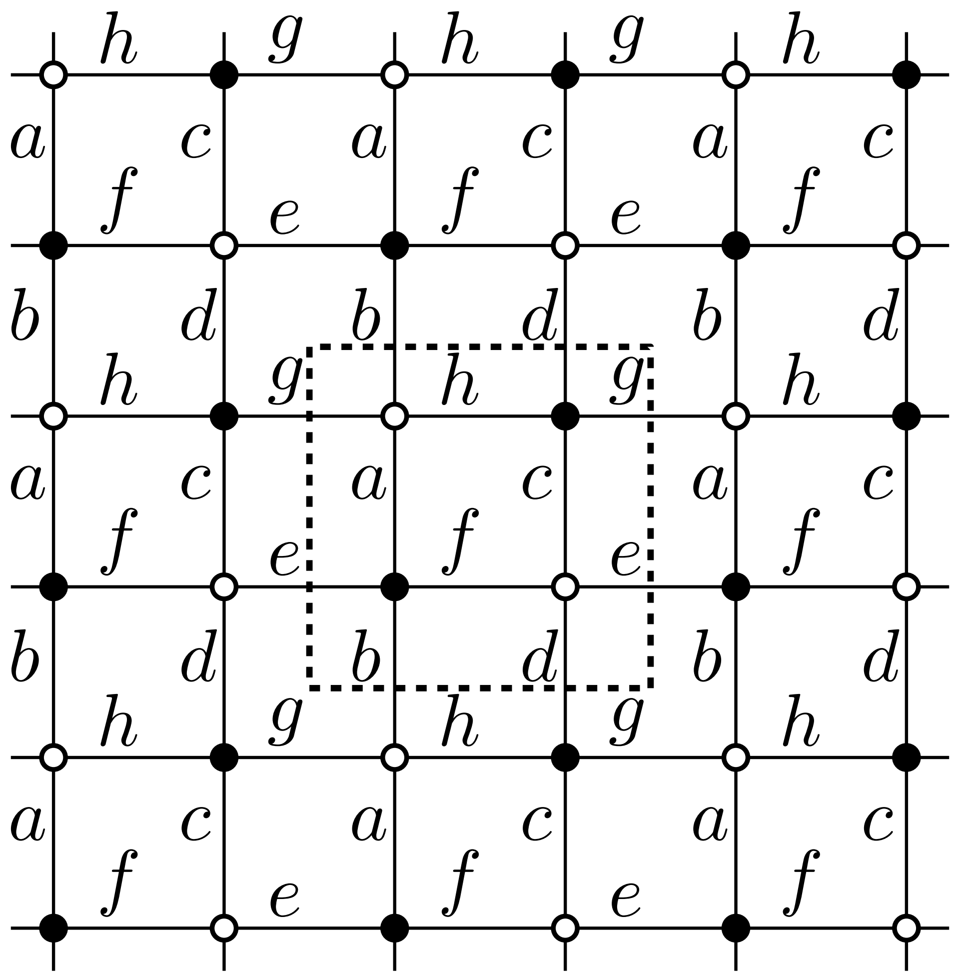
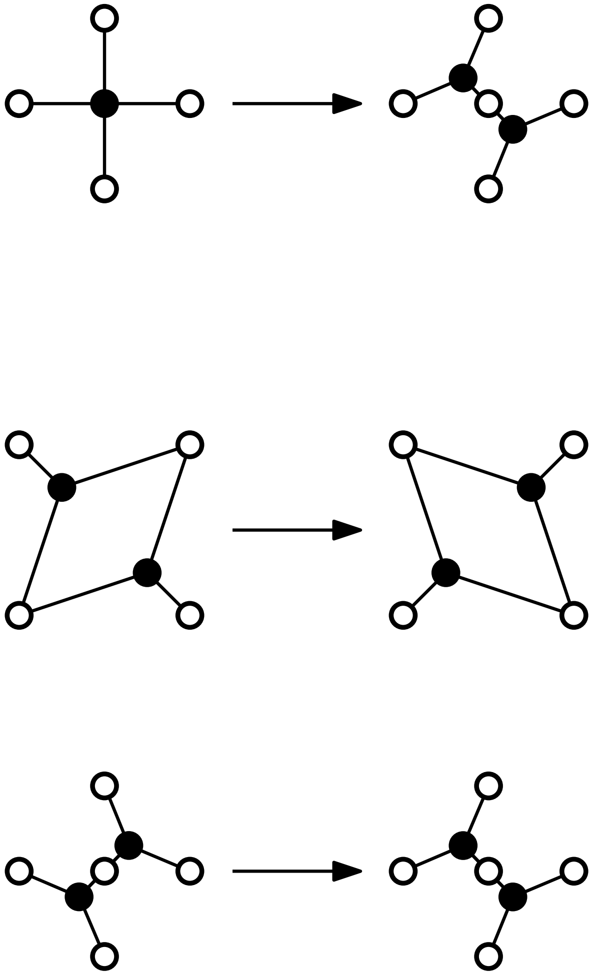
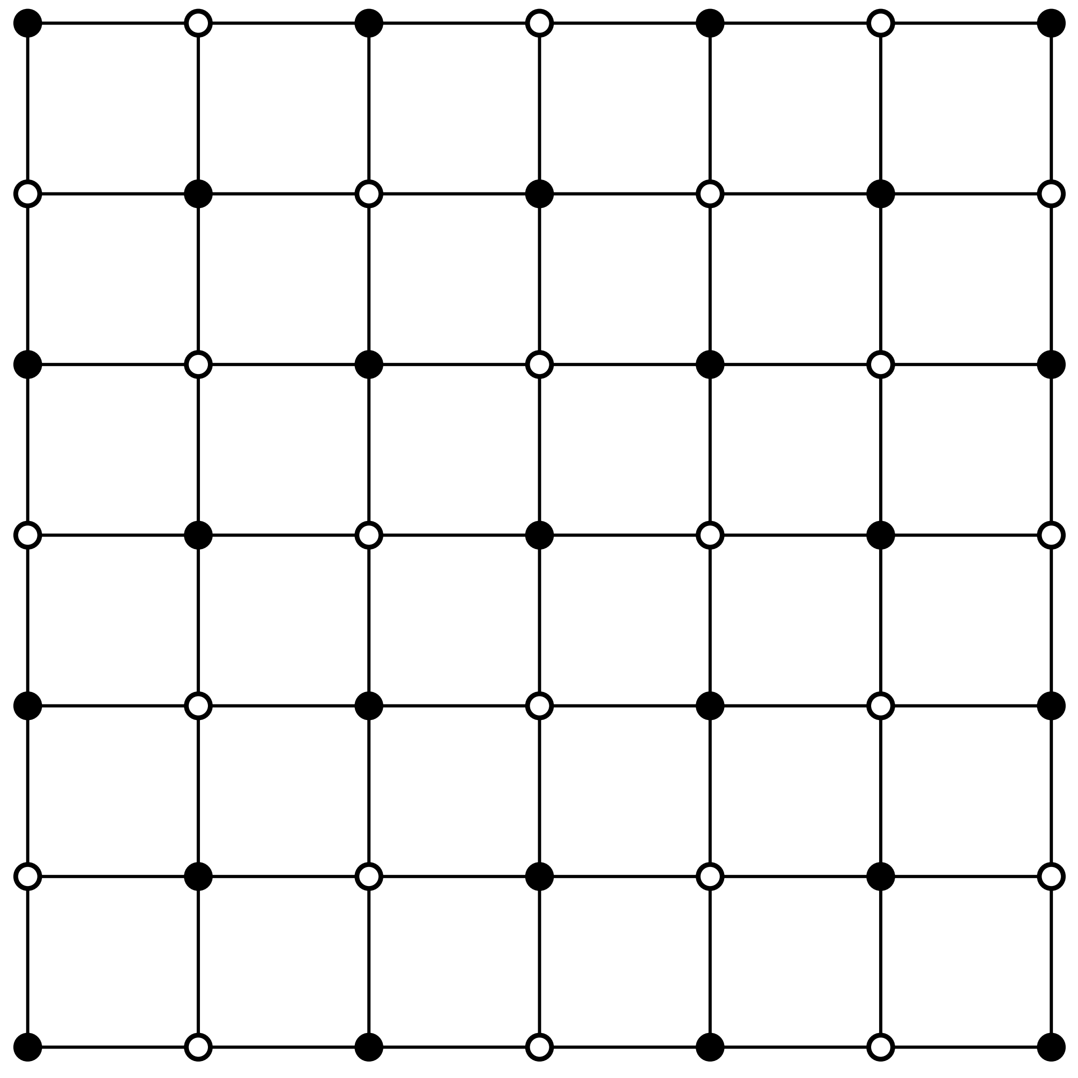
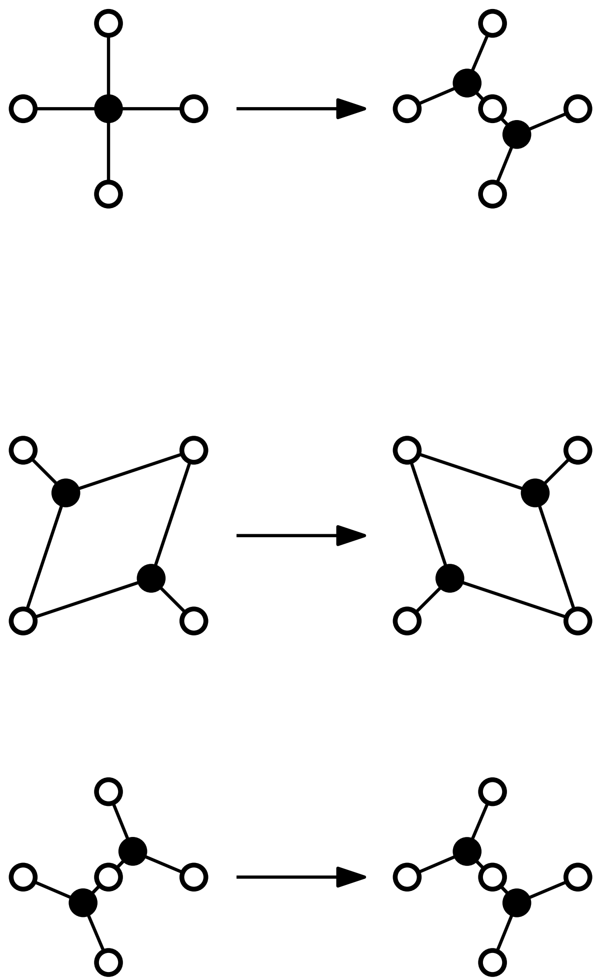


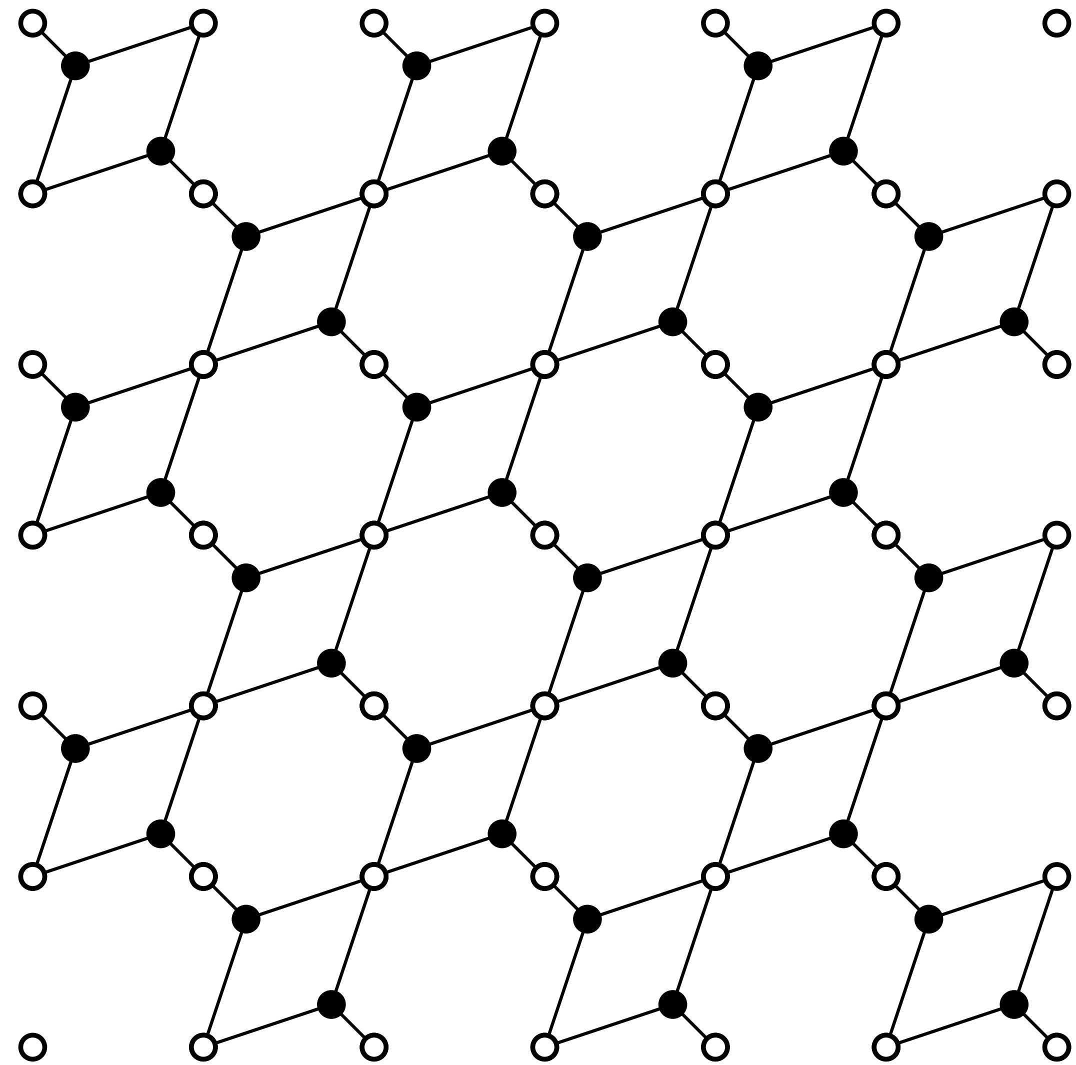
Square grid example





Square grid example





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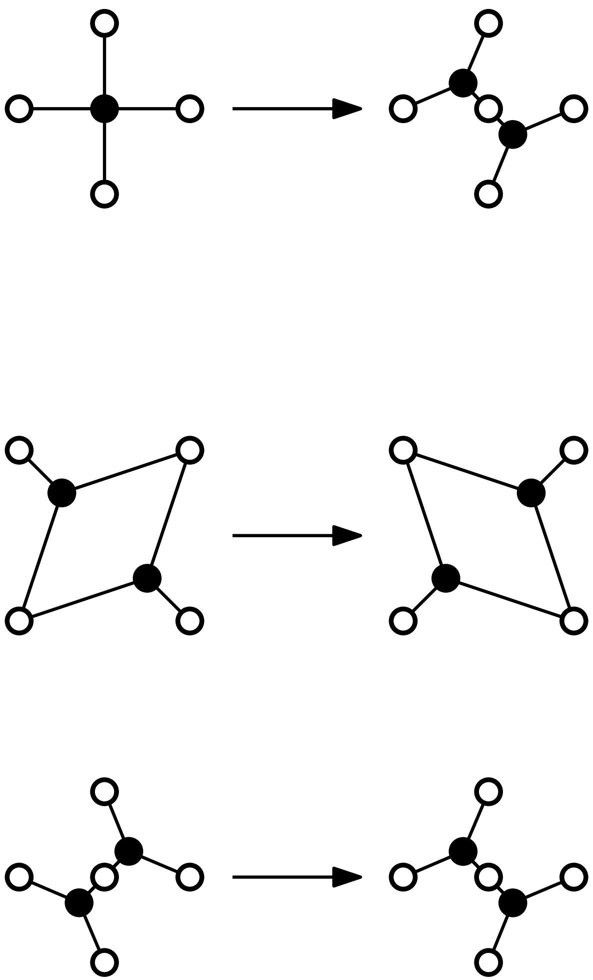

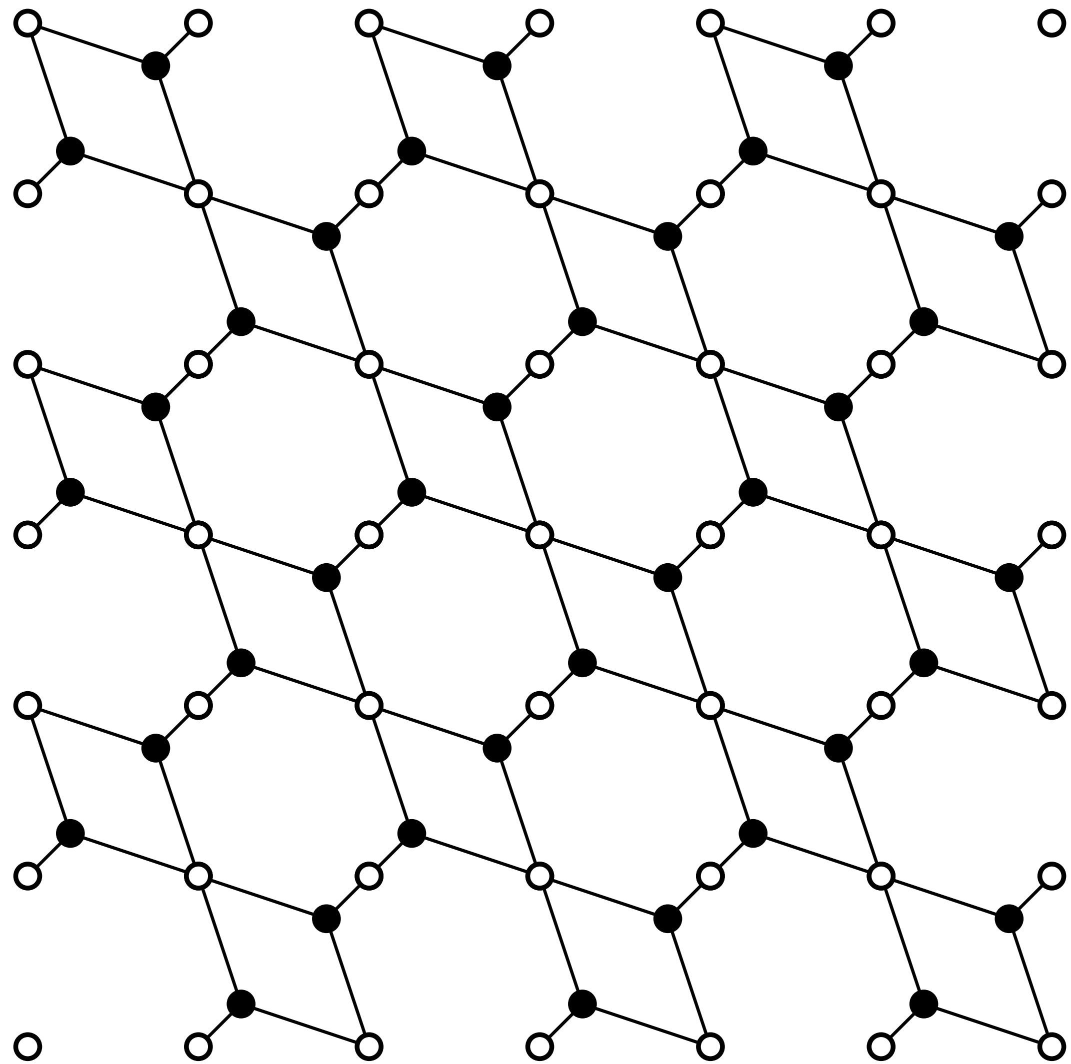

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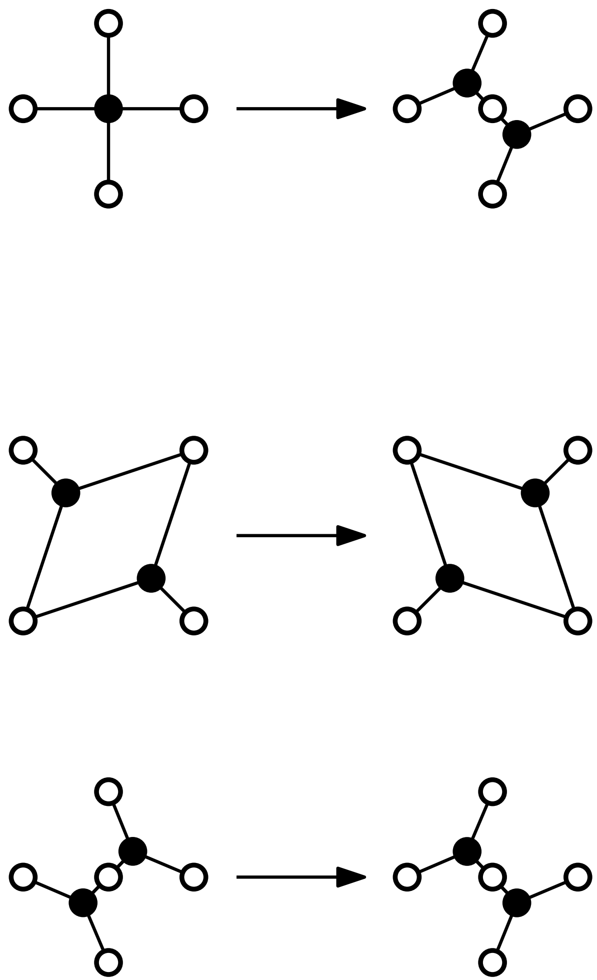

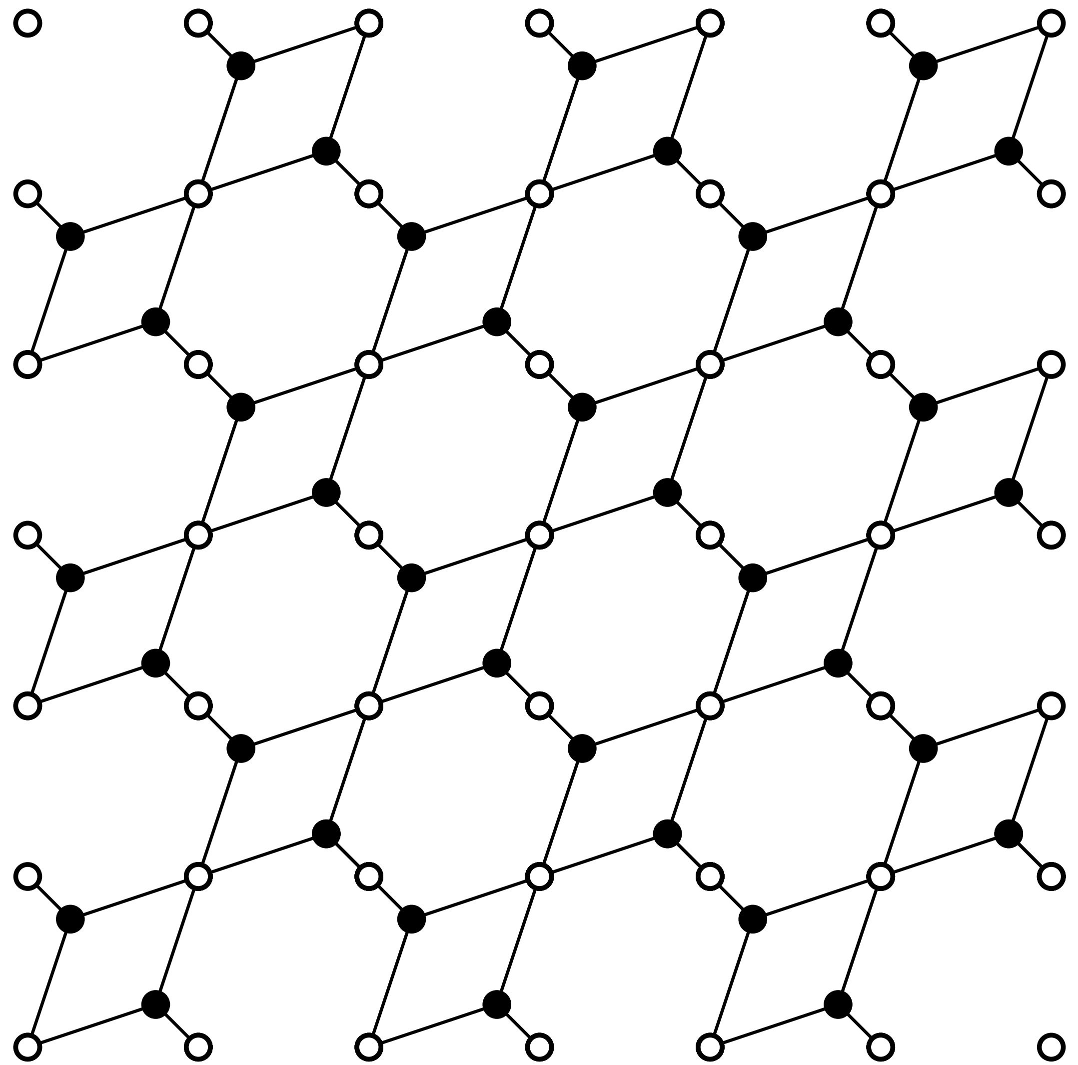

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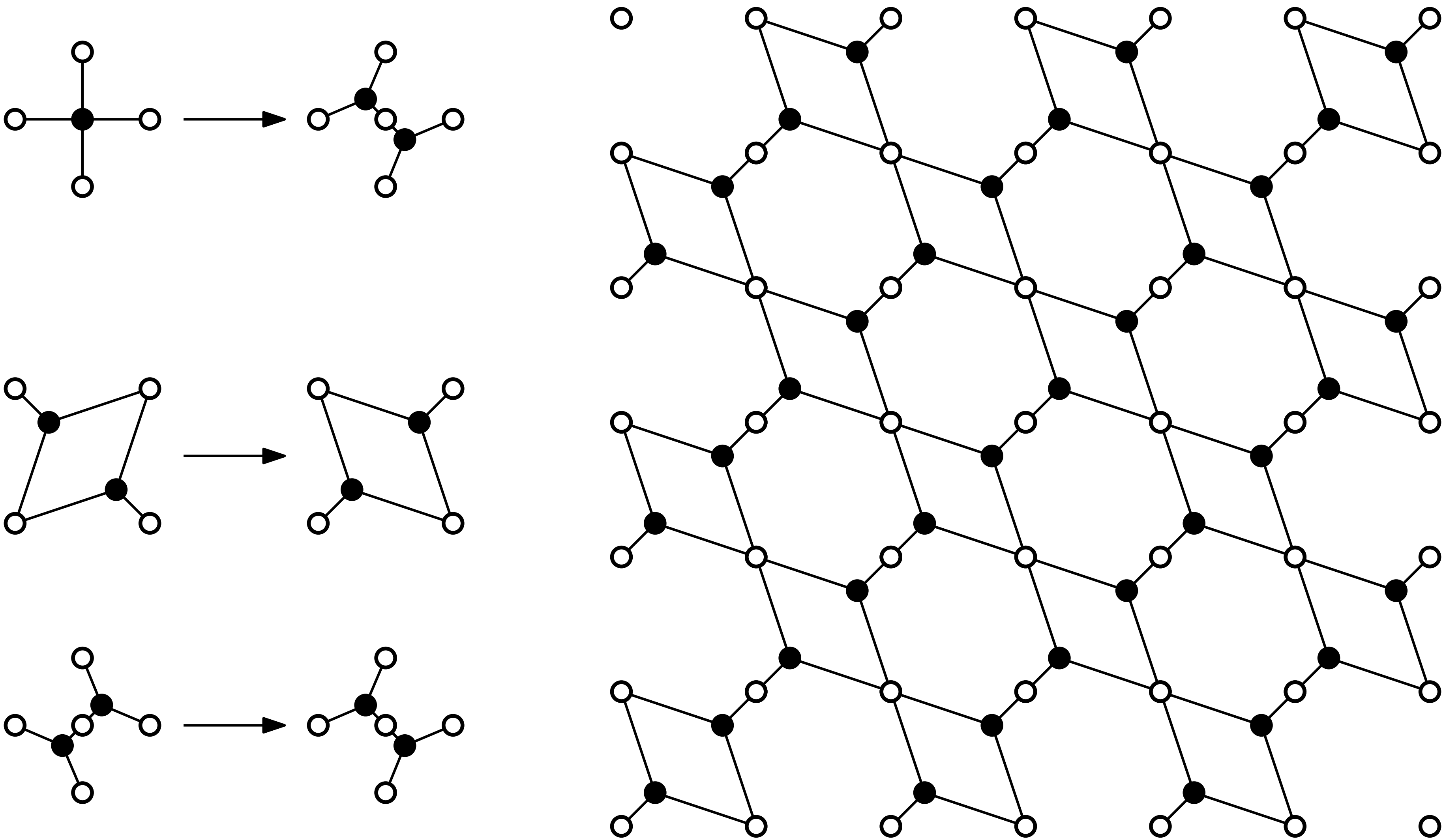

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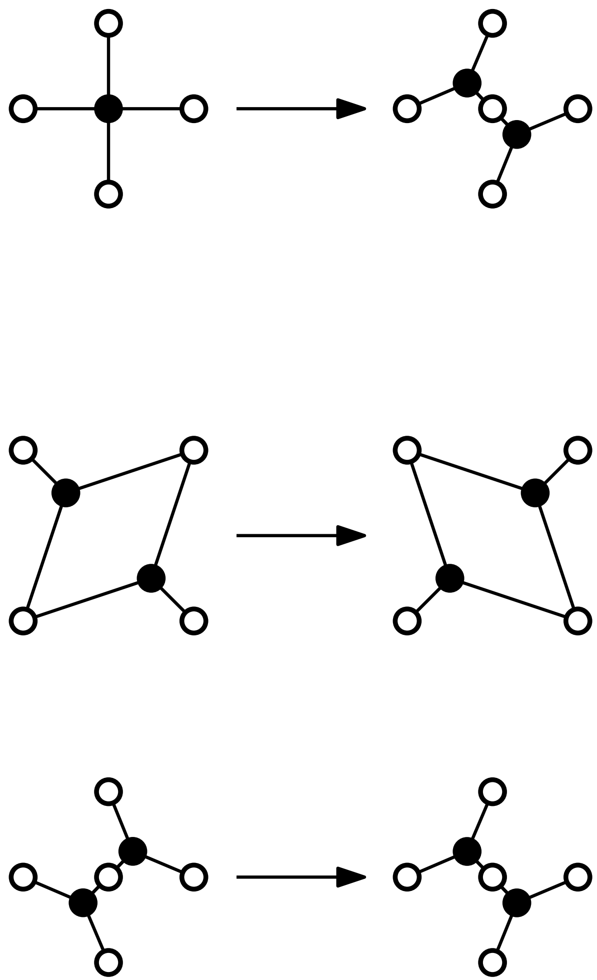
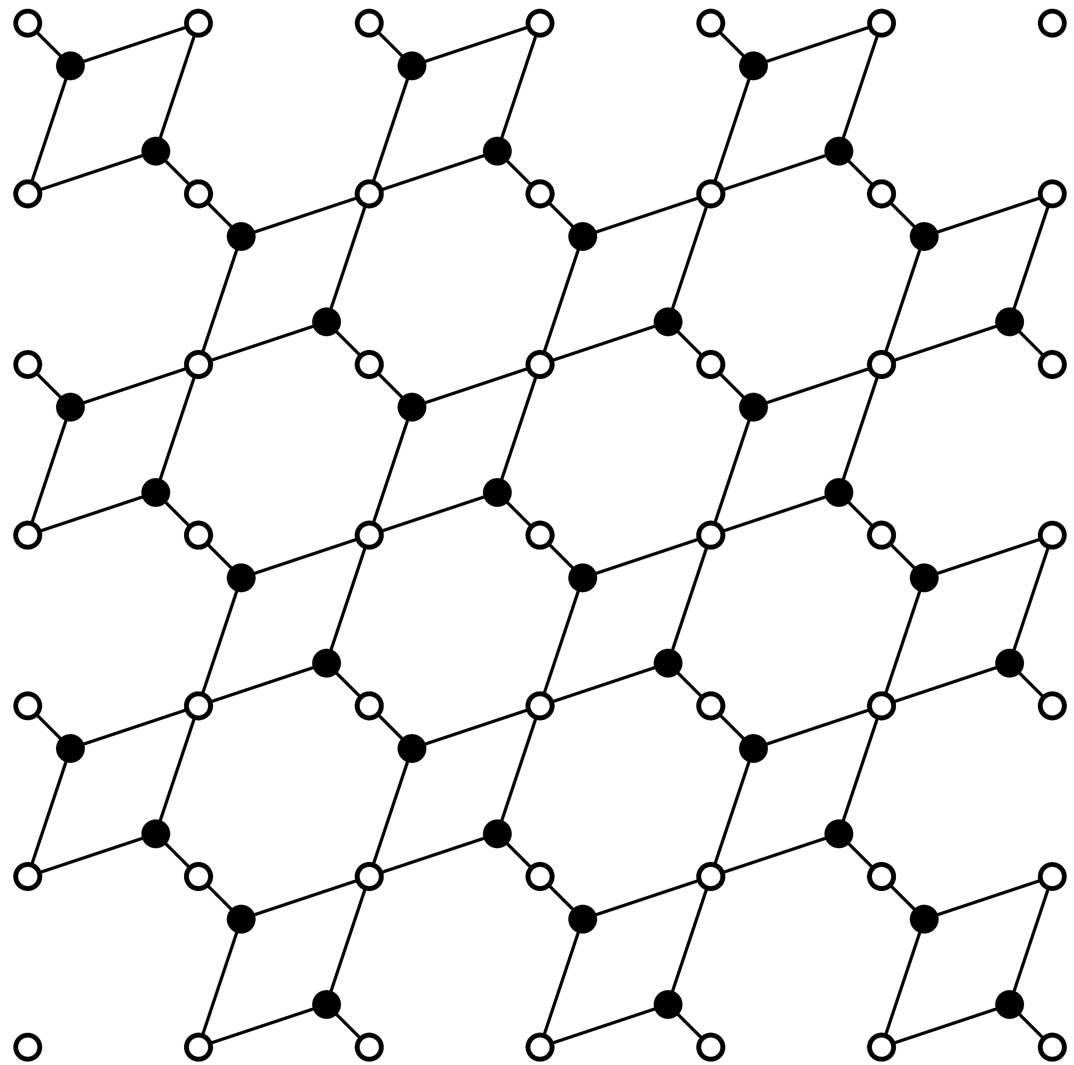




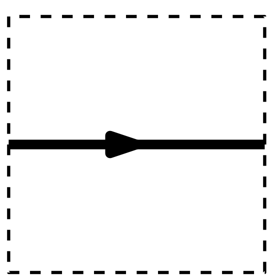
- For any given bipartite graph on the torus, this discrete-time dynamics is integrable, in the sense that it has "enough" conserved quantities.

- These conserved quantities have a particular structure with respect to a Poisson bracket. In the right coordinates, the motion is translation on some high-dimensional torus (Goncharov-Kenyon '13, Fock-Marshakov '16, Vichitkunakorn '18 and George-Inchiostro '22).

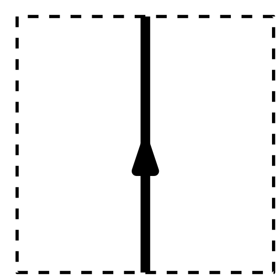
- The conserved quantities are partition functions for dimer covers on the torus with prescribed homology.

Homology group of the torus

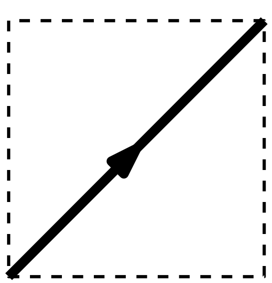
- One associates to a collection of directed loops on the torus $\mathbb{T}$ its *homology*, which measures how many times it winds around each direction of the torus.



homology $(1, 0)$



homology $(0, 1)$



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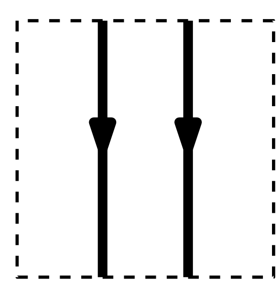
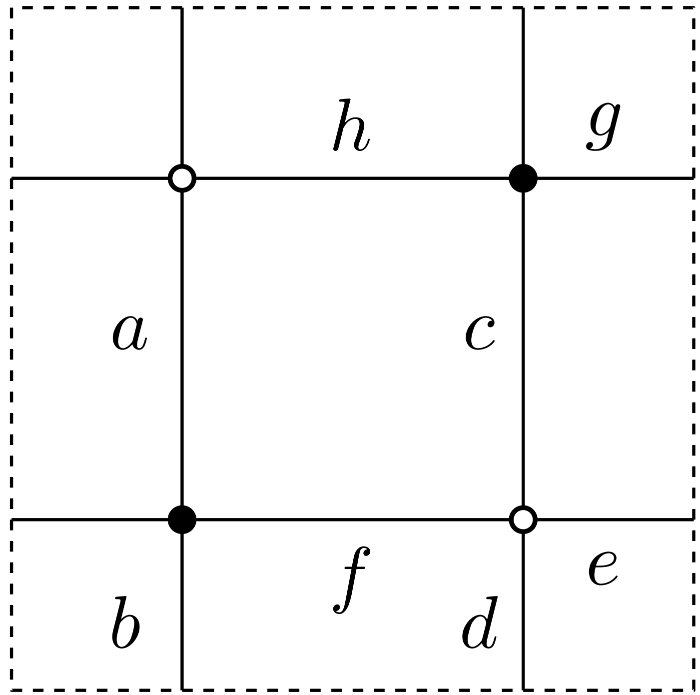
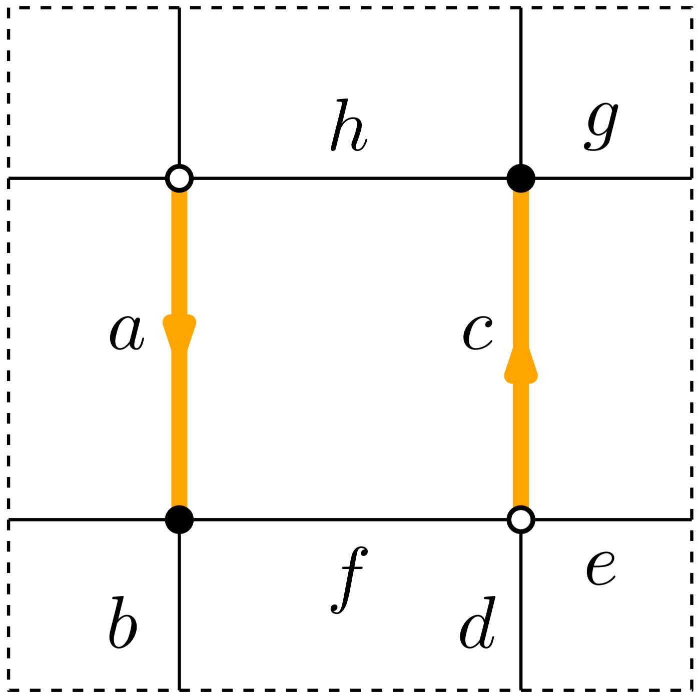


homology $(0, -2)$



- Pick a reference dimer covering (orange) and orient it from white vertices to black vertices.

- An arbitrary dimer covering (blue) gives rise to directed loops by orienting it from black to white and concatenating it with the reference dimer covering.



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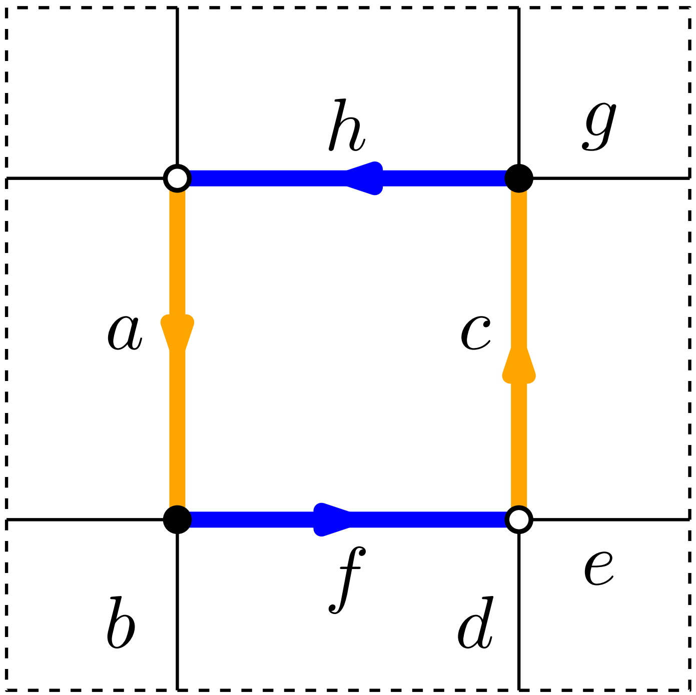

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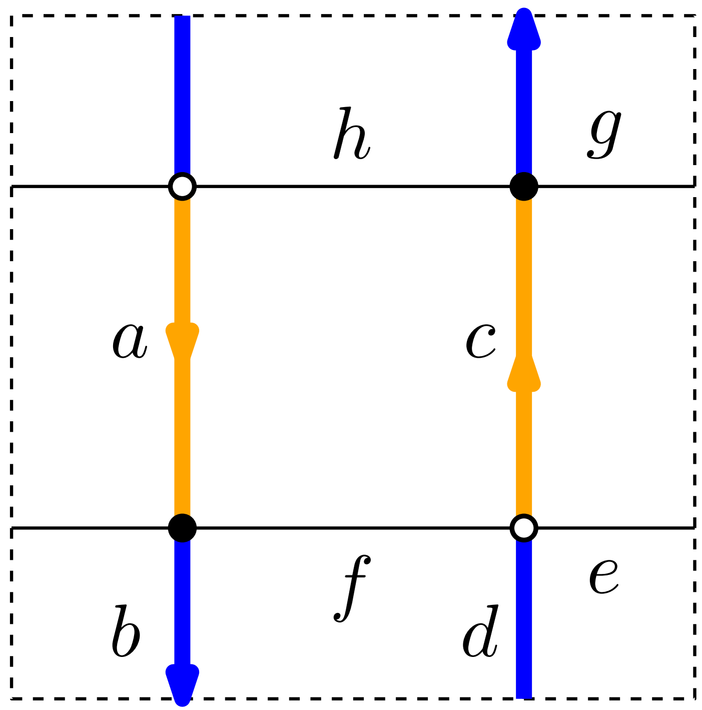

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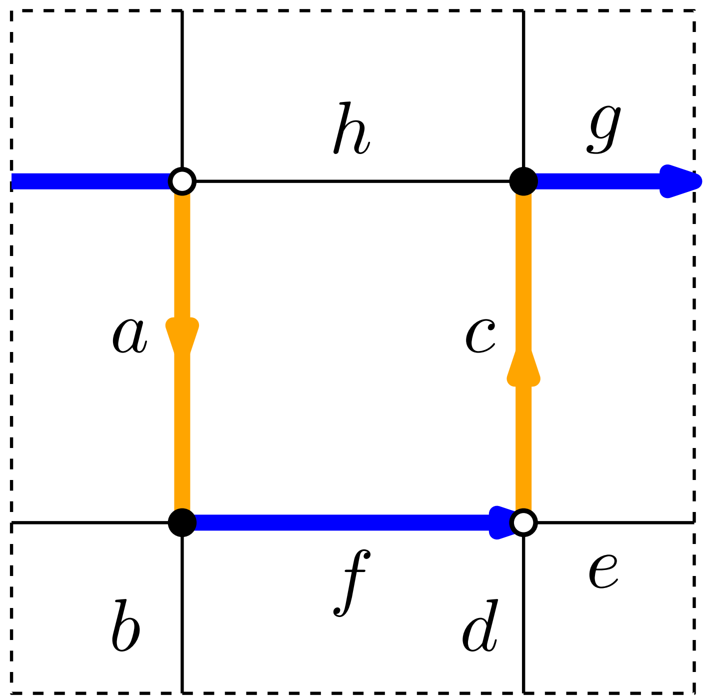

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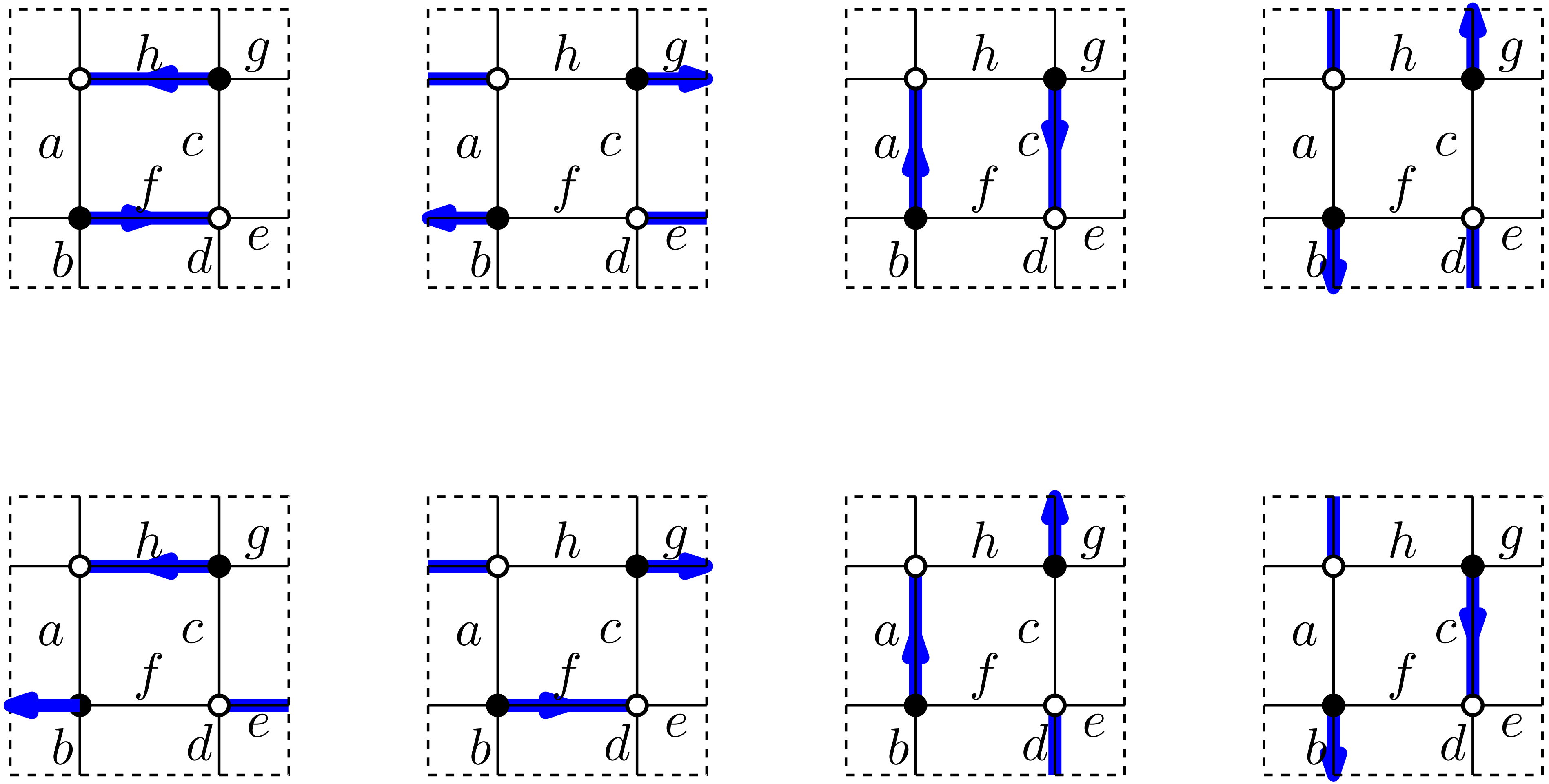
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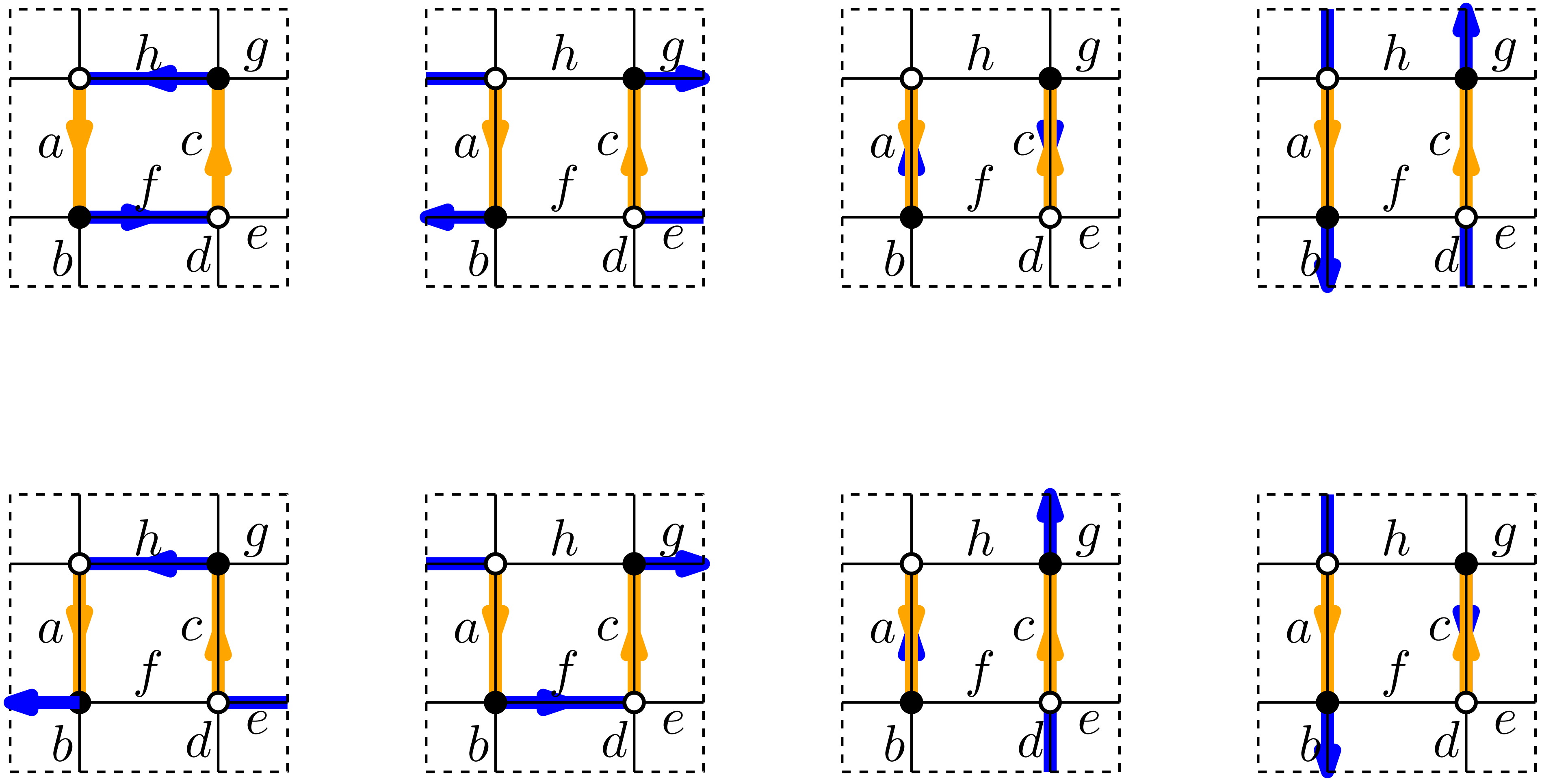
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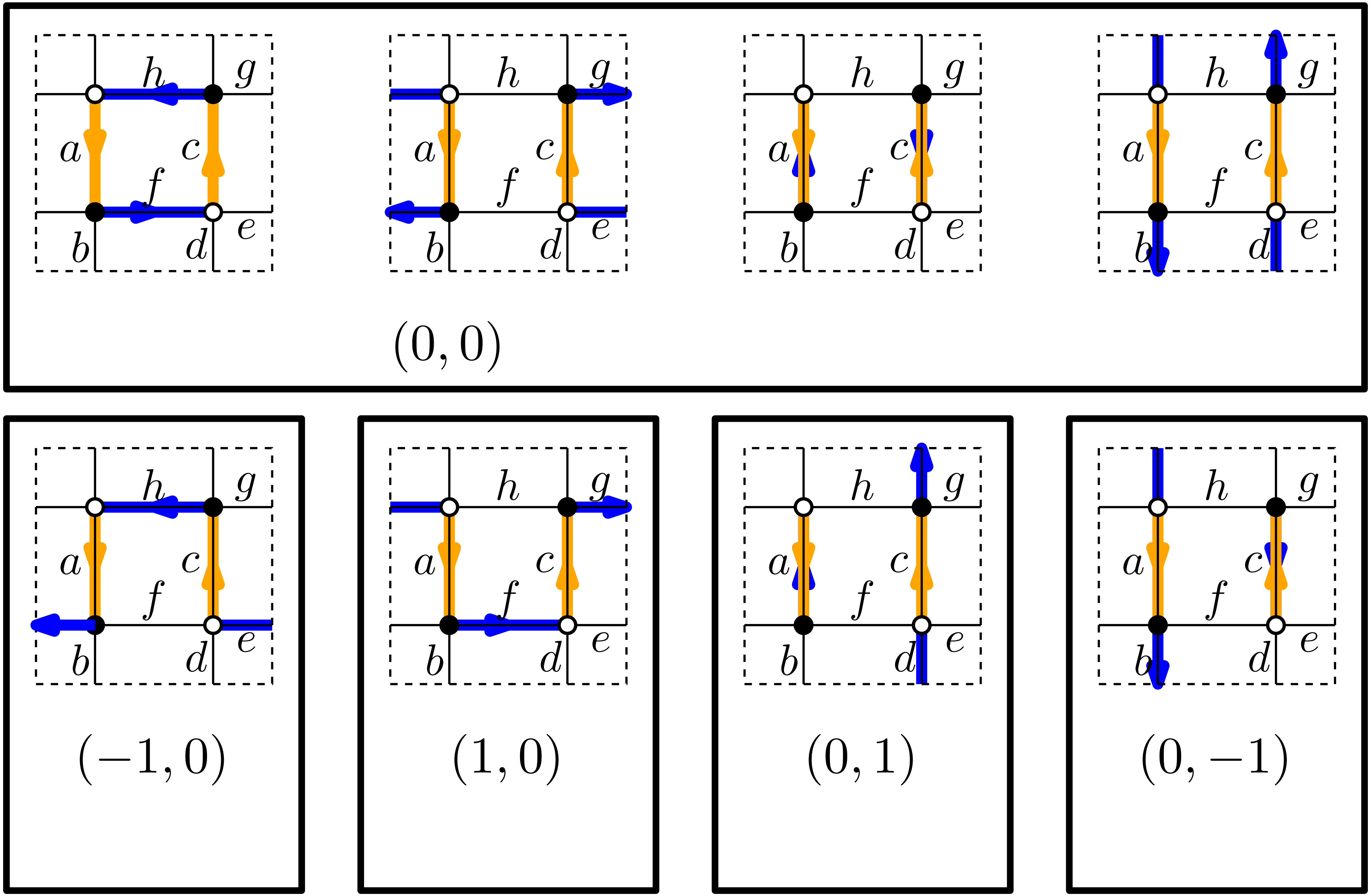

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(−1, 0)
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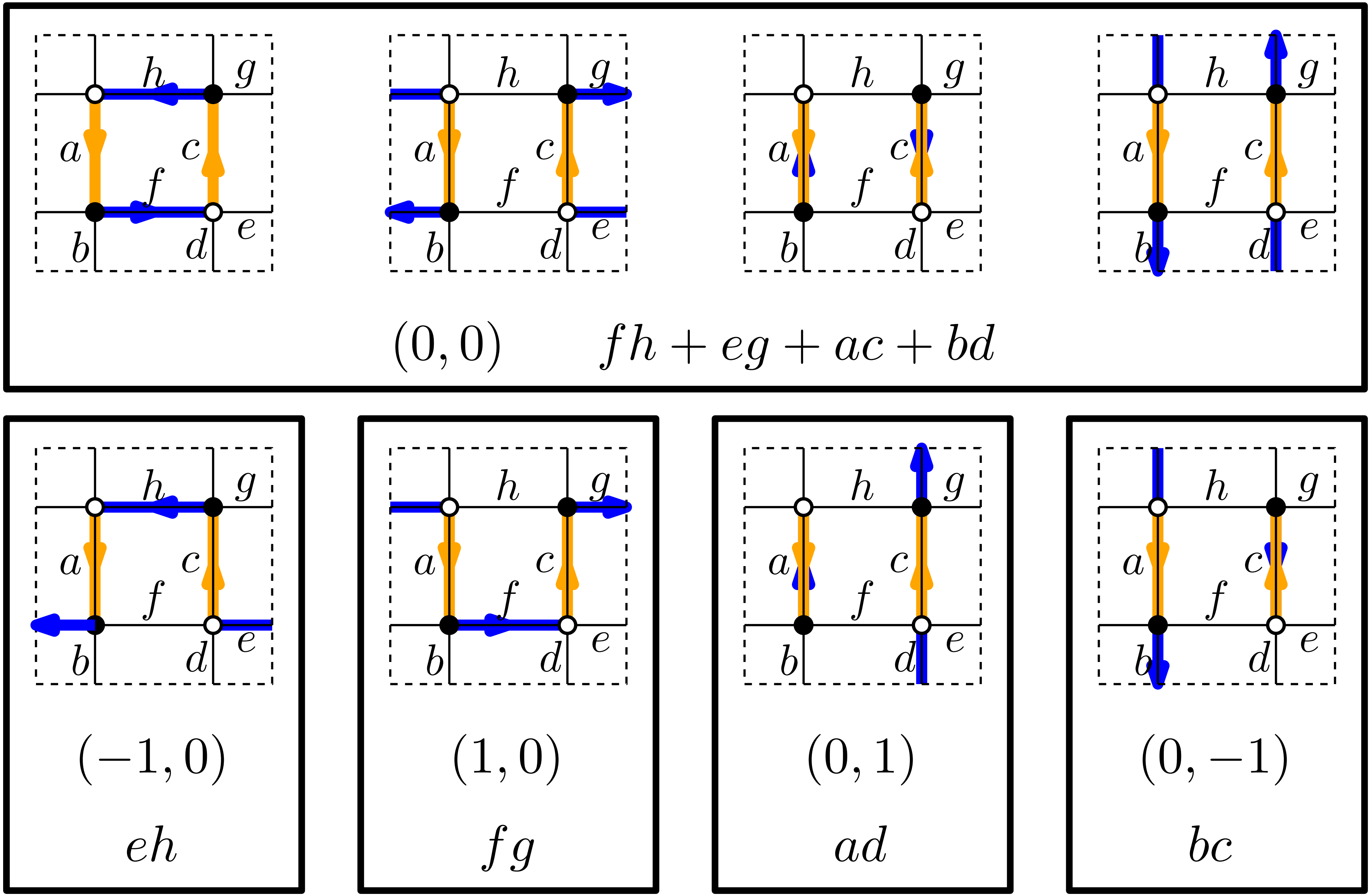
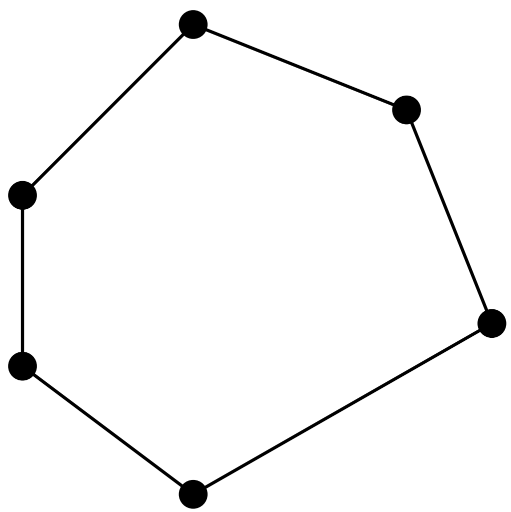
(0, 1)
(0, −1)



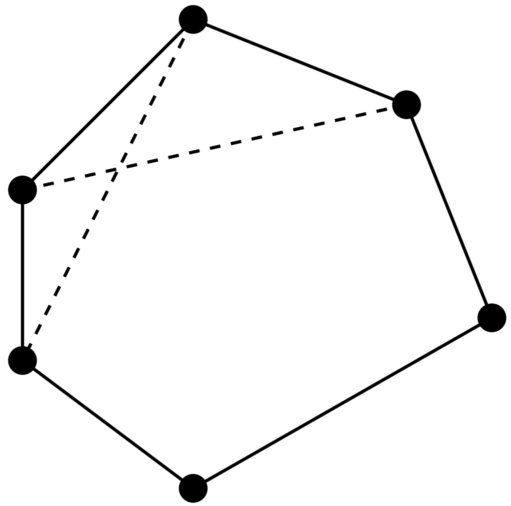
h g a c f b d e
(0, 0) fh + eg + ac + bd
(−1, 0) eh
(1, 0) fg
(0, 1) ad
(0, −1) bc

2 The pentagram map



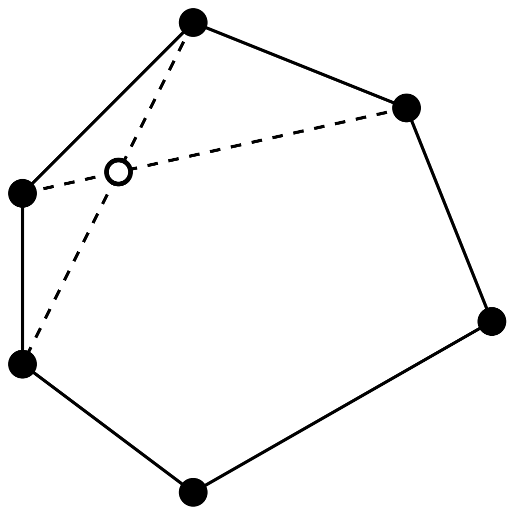
$n = 6$

- points at time 0



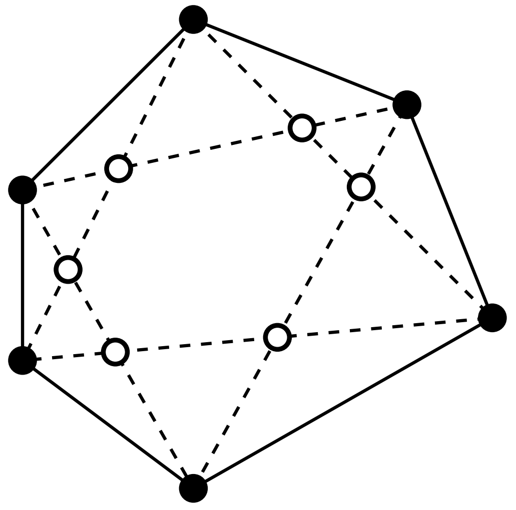
$n = 6$

- points at time 0



$n = 6$

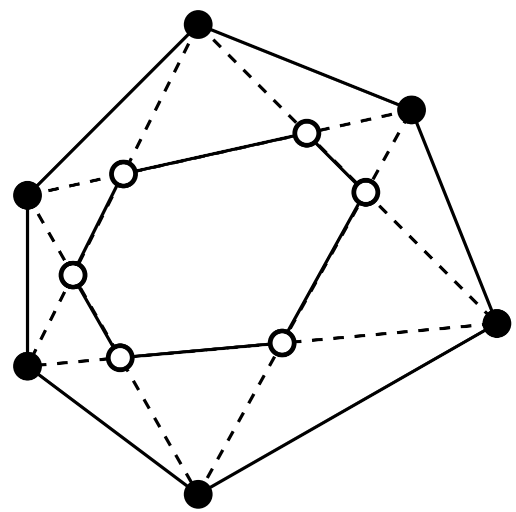
- points at time 0
- points at time 1



$n = 6$

● points at time 0

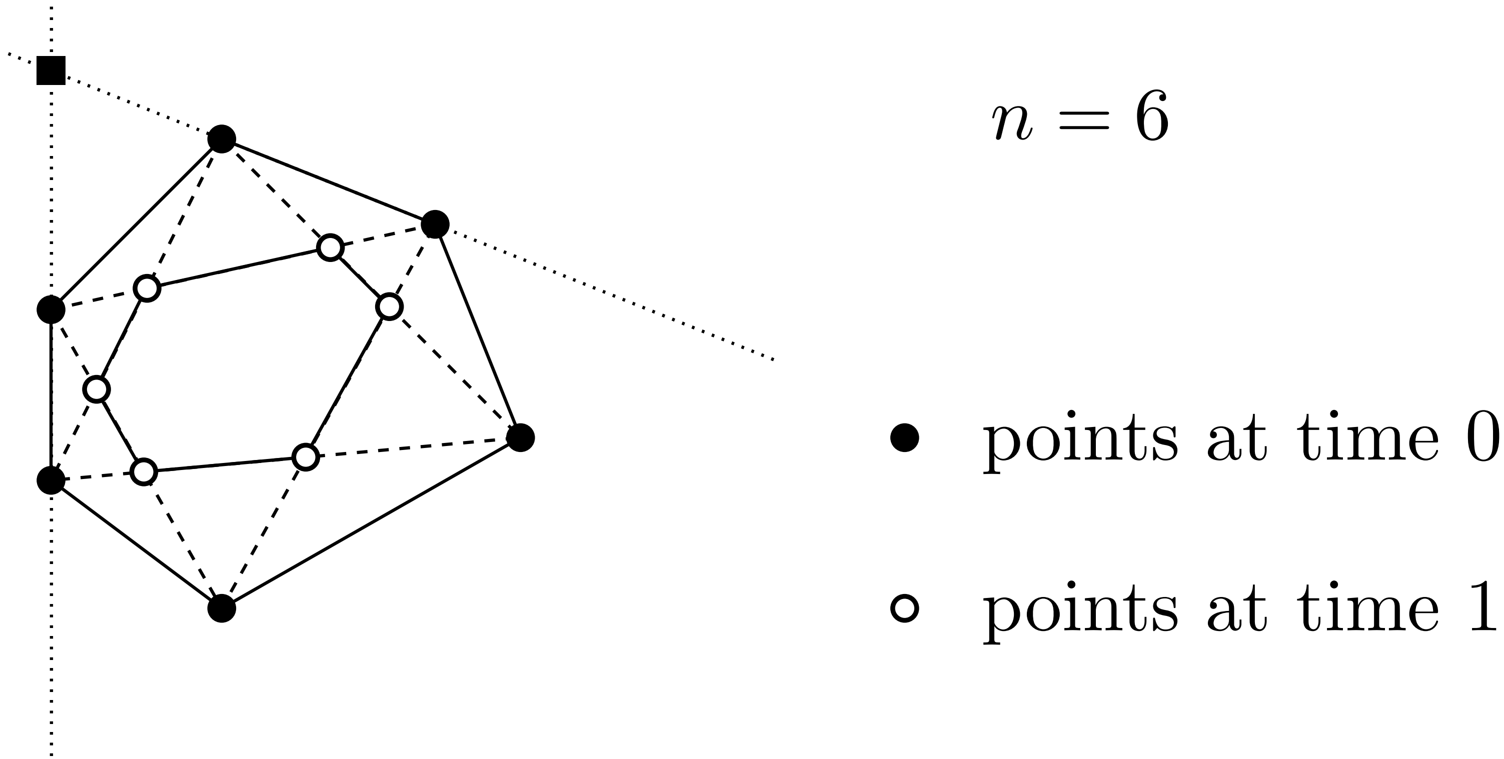
○ points at time 1

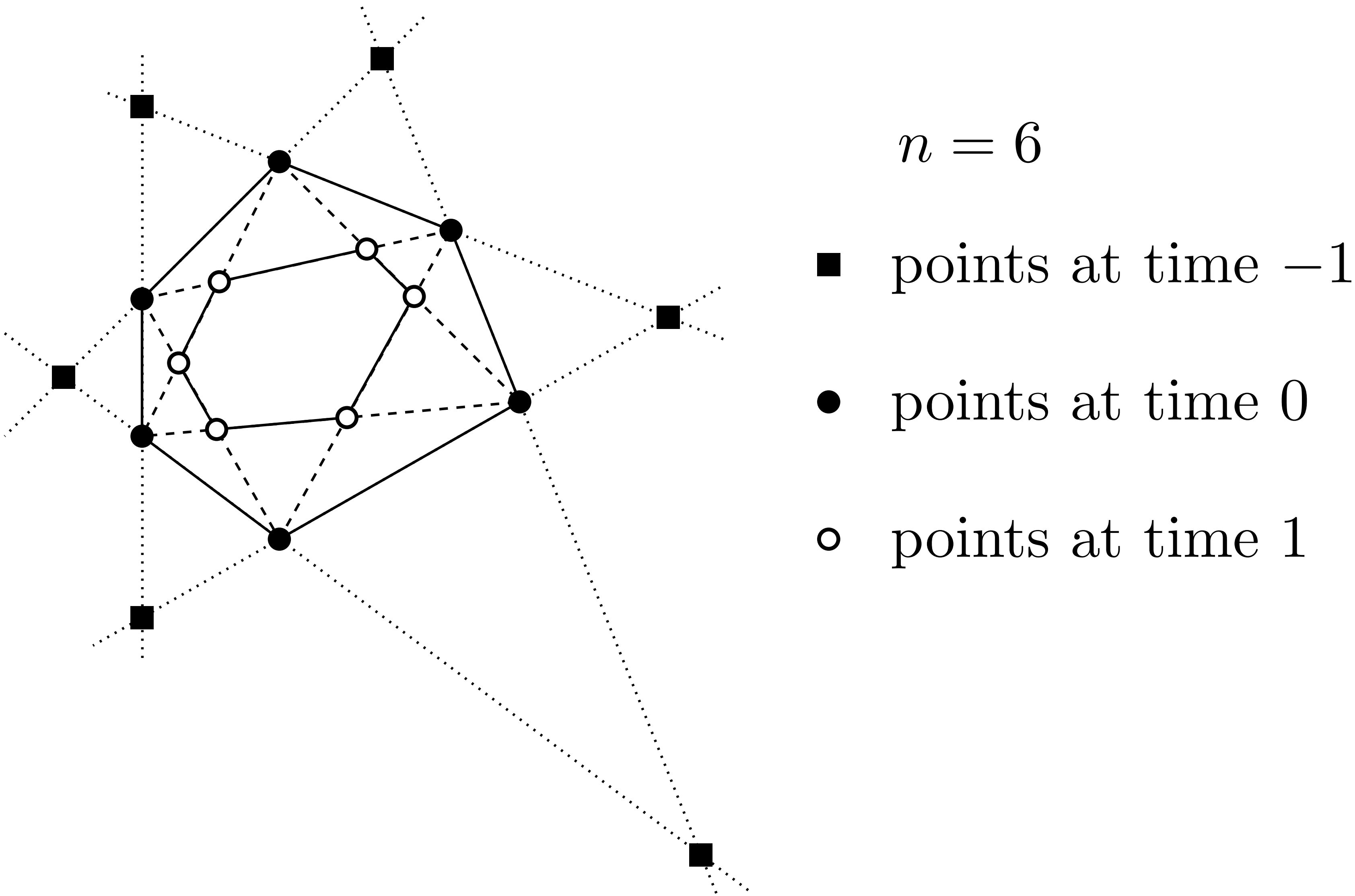


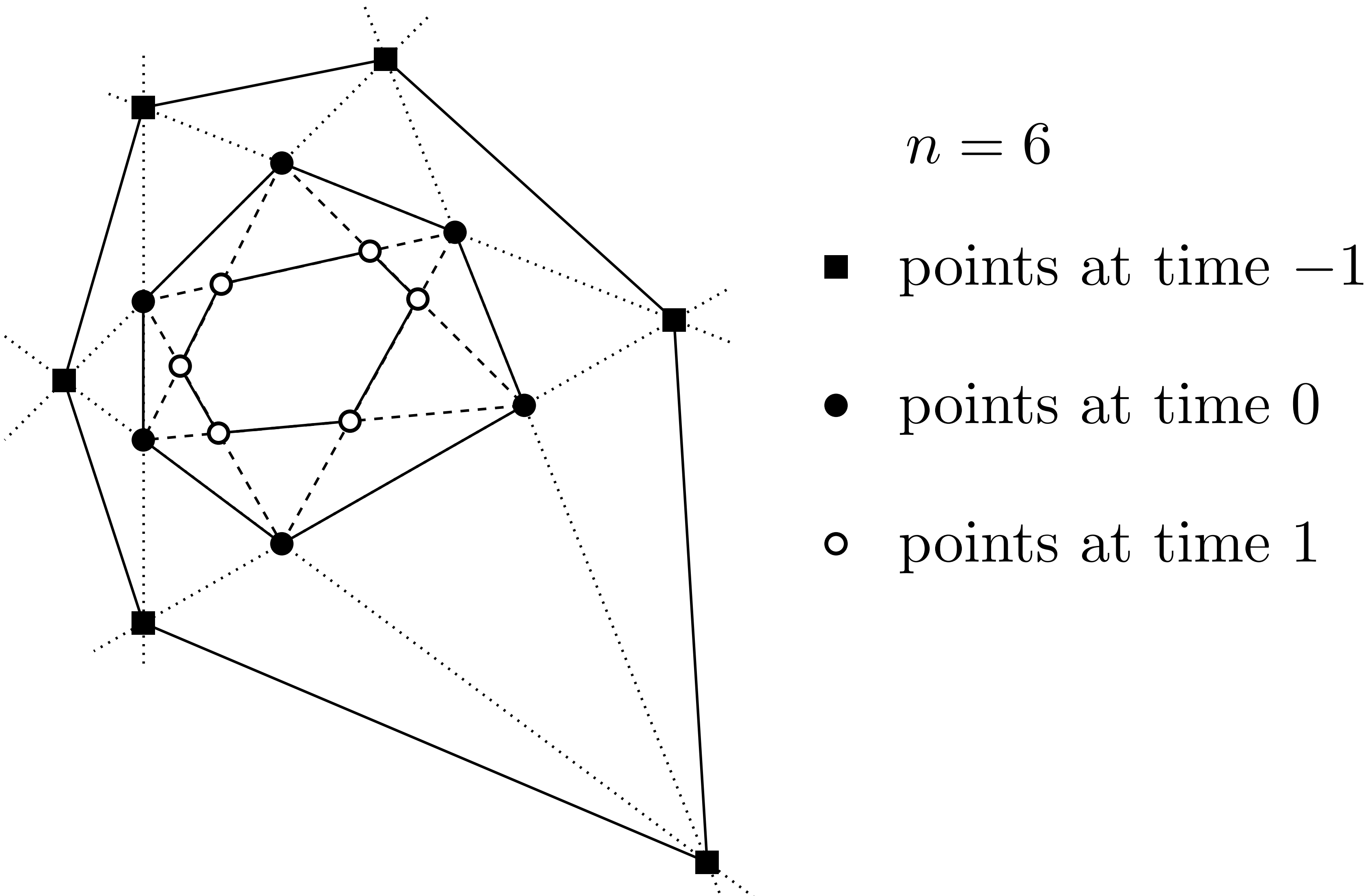
$n = 6$

● points at time 0

○ points at time 1


$n = 6$
points at time 0
points at time 1


$n = 6$
points at time -1
points at time 0
points at time 1



- Discrete-time dynamics on the space of n-gons with vertices in $\mathbb{C}P^2$, considered up to $PGL_3(\mathbb{C})$ (Schwartz '92).

Reminders on projective geometry

- Fix $m \geq 1$. Points of the projective space $\mathbb{C}P^m$ are defined to be lines in the vector space $\mathbb{C}^{m+1}$.

- If v is a vector in $\mathbb{C}^{m+1}$ and P is the point of $\mathbb{C}P^m$ associated with the line $\mathbb{C}v$, then v is called a *lift* of P.

- A line in $\mathbb{C}P^m$ is defined to be a plane in $\mathbb{C}^{m+1}$.

Reminders on projective geometry

- Any $A \in GL_{m+1}(\mathbb{C})$ acts on the lines of $\mathbb{C}^{m+1}$. For any $\lambda \in \mathbb{C}$, λA has the same action as A on these lines.

- The automorphisms of $\mathbb{C}P^m$ are given by the elements of $PGL_{m+1}(\mathbb{C})$, which is the quotient of $GL_{m+1}(\mathbb{C})$ by the subgroup of scalar matrices.

- An affine chart of $\mathbb{C}P^m$ is given by the affine space $\mathbb{C}^m$.

Reminders on projective geometry

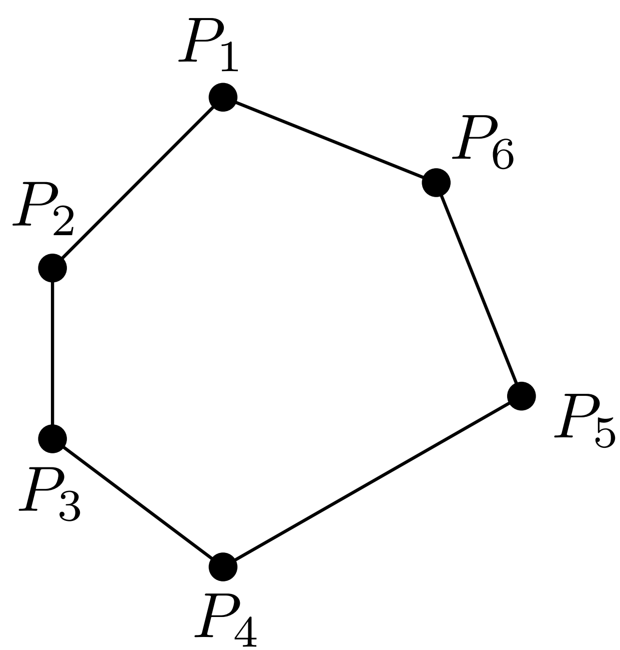
- If $(P_1, P_2, \ldots, P_{2k})$ are $2k$ points in $\mathbb{C}P^m$ such that each P_{2i} lies on the line $(P_{2i-1}P_{2i+1})$, their *multi-ratio* is defined by
$$\mathrm{mr}(P_1, P_2, \ldots, P_{2k}) = \frac{(P_1-P_2)(P_3-P_4)\cdots(P_{2k-1}-P_{2k})}{(P_2-P_3)(P_4-P_5)\cdots(P_{2k}-P_1)}.$$

- The definition is independent of the affine chart chosen to compute the ratios along each line.

- When $k = 2$, the multi-ratio of four aligned points is called their *cross-ratio*.

Twisted polygons

- A closed n-gon can be seen as a bi-infinite sequence of points $(\ldots, P_{-1}, P_0, P_1, P_2, \ldots)$ in $\mathbb{C}P^2$ which is n-periodic, i.e. such that for every $i \in \mathbb{Z}$, $P_{i+n} = P_i$.



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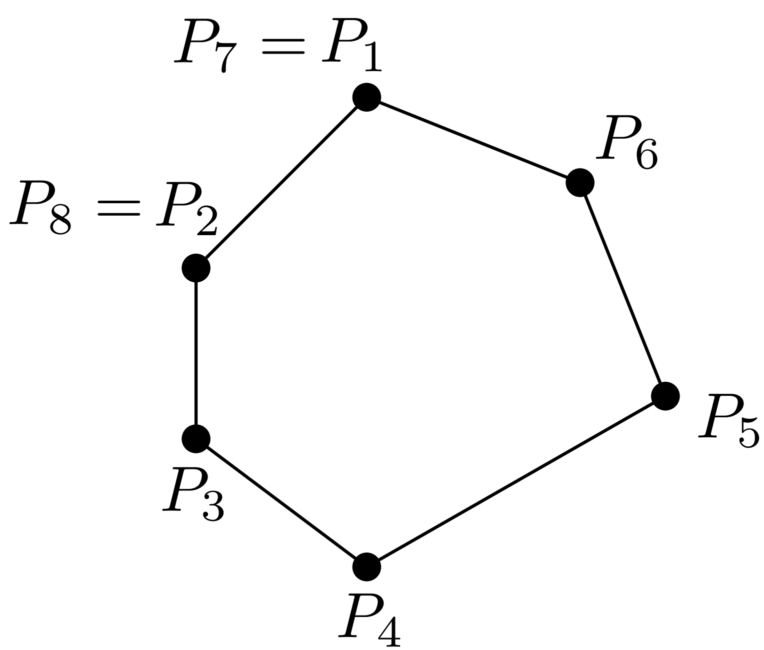
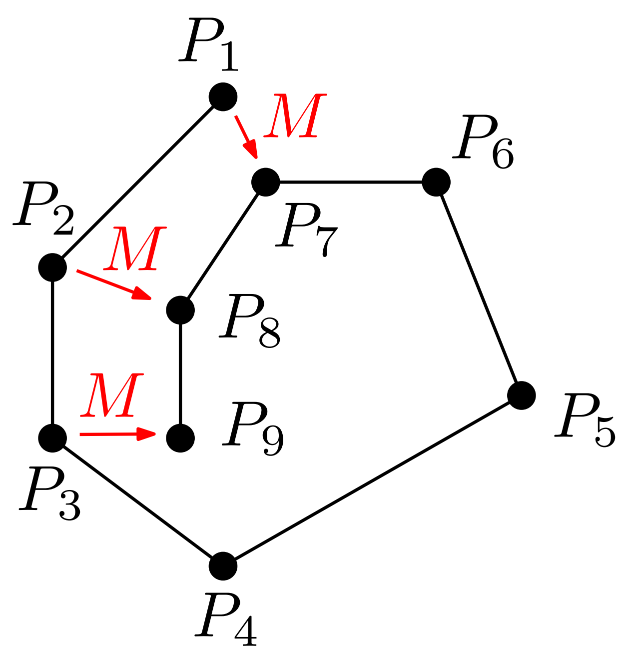


Twisted polygons

- If $M \in PGL_3(\mathbb{C})$, a *twisted n-gon with monodromy M* is defined as a bi-infinite sequence of points $(\ldots, P_{-1}, P_0, P_1, P_2, \ldots)$ in $\mathbb{C}P^2$ such that for every $i \in \mathbb{Z}$, $P_{i+n} = M.P_i$.



Twisted polygons

- The pentagram map is defined on the space of twisted n-gons considered up to the action of $PGL_3(\mathbb{C})$: $(P_i)_{i\in\mathbb{Z}}$ is identified with $(A.P_i)_{i\in\mathbb{Z}}$ if $A \in PGL_3(\mathbb{C})$.

- The monodromy M of a twisted n-gon is thus well-defined only up to conjugation: M is identified with AMA^{-1} if $A \in PGL_3(\mathbb{C})$.

Results on the pentagram map

- The dynamics is integrable, with explicit formulas for the conserved quantities (Ovsienko-Schwartz-Tabachnikov '10, Soloviev '13). This follows from the conservation of the monodromy.

- However, the majority of these conserved quantities lack a geometric interpretation.

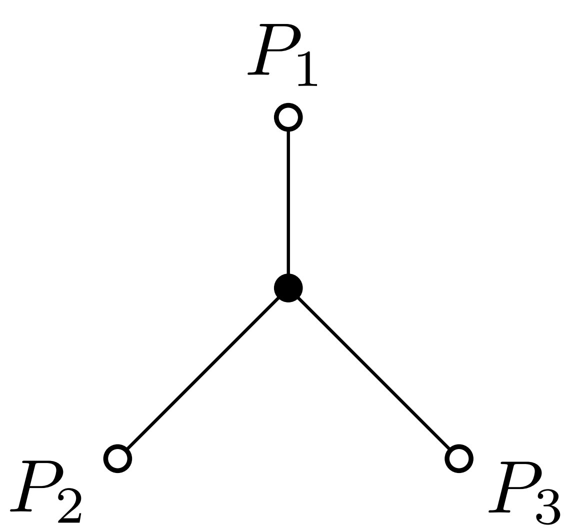
- Certain multi-ratios of points of two consecutive generations (times t and $t+1$) evolve like mutations of coefficient variables of cluster algebras (Glick '11).

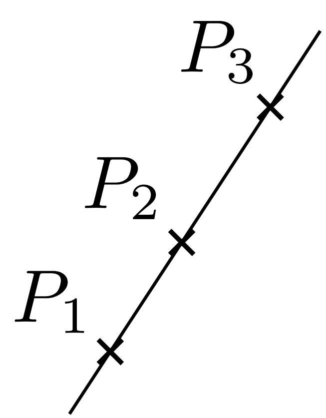
3 Triple crossing diagram maps

Following mostly Affolter-Glick-Pylyavskyy-R. '19

- A *triple crossing diagram* (TCD) is a bipartite graph such that all the black vertices have degree 3.

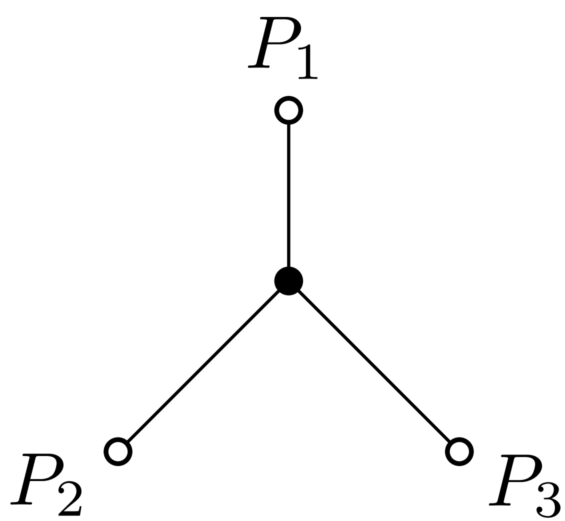
- Let $m \geq 1$ and Γ be a TCD with white vertex set W and black vertex set B. A *TCD map* is a map from W to $\mathbb{C}P^m$ such that for any $b \in B$, the three vertices around b are mapped to three collinear points.





$P_1, P_2, P_3 \in \mathbb{C}P^m$

- One can associate edge variables to a TCD map by lifting the points $P_i \in \mathbb{C}P^m$ to vectors $v_i \in \mathbb{C}^{m+1}$. These edge variables are only well-defined up to gauge transformations.



$$P_1, P_2, P_3 \in \mathbb{C}P^m \text{ collinear}$$

$$\Big\downarrow$$

$$v_1, v_2, v_3 \in \mathbb{C}^{m+1} \text{ coplanar}$$

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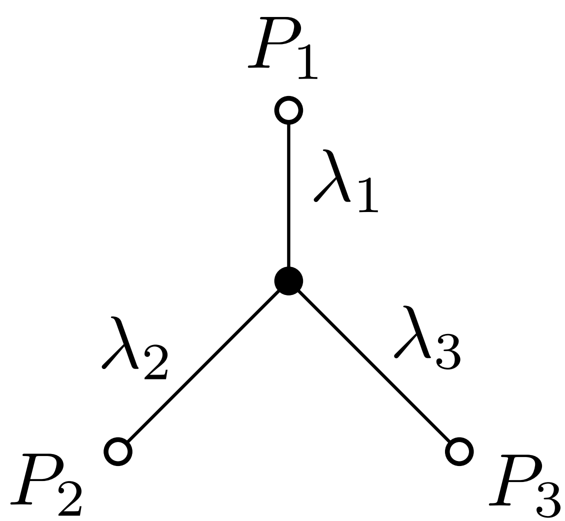

$$P_1, P_2, P_3 \in \mathbb{C}P^m \text{ collinear}$$

$$\downarrow$$

$$v_1, v_2, v_3 \in \mathbb{C}^{m+1} \text{ coplanar}$$

There exist $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{C}$ such that $\lambda_1 v_1 + \lambda_2 v_2 + \lambda_3 v_3 = 0$.

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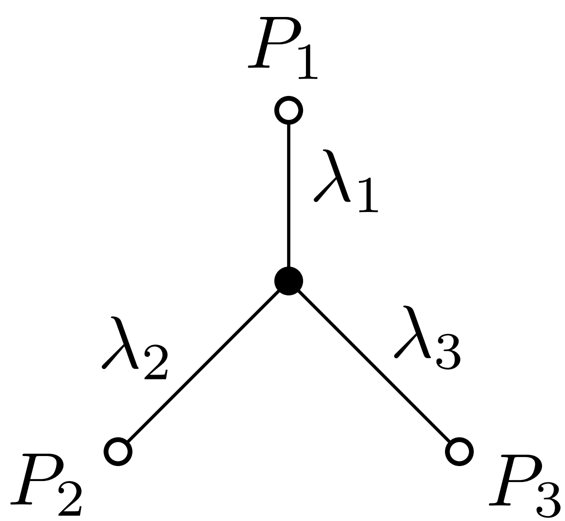

$P_1, P_2, P_3 \in \mathbb{C}P^m$ collinear

$$\Big\downarrow$$

$v_1, v_2, v_3 \in \mathbb{C}^{m+1}$ coplanar

There exist $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{C}$ such that $\lambda_1 v_1 + \lambda_2 v_2 + \lambda_3 v_3 = 0$.

Gauge at a white vertex: choose a different lift.
Gauge at a black vertex: multiply the linear combination by an overall factor.

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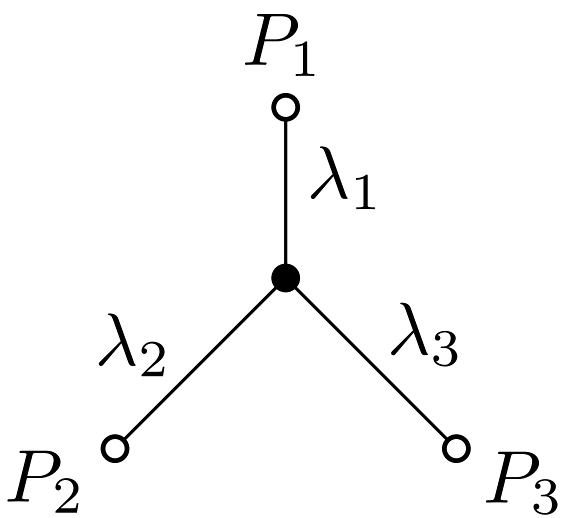
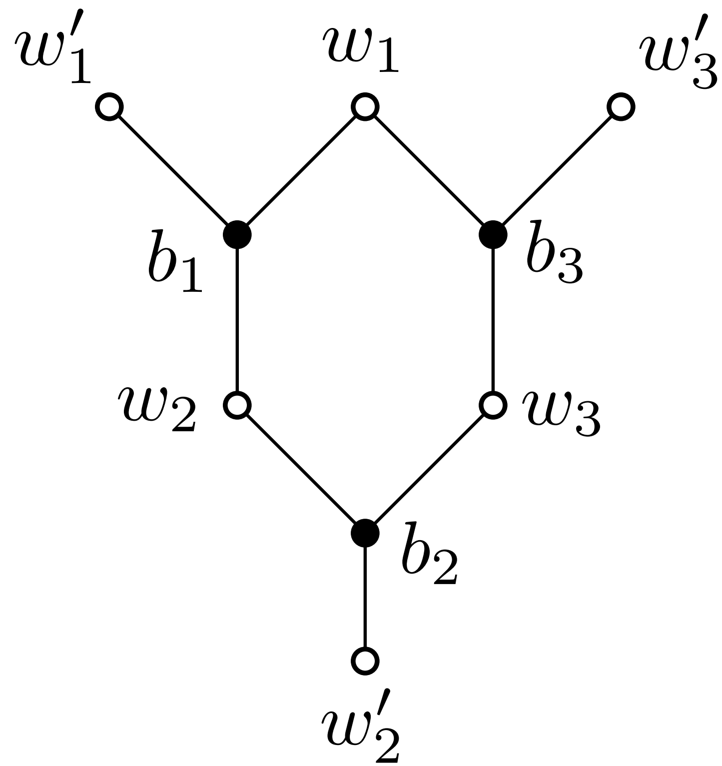


$$\begin{array}{c} P_1, P_2, P_3 \in \mathbb{C}P^m \text{ collinear} \\ \Big\downarrow \\ v_1, v_2, v_3 \in \mathbb{C}^{m+1} \text{ coplanar} \end{array}$$

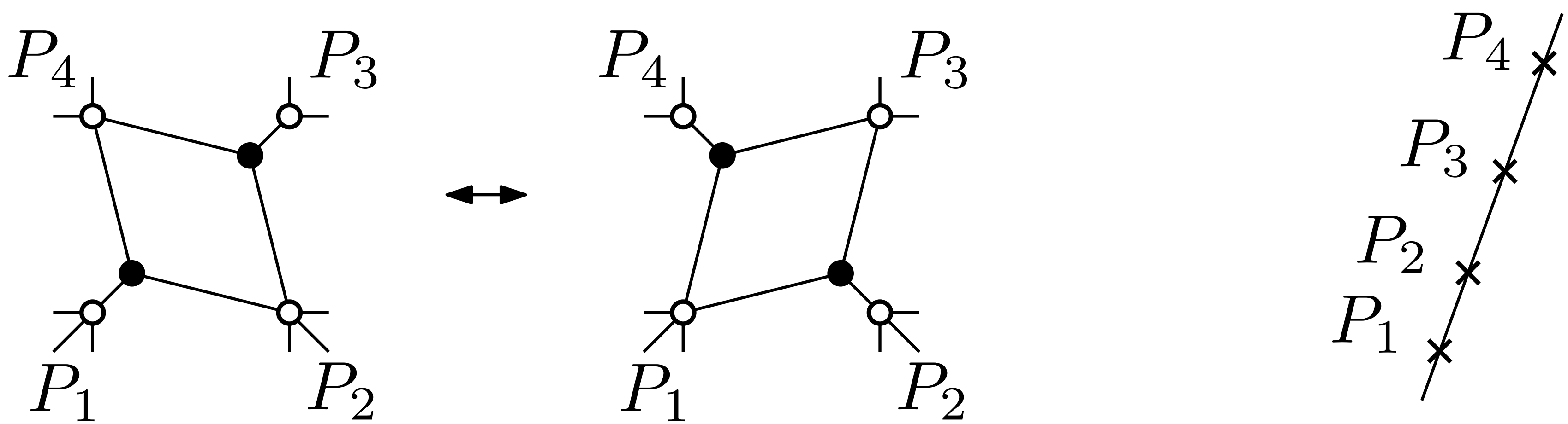
- These edge variables will play the role of Kasteleyn edge weights when looking at the underlying dimer model.

- For any face with vertices $(w_1, b_1, w_2, b_2, \ldots, w_k, b_k)$, if we call w_i' the other white vertex adjacent to b_i, then the alternating product of edge variables around the face equals $\mathrm{mr}(P_{w_1}, P_{w_1'}, P_{w_2}, P_{w_2'}, \ldots, P_{w_k}, P_{w_k'})$.

- This provides a geometric definition for face weights.



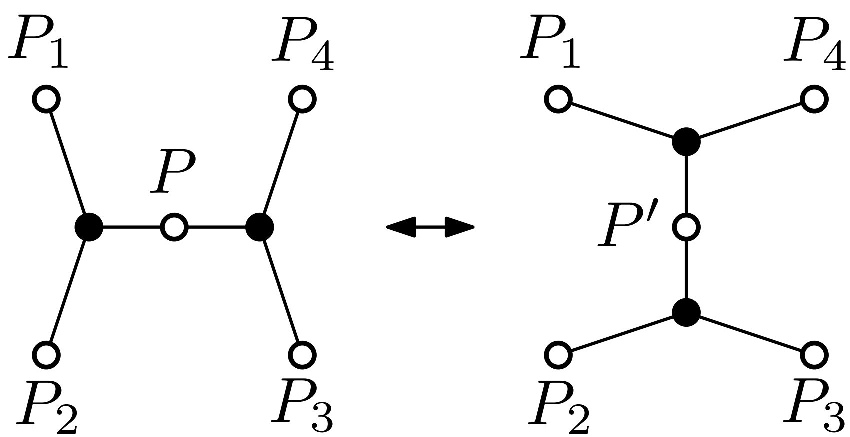
Spider move for TCD maps

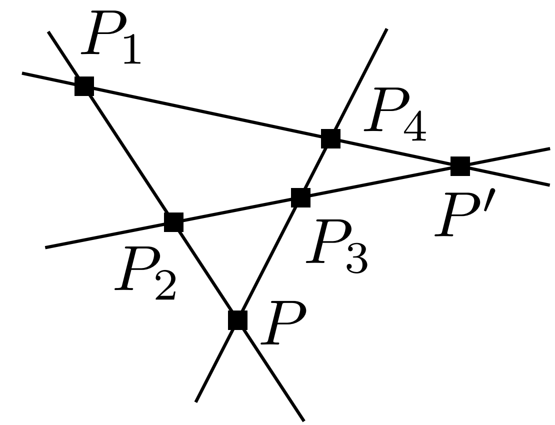


- The points of the TCD maps do not change, but the face weights change. This is a reparametrization.

- The face weights change according to the same formula as for the dimer spider move.

Resplit for TCD maps

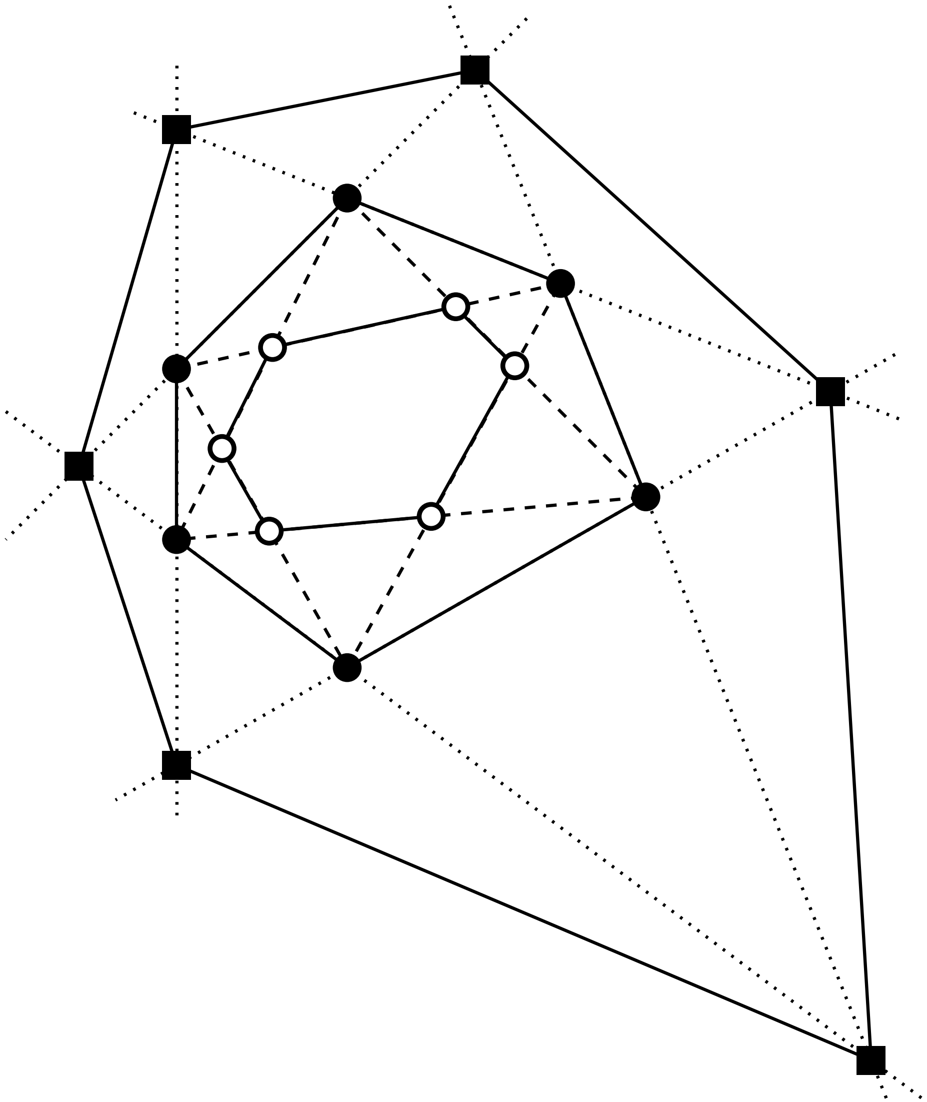




- If the points are in $\mathbb{C}P^m$ with $m \geq 2$, the new point P' or P is determined by Menelaus theorem for complete quadrilaterals:

$$\mathrm{mr}(P_1, P, P_2, P_3, P', P_4) = -1.$$

- One point changes but face weights don't change.



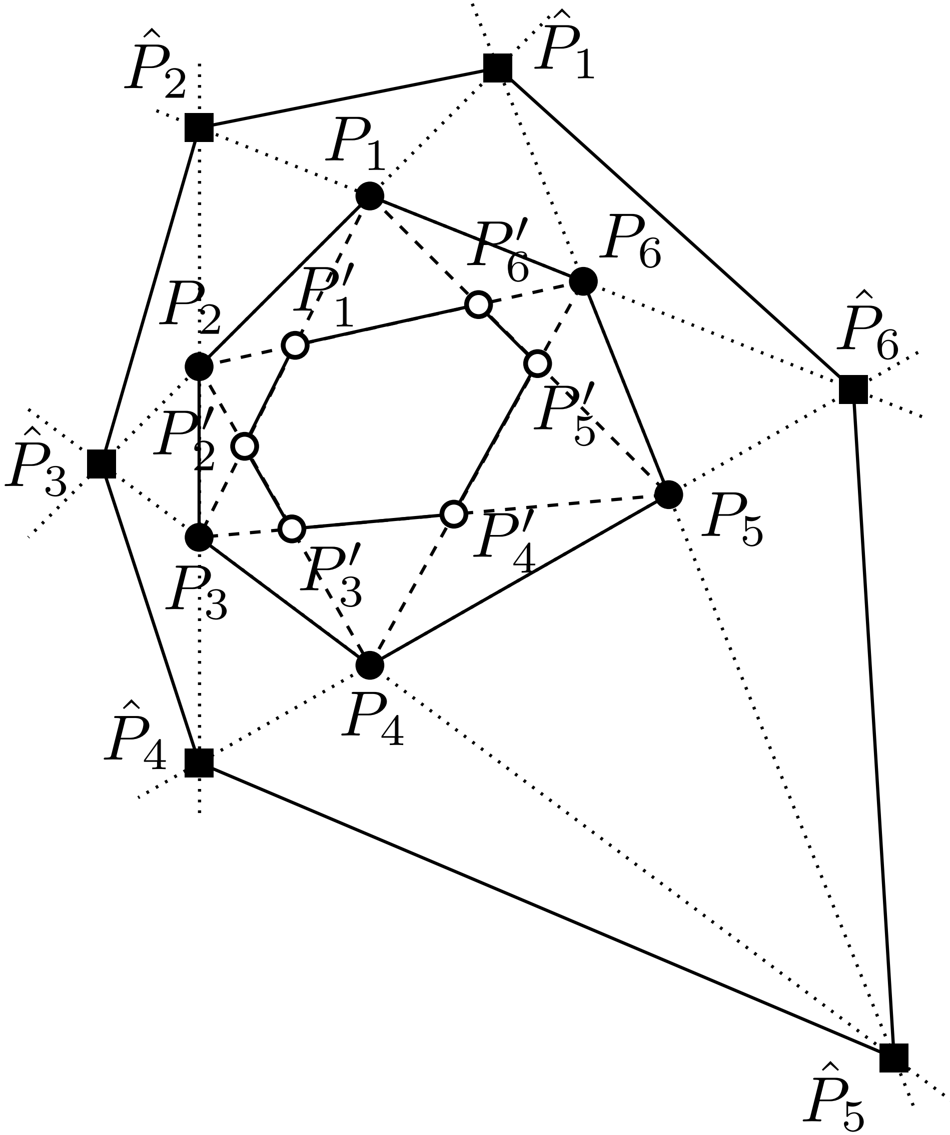
$n = 6$

■ points at time -1

● points at time 0

○ points at time 1

TCD map for the pentagram map



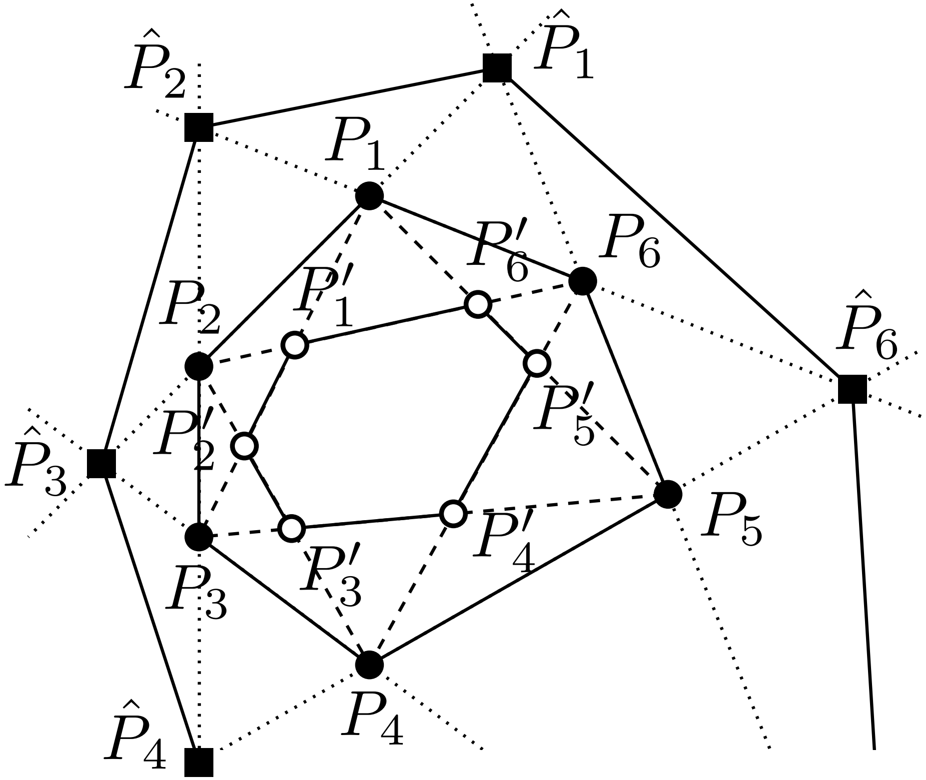
$n = 6$

■ points at time -1 $\hat{P}_i$

● points at time 0 P_i

○ points at time 1 P'_i

TCD map for the pentagram map

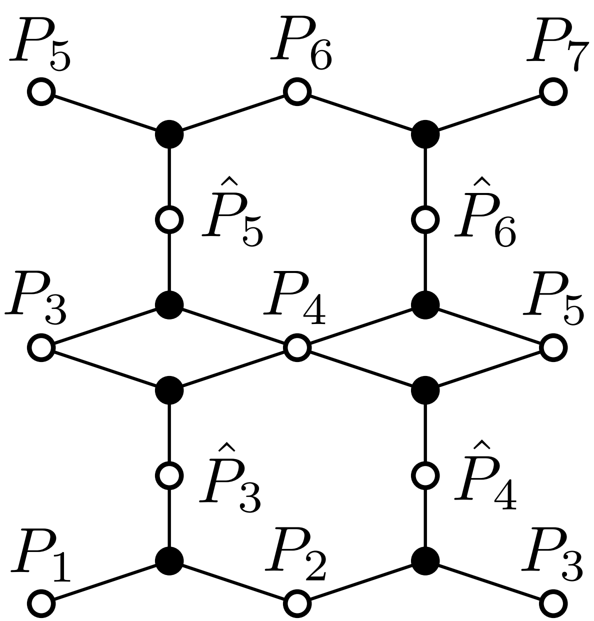


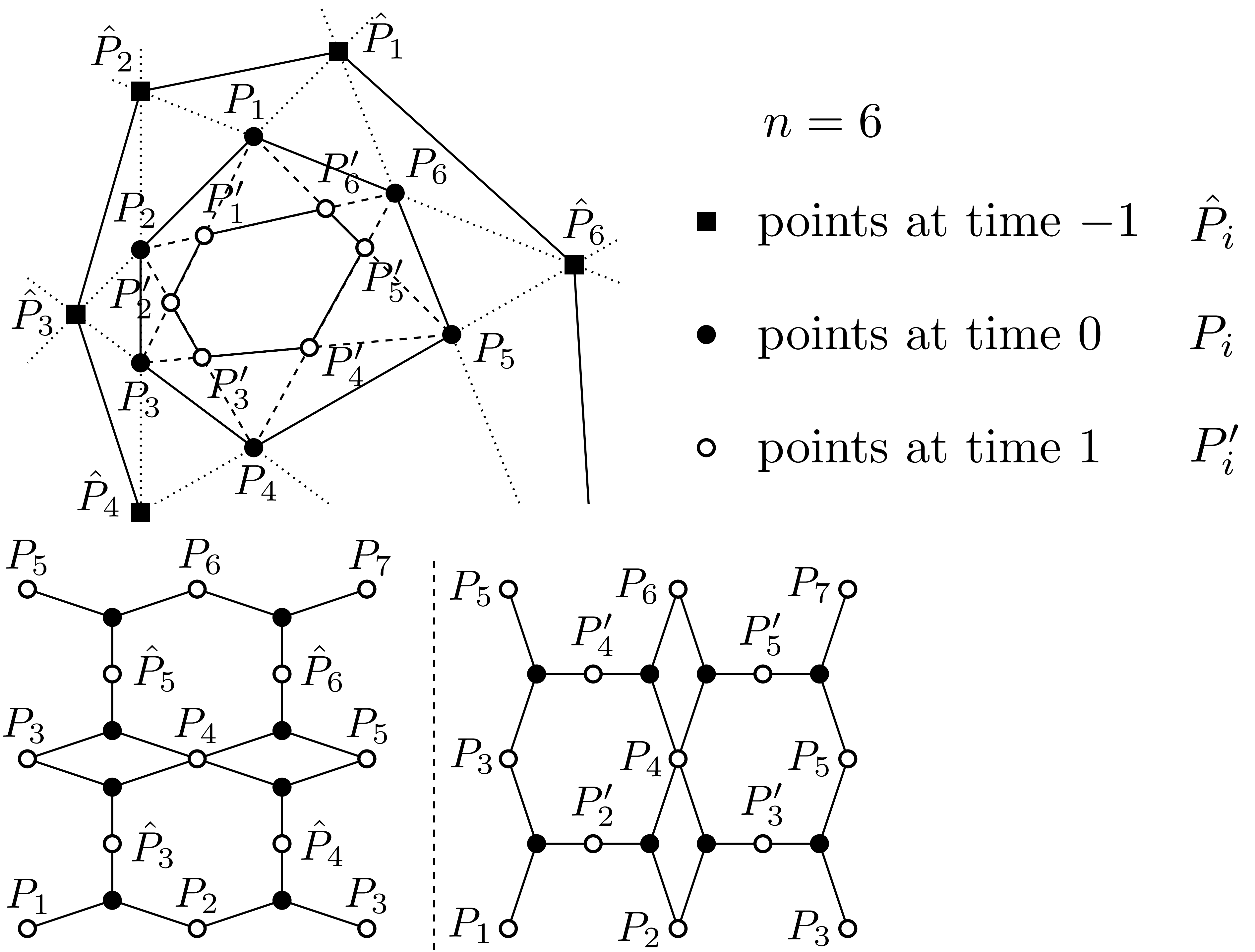
$n = 6$

■ points at time -1 $\hat{P}_i$

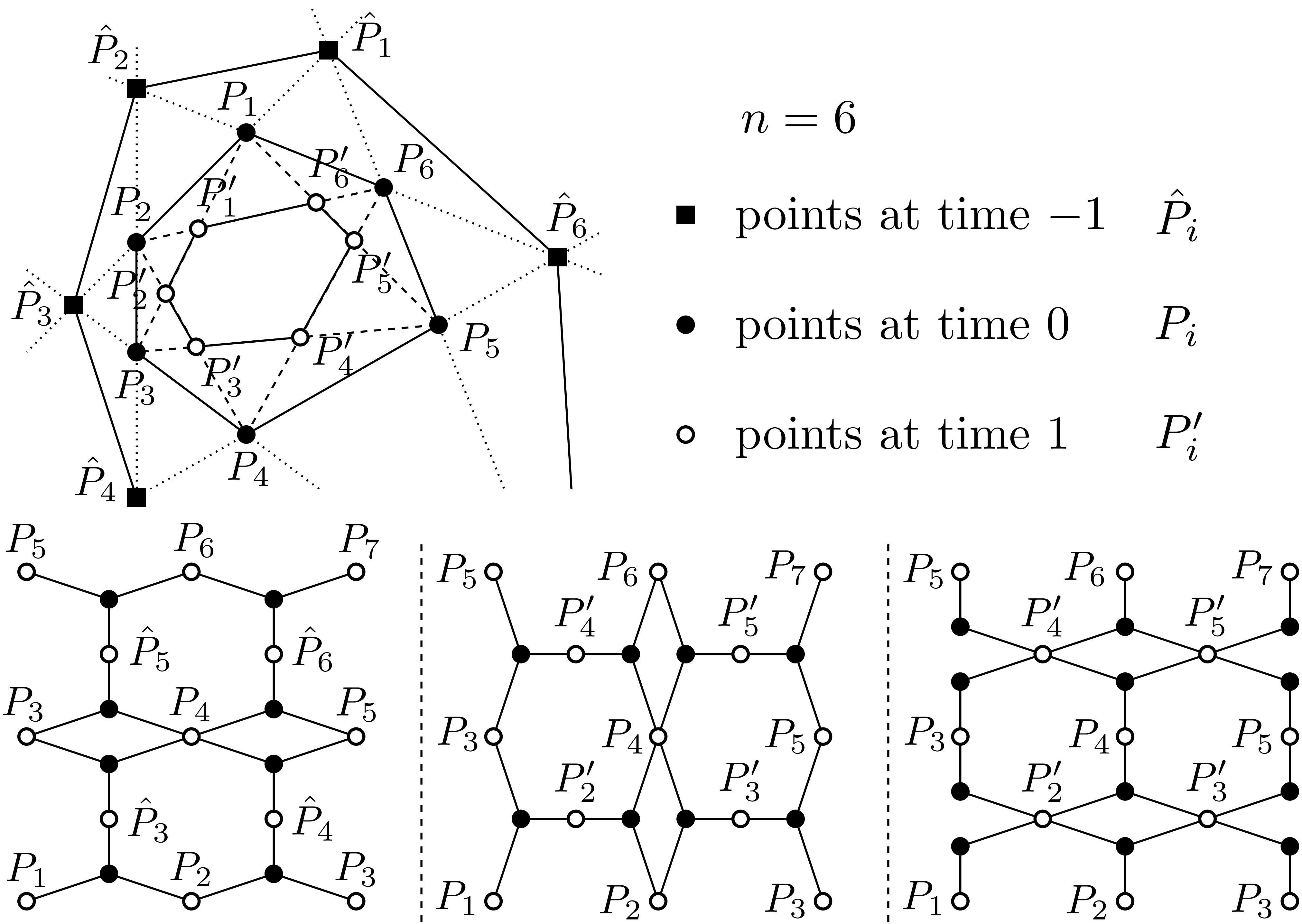
● points at time 0 P_i

○ points at time 1 P'_i





$n = 6$
points at time -1 $\hat{P}_i$
points at time 0 P_i
points at time 1 P'_i
$\hat{P}_1$ $\hat{P}_2$ $\hat{P}_3$ $\hat{P}_4$ $\hat{P}_6$
P_1 P_2 P_3 P_4 P_5 P_6
P'_1 P'_2 P'_3 P'_4 P'_5 P'_6
P_5 P_6 P_7 $\hat{P}_5$ $\hat{P}_6$ P_3 P_4 P_5 $\hat{P}_3$ $\hat{P}_4$ P_1 P_2 P_3
P_5 P_6 P_7 P'_4 P'_5 P_3 P_4 P_5 P'_2 P'_3 P_1 P_2 P_3



$n = 6$
points at time -1 $\hat{P}_i$
points at time 0 P_i
points at time 1 P'_i
$\hat{P}_1$ $\hat{P}_2$ $\hat{P}_3$ $\hat{P}_4$ $\hat{P}_5$ $\hat{P}_6$
P_1 P_2 P_3 P_4 P_5 P_6 P_7
P'_1 P'_2 P'_3 P'_4 P'_5 P'_6

Results of Affolter-George-R. '22

- We recover the pentagram cluster variables of Glick as dimer face weights.

- We recover the pentagram conserved quantities of Ovsienko-Schwarz-Tabachnikov as conserved quantities for the dimer dynamics.

- To do so, we build a general framework for twisted TCD maps living on infinite cylinders.

Results of Affolter-George-R. '22

- The monodromy of twisted polygons (generating function of the conserved quantities for the geometric dynamics) gets identified with the polynomial defining the dimer spectral curve (generating function of the conserved quantities of the dimer dynamics).

- We apply the general framework to obtain similar results for *cross-ratio dynamics* on polygons.

Conclusion and outlook

- Dimer integrable systems possess a unified theory, valid for any bipartite graph on the torus.

- Taking TCD maps for different choices of graphs, we recover a wealth of examples, coming from either geometric dynamics or from discrete differential geometry, which previously didn't form a unified theory (Affolter-Glick-Pylyavskyy-R. '19).

- We find multiple cluster algebra structures for TCD maps, which are related by natural geometric operations. This creates new connections between geometric systems (Affolter-Glick-R. '22).

Early announcement

- “Geometry, statistical mechanics, integrability” long program

- Location: Institute for Pure and Applied Mathematics (IPAM) in Los Angeles

- Dates: 11 March - 14 June 2024

- Organizing committee: Dmitry Chelkak, Jan de Gier, Vadim Gorin, Richard Kenyon, Greta Panova, Sanjay Ramassamy and Marianna Russkikh

N. Affolter, M. Glick, P. Pylyavskyy, S. Ramassamy, Vector-relation configurations and plabic graphs, arXiv:1908.06959.

N. Affolter, T. George, S. Ramassamy, Cross-ratio dynamics and the dimer cluster integrable system, arXiv:2108.12692.
Version 2 (with a new title) available soon !

N. Affolter, M. Glick, and S. Ramassamy, Triple crossing diagram maps and multiple cluster structures. In preparation.

Grazie per la vostra attenzione!