Ballistic macroscopic fluctuation theory
for
Integrable systems

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Based on collaboration with B. Doyon, G. Perfetto, and T. Sasamoto
Inhomogeneous dynamics in quantum many-body systems

$t = 0 \quad |\Psi(0)\rangle = T_L \quad T_R$

t $\to \infty \quad |\Psi\rangle_{NESS} = J_{\text{particle}}, J_{\text{energy}}, \cdots$

$|\Psi\rangle_{NESS} = \lim_{t \to \infty} |\Psi(t)\rangle$

How can we characterise the persistent (ballistic) currents?
• One of the most important quantities is the average of the currents 
\[ \lim_{T \to \infty} \frac{1}{T} \langle \Psi(0) \mid \hat{J}(T) \mid \Psi(0) \rangle \]

• But of course that’s not the only thing that describes the properties of currents! More intricate information is encoded in
  
  correlations pertaining to rare fluctuations

\[ \lim_{T \to \infty} \frac{1}{T} \langle \Psi(0) \mid \hat{J}(T) \hat{J}(T) \mid \Psi(0) \rangle, \lim_{T \to \infty} \frac{1}{T} \langle \Psi(0) \mid \hat{J}(T) \hat{J}(T) \hat{J}(T) \mid \Psi(0) \rangle, \ldots \]

• They are known as (scaled) cumulants, and capture the statistics of currents, e.g. variance, skewness, etc.
More formally

\[
F(\lambda) := \lim_{T \to \infty} \frac{1}{T} \log \langle e^{\lambda \hat{J}(T)} \rangle, \quad \langle \cdot \rangle := \langle \Psi(0) | \cdot | \Psi(0) \rangle
\]

The nontrivial \( F(\lambda) \) exists when the probability \( \mathbb{P}(\hat{J}(T) = Tj) \) follows the large deviation principle \( \mathbb{P}(\hat{J}(T) = Tj) \approx e^{-TI(j)} \) with a convex large deviation function \( I(j) \).

How would we compute \( F(\lambda) \) for ballistic many-body systems?

The ballistic fluctuation theory (BFT) was devised for this purpose but it is restricted to the homogeneous initial condition

[Doyon and Myer, 2020; Myers, Bhaeese, Harris, and Doyon, 2020]

We propose a general framework to describe the physics pertaining to (rare) fluctuations. It can be seen as the ballistic version of the diffusive macroscopic fluctuation theory (MFT). Its general idea was also introduced in Benjamin’s talk.
Hydrodynamics and local relaxation

- We consider a many-body system with $N$ conservation laws, which support ballistic transport. Let us start with recalling the principle of Euler hydrodynamics

- Suppose the initial condition is given by some local (generalised) Gibbs ensemble with the statistical average

$$\langle \cdot \rangle = \frac{1}{Z} \text{Tr} \left[ \exp \left( - \sum_{i=0}^{N-1} \int dx \beta^{i}(x/\ell) \hat{q}_{i}(x,0) \right) \cdot \right]$$

- The primary objects in hydrodynamics are the space-time averaged mesoscopic observables

$$\bar{\sigma}(x, t) = \frac{1}{vL^{2}} \int_{-L/2}^{L/2} dy \int_{-vL/2}^{vL/2} ds \hat{o}(x + y, t + s)$$

- Note that we eventually take $1 \ll L/\ell_{\text{micro}} \ll \ell$
• For instance hydrodynamics predicts that

$$\lim_{\ell \to \infty} \langle \bar{\sigma}(\ell x, \ell t) \rangle_\ell = \langle \hat{\sigma} \rangle_{\bar{\beta}(x,t)}, \quad \langle \cdot \rangle_{\bar{\beta}} := \frac{1}{Z} \text{Tr} \left[ e^{-\sum_{i=0}^{N-1} \beta_i Q_i \cdot} \right]$$

• Note that $\beta_i(x, t)$ characterise the fluid cell at the (macroscopic) space-time $(\ell x, \ell t)$, so their arguments $(x, t)$ are macroscopic.

• The above equality implies that the local equilibrium average of the (mesoscopic) observable at the macroscopic point $(\ell x, \ell t)$ is given by its equilibrium average with the spatio-temporal Lagrange multipliers $\beta_i(x, t)$.

• On the Euler scale the average of any local operator can be thought of as a functional of $q(x, t) := \langle \hat{q} \rangle_{\bar{\beta}(x,t)}$, i.e.

$$o(x, t) = \langle \hat{\sigma} \rangle[q(x, t)]$$

Local relaxation of averages
• In terms of \(\mathbb{q}(x, t)\) and \(\mathbb{j}(x, t)\), the Euler hydrodynamic equation reads \(\partial_t q_i(x, t) + \partial_x j_i(x, t) = 0\), or equivalently,

\[
\partial_t q_i(x, t) + A^j_i[q(x, t)] \partial_x q_j(x, t) = 0,
\]

\[
A^j_i[q] := \frac{\partial j_j[q]}{\partial q_i} = \frac{\partial j_j}{\partial \beta^k} C^{ki}
\]

• Here \(C^{ki} = (C^{-1})_{ki}\) with the susceptibility matrix \(C_{ij} = -\frac{\partial q_j}{\partial \beta^i} = \int d\mathbb{x} \langle \mathbb{q}_i(x, 0) \mathbb{q}_j(0, 0) \rangle_{\beta}\)

• One can also write down a hydro equation for the Lagrange multipliers

\[
\partial_t \beta^i(x, t) + A^i_j[\beta(x, t)] \partial_x \beta^j(x, t) = 0
\]

\[
\text{Positivity of } C \text{ implies a bijection } q \leftrightarrow \beta
\]

• Hydro is all about the dynamics of averages (and equilibrium correlation functions obtained out of them). In order to access the quantities that are strongly affected by the (rare) fluctuations, one also has to know how the fluctuations (regardless of their origins) propagate in time.
Initial fluctuations

• To consider the propagation of fluctuations, one has to first characterise the fluctuations of the initial condition

• We are interested in the fluctuations of the mesoscopic variables \( q_i(x,0) \equiv \bar{q}_i(x,0) \). Its correlations can be obtained from a measure

\[
d\mathbb{P}_{\text{ini}}[q(\cdot,0)] = d\mu[q(\cdot,0)] e^{-\mathcal{F}[q(\cdot,0)]},\]

where \( e^{-\mathcal{F}[q(\cdot,0)]} \) is the probability of observing the density profile \( q_i(x,0) \) given the initial background density \( \bar{q}_{i,\text{ini}}(x) \)

\[
\mathcal{F}[q(\cdot,0)] = \int_{\mathbb{R}} dx \left( \beta_{\text{ini}}^i(x)(q_{i}(x,0) - \bar{q}_{i,\text{ini}}(x)) + s[q_{i,\text{ini}}(x)] - s[q(x,0)] \right).
\]

• Defining \( \beta(x,0) \) via \( q_i(x,0) = q_i[\beta(x,0)] \), the saddle point of \( d\mathbb{P}_{\text{ini}}[q(\cdot,0)] \) gives the \( \beta^i(x,0) = \beta^i_{\text{ini}}(x) \)

• How do we generalise \( d\mathbb{P}_{\text{ini}}[q(\cdot,0)] \) to \( d\mathbb{P}_{\text{ini}}[q(\cdot,\cdot)] \)?
Local relaxation of fluctuations

• The idea of **local relaxation of fluctuations** constitutes the cornerstone of the BMFT. This is the only **assumption** we make.

Mesoscopic means of local observables (coarse-grained observables), \( \overline{o}(\ell x, \ell t) \),
do not fluctuate **independently** but are **fixed** functionals of conserved densities, i.e.
\[
\overline{o}(\ell x, \ell t) = o[\overline{q}(x, t)]
\]

• This is closely related to local relaxation of averages \( o(x, t) = \langle \hat{o} \rangle[\overline{q}(x, t)] \). To fix \( o[\cdot] \), we can determine it by taking \( t = 0 \) and invoke
\[
\langle \hat{o} \rangle_{\beta(x,0)} = \int_{(\mathbb{R})} d\mathbb{P}_{ini}[\overline{q}(\cdot,0)] o[\overline{q}(x,0)]
\]
(because \( o(x,0) = o[\overline{q}(x,0)] \) according to the ansatz)

• The saddle point of it yields \( o[\overline{q}(x,0)] = o[\overline{q}(x,0)] \), i.e. \( \overline{o}(\ell x, \ell t) = o[\overline{q}(x, t)] \).

\[
\langle o(x, t) \rangle_\ell = o[\overline{q}(x, t)] \quad o(x, t) = o[\overline{q}(x, t)]
\]

Local relaxation of averages \quad Local relaxation of fluctuations

\( \ell \to \infty \)
• Having this assumption with $j_i(x,t) = \bar{j}_i(\ell x, \ell t) = j_i[q(x,t)]$, the measure $d\mathbb{P}_{\text{ini}}[q(\cdot, \cdot)]$ we use for the BMFT is given by

\[
\text{d}\mathbb{P}[q(\cdot, \cdot)] = \text{d}\mu[q(\cdot, \cdot)] e^{-\ell \mathcal{F}[q(\cdot,0)]} \delta[\partial_t q + \partial_x j[q]]
\]

which we identify with $\langle \cdot \rangle_{\ell}$.

• With this measure we write the BMFT average as

\[
\langle \langle \cdot \rangle \rangle_{\ell} = \frac{1}{Z} \int_{(S)} \text{d}\mu[q(\cdot, \cdot)] e^{-\ell \mathcal{F}[q(\cdot,0)]} \delta[\partial_t q + \partial_x j[q]] \cdot, \quad S := \mathbb{R} \times [0,T]
\]

which we identify with $\langle \cdot \rangle_{\ell}$.

• Closely related to the conventional MFT ansatz

\[
d\mathbb{P}[q(\cdot, \cdot)] = \text{d}\mu[q(\cdot, \cdot)] \text{d}\mu[\eta(\cdot, \cdot)] e^{-\ell \mathcal{F}[q(\cdot,0)]} \delta[\partial_t q + \partial_x j_{\text{diff}}] \cdot, \quad j_{\text{diff},i} := - \mathcal{D}_i \partial_q q_j + \eta_i
\]
• Using the ansatz for the measure, one can compute any average of observables. In particular we’re interested in two objects:

\[
S_{\hat{o}_1,\ldots,\hat{o}_n}(x_1, t_1; \ldots; x_n, t_n) := \lim_{\ell \to \infty} \ell^{n-1} \langle \hat{o}_1(\ell x_1, \ell t_1) \cdots \hat{o}_n(\ell x_n, \ell t_n) \rangle_c^c
\]

\[
F(\lambda, T) := \lim_{\ell \to \infty} \frac{1}{\ell T} \log \langle e^{\lambda T J_{\text{cl}}(T)} \rangle_{\ell}
\]

Euler scale dynamical correlation functions

Scaled cumulant generating function

• The BMFT allows us to evaluate these quantities via the path-integrals

\[
S_{\hat{o}_1,\ldots,\hat{o}_n}(x_1, t_1; \ldots; x_n, t_n) = \lim_{\ell \to \infty} \ell^{n-1} \langle \hat{o}_1[q(x_1, t_1)] \cdots \hat{o}_n[q(x_n, t_n)] \rangle_c^c
\]

\[
F(\lambda, T) = \lim_{\ell \to \infty} \frac{1}{\ell T} \log \langle e^{\lambda T J_{\text{cl}}(T)} \rangle_{\ell}
\]

• The power of the (B)MFT is that we don’t even have to evaluate the path-integrals, as they turn out to be dominated by their saddle points when \(\ell' \to \infty\)!
Current fluctuations from the BMFT

• Let us first look into the current fluctuations, i.e. the evaluation of $F(\lambda, T)$

$$
\langle e^{\lambda J_i(T)} \rangle = \int_{(S)} d\mu[q(\cdot, \cdot)] e^{\lambda J_i(T)} e^{-\ell \mathcal{F}[q(\cdot,0)]} \delta(\partial_t q + \partial_x j[q])
$$

$$
=: \int_{(S)} d\mu[q(\cdot, \cdot)] \mu[H(\cdot, \cdot)] e^{-\ell S_{\text{curr}}[q, H]}
$$

with $S_{\text{curr}}[q, H] := -\lambda J_i(T) + \mathcal{F}[q(\cdot,0)] + \int_S dtdx H^i(\partial_t q_i + \partial_x j_i[q])$

• Clearly the path-integral is dominated by $q$ and $H$ such that $\delta S_{\text{curr}} = 0$, which can be obtained by solving the MFT equations

$$
\begin{align*}
\lambda \delta_{\ell_i}^i \Theta(x) - \beta_i(x,0) + \beta^{i}_{\text{ini}}(x) - H^i(x,0) &= 0, \\
\lambda \delta_{\ell_i}^i \Theta(x) - H^i(x, T) &= 0, \\
\partial_t \beta^i(x, t) + A_j^i[\beta(x, t)] \partial_x \beta^j(x, t) &= 0, \\
\partial_t H^i(x, t) + A_j^i[\beta(x, t)] \partial_x H^j(x, t) &= 0.
\end{align*}
$$
• Once we have the solution we can compute the SCGF as $F(\lambda, T) = \int_0^\lambda d\lambda' \int_0^T dt \, j_i^{(\lambda)}(0, t)/T$

• In general it is hard to solve the equations, as the solution tends to be not unique due to the appearance of shocks (e.g. the TASEP). Some progress has been made for the TASEP by taking the totally asymmetric limit of the MFT for the WASEP [Bodineau and Derrida, 2006]

• Fortunately no shock appears in integrable systems, which turns out to be a great help in doing the BMFT for integrable systems.

• Surprisingly, the structure of the MFT equations implies the Gallavotti-Cohen fluctuation theorem (GCFT). Suppose the initial condition is a step-initial condition $\beta_{\text{ini}}^i(x) = \delta^i_i, (\beta_L \Theta(-x) + \beta_R \Theta(x))$. Then the GCFT claims the symmetry

$$ F(\lambda) = F(\beta_L - \beta_R - \lambda), \quad F(\lambda) := F(\lambda, 1) $$

• It turns out that the symmetry follows from the fact that $q_i^{(\lambda)}(x, t)$ and its time-reversal counterpart with the counting field $\tilde{\lambda} = \beta_L - \beta_R - \lambda$ satisfy the same MFT equation
Dynamical correlation functions from the BMFT

• In a similar way one can also compute the Euler-scale dynamical correlation function

\[
S_{\hat{q}_1, \hat{q}_2}(x_1, t_1; x_2, t_2) = \frac{d^2}{\ell d\lambda_1 d\lambda_2} \log \left[ \int \left[ \frac{d\mu}{(S)} \exp[\ell (\lambda_1 q_{t_1}(x_1, t_1) + \lambda_2 q_{t_2}(x_2, t_2))]e^{\epsilon \mathcal{F}[\hat{q}(\cdot, 0)]} \delta(\partial_q + \partial \mathcal{J}[q]) \right] \right]_{\lambda_1 = \lambda_2 = 0}
\]

\[
= \frac{d^2}{\ell d\lambda_1 d\lambda_2} \log \left[ \int \left[ \frac{d\mu}{(S)} \exp[\mathcal{F}[\hat{q}(\cdot, 0)] + \int_\mathbb{S} d\mathcal{S} d\mu \left[ q_{t_1}(x_1, t_1) + \mathcal{J}[q(x, t)] \right] \right] \right],
\]

with \( S_{\text{corr}}[q, H] := - (\lambda_1 q_{t_1}(x_1, t_1) + \lambda_2 q_{t_2}(x_2, t_2)) + \mathcal{F}[q(\cdot, 0)] + \int_\mathbb{S} d\mathcal{S} d\mu \left[ q_{t_1}(x_1, t_1) + \mathcal{J}[q(x, t)] \right] \).

• As in the case of the SCGF, one can compute \( S_{\hat{q}_1, \hat{q}_2}(x_1, t_1; x_2, t_2) \) out of the solution of the MFT equation involving \( \lambda_1 \) and \( \lambda_2 \).

\[
S_{\hat{q}_1, \hat{q}_2}(x_1, t_1; x_2, t_2) = - \frac{d^2}{d\lambda_1 d\lambda_2} S_{\text{corr}}^{sp} \bigg|_{\lambda_1 = \lambda_2 = 0}
\]

• Since \( S_{\text{corr}}^{sp} \) is the saddle point action, we have \( \partial_{\lambda_2} S_{\text{corr}}^{sp} = - q_{t_2}(x_2, t_2) \). Hence

\[
S_{\hat{q}_1, \hat{q}_2}(x_1, t_1; x_2, t_2) = \frac{d}{d\lambda} q_{t_2}(x_2, t_2) \bigg|_{\lambda = \lambda_1}
\]
• The associated MFT equation is then

\[ \beta^i(x,0) - \beta_{i}\text{ini}(x) + H^i(x,0) = 0, \]
\[ H^i(x,T) = 0, \]
\[ \partial_t \beta^i + A_j^i \partial_x \beta^j = 0, \]
\[ \partial_t H^i + A_j^i \partial_x H^j + \lambda \delta^i_{i1} \delta(x-x_1) \delta(t-t_1) = 0. \]

• The solution of the MFT equation actually predicts the existence of the long-range correlation amongst the fluid cells on the same time slice \( t_1 = t_2 = t \), i.e.

\[ \lim_{\ell \to \infty} \langle \bar{q}_{i1}(\ell x_1, \ell t) \bar{q}_{i2}(\ell x_2, \ell t) \rangle_\ell = S_{\hat{q}_{i1} \hat{q}_{i2}}(x_1, t; x_2, t) = C_{i1i2}(x_1, t) \delta(x_1 - x_2) + E_{i1i2}(x_1, x_2; t) \]

\[ \langle \cdot \rangle_\ell \neq \frac{1}{Z} \text{Tr} \left[ \exp \left[ - \sum_{i=0}^{N-1} \int \mathbb{R} \ dx \beta^i(x/\ell, t/\ell) \hat{q}_i(x,0) \right] \cdot \right] \text{ for higher point functions} \]

• Even if there is no long-range correlation initially, it could build up by the coupling between normal modes on an inhomogeneous background.
• Note that the long-range correlations observed here is of purely hydrodynamic nature

For instance it is readily seen that TASEP cannot develop long-range correlations based on this mechanism.

Long-range correlations have also been observed in NESS in the partitioning protocol for different models e.g. free 1d Klein-Gordon model, Lieb-Liniger model where \( \langle \hat{q}_0(x,0)\hat{q}(0,0) \rangle_{\text{NESS}}^c \) decays as \( 1/x^2 \) [Doyon, Lucas, Schalm, and Bhaseen, 2015; De Nardis and Panfil, 2018]. But this has a different origin, which is a non-locality of the NESS density operator. A similar observation was also made for the symmetric simple exclusion process [Derrida, 2007].

• We are going to compute these quantities explicitly for integrable systems
Generalised hydrodynamics

- In integrable systems thermodynamics as well as the dynamics of the system are dictated by the scattering data: particle species, dispersion relation, and the two-body S-matrix.

- We shall consider a diagonally-scattering integrable model with a single species defined on a line. Generalisations to more complicated models are straightforward.

- A kinetic intuition behind GHD is that on a hydrodynamic scale, quasi-particles in integrable systems behave pretty much like tracer particles of hard-rods. [Boldrighini, Dobrushin, and Sukhov, 1983; Spohn, 1991; Doyon and Spohn, 2017; Doyon, TY, and Caux, 2018]

- This underlying similarity of kinetics among integrable systems amounts to universal structures of hydrodynamic equations.

- An exact expression of the current average turns out to be instrumental in GHD. [See reviews: Borsi, Pozsgay, and Pristyák, 2021; Cortés Cubero, TY, and Spohn, 2021]
• On the Euler scale, quasi-particles in integrable systems are transported according to the GHD equation

\[ \partial_t q(x, t) + \partial_x j(x, t) = 0 \iff \partial_t \rho_\theta(x, t) + \partial_x (v_\theta^{\text{eff}}(x, t) \rho_\theta(x, t)) = 0 \]

· In MFT it is in fact more convenient to work with the Lagrange multipliers \( \rho^\theta \) (we are considering a GGE \( q \sim e^{-\rho^\theta \hat{Q}} \))

\[ \partial_t \rho^\theta(x, t) + A_\phi^\theta [\rho_\phi(x, t)] \partial_x \rho^\phi(x, t) = 0 \]

\( \text{Note } C_{ij} = -\frac{\partial \rho_i}{\partial \rho_j} \text{ and } AC = CA^T \text{ with } A_\phi^\theta := \frac{\partial (\rho_\theta v_\theta^{\text{eff}})}{\partial \rho_\phi} \)

• To solve initial value problems we shall use the GHD equation in terms of the normal mode

\[ \partial_t \epsilon_\theta(x, t) + v_\theta^{\text{eff}}(x, t) \partial_x \epsilon_\theta(x, t) = 0 \]

• To go from the equation for \( \rho^\theta \) to that for \( \epsilon_\theta \), we used \( (R^{-1})_\phi \partial_{t,x} \rho^\phi = \partial_{t,x} \epsilon^\theta \) where \( R = 1 - nT \) diagonalises the flux matrix: \( RAR^{-1} = \text{diag } v^{\text{eff}} \).

• One of the crucial properties of the GHD equation is that its solutions always involve neither shocks nor rarefaction waves but only contact discontinuities (CD). CDs can be thought of as shocks without entropy production.
Initial value problems in GHD [Doyon, Spohn, and TY, 2017; Doyon 2020]

• Let us start with recalling how the method of characteristics works in a simple case: \( \partial_t \rho + v(\rho) \partial_x \rho = 0 \) with \( \rho(x,0) = \rho_0(x) \).

• For each \( x = u \) at \( t = 0 \), we have the characteristic curve \( \Gamma_u \) along which \( \rho(x,t) \) is constant: \( \frac{dx(u,t)}{dt} = v(\rho(x(u,t),t)) \). This implies

\[
\frac{d}{dt} \rho(x,t) = \frac{\partial}{\partial t} \rho(x,t) + \frac{dx}{dt} \frac{\partial}{\partial x} \rho(x,t) = \frac{\partial}{\partial t} \rho(x,t) + v(\rho) \frac{\partial}{\partial x} \rho(x,t) = 0
\]

• Furthermore the characteristic curve is straight because clearly \( \frac{d^2x}{dt^2} = 0 \). The equation of characteristics can be solved as \( \frac{dx}{dt} = v(\rho(x,t)) = v(\rho(u,0)) = v(\rho_0(u)) \), i.e.

\[
x = v(\rho_0(u)) t + u
\]

• Having \( u(x,t) \) by solving the equation, we obtain \( \rho(x,t) = \rho(u(x,t),0) = \rho_0(u(x,t)) \).

• We want to do the same for GHD.
• The characteristic curve in GHD is defined by \( \frac{dx_\theta(u,t)}{dt} = v_\theta^{\text{eff}}[\epsilon_\theta(x_\theta(u,t), t)] \), which immediately implies \( \frac{dc_\theta(x_\theta(u,t), t)}{dt} = 0 \) with \( x_\theta(u,0) = u \).

• The characteristic curve is not straight, i.e. \( \frac{d^2x_\theta(u,t)}{dt^2} \neq 0 \). One gets \( c_\theta(x_\theta(u,t), t) = c_\theta(x_\theta(u,0), 0) = c_\theta(u,0) \).

• In fact it is more convenient to fix the space time \( (x', t') \) and then construct a characteristic curve that passes \( x = u_\theta(x', t') \) at \( t = 0 \).

• We thus redefine \( u = u_\theta(x, t), x_\theta(u_\theta(x, t), t) = x \) with which we have

\[
\epsilon_\theta(x, t) = \epsilon_\theta(u_\theta(x, t), 0)
\]

• How do we determine \( u_\theta(x, t) \)? Clearly the characteristic curves being not straight isn’t helpful.

• The following observation makes things simpler: by changing the state-dependent coordinate change we get

\[
\partial_t \hat{c}_\theta(q, t) + v_\theta^- \partial_x \hat{c}_\theta(q, t) = 0, \quad v_\theta^- := v_\theta^{\text{eff}} \mathcal{K}_\theta[n^-]
\]

[Doyon, Spohn, and TY, 2017]

• Here \( \hat{c}_\theta(q, t) \) is defined by \( \hat{c}_\theta(q_\theta(x, t), t) = c_\theta(x, t) \) with

\[
q_\theta(x, t) := \int_{x_\theta}^{x} dy \mathcal{K}_\theta[\epsilon(y, t)], \quad \mathcal{K}_\theta[\epsilon] := \frac{(P_\theta^t)^{dr}[\epsilon]}{p_\theta^t}, \quad h_\theta^{dr} := (R^{-T})_\theta \phi h_\phi
\]
Quasi-particles are now transported freely according to the above equation but on the state-dependent phase space $dq d\theta = \mathcal{H}_\theta[e(x)] dx d\theta$. The asymptotic coordinate $x_0$ is chosen so that $\rho_\theta(x, t) = \rho_\theta$ for all $x_0 \leq x$ at any time $t \in [0, T]$.

The equation is trivially solved by $\hat{e}_\theta(q, t) = \hat{e}(q - \nu_\theta t, 0)$. Using the definition of $u_\theta(x, t)$ i.e. $e_\theta(x, t) = e_\theta(u_\theta(x, t), 0)$ and $\hat{e}_\theta(q_\theta(x, t), t) = e_\theta(x, t)$, it immediately follows that

$$\hat{e}_\theta(q_\theta(u_\theta(x, t), 0), 0) = \hat{e}(q_\theta(x, t) - \nu_\theta t, 0)$$

We thus get the solution of the characteristics in GHD, which also determines $u_\theta(x, t)$

$$q_\theta(x, t) = v_\theta t + q_\theta(u_\theta(x, t), 0)$$

In the $q$-coordinate space the characteristic lines are all straight. Also importantly they share the same velocity $v_\theta$, which is in accordance with the fact that there is no shock in GHD.

Alternatively the above solution can also be written as

$$\int_{x_0}^{u_\theta(x, t)} dy \mathcal{H}_\theta[e_0(y)] + v_\theta t = \int_{x_0}^{x} dy \mathcal{H}_\theta[e_\theta(y, t)]$$
Current fluctuations in integrable systems

- We simply adopt the BMFT we formulated to (quantum) integrable systems. We want to compute
  \[ F(\lambda, T) = \lim_{\ell \to \infty} \log \langle e^{\lambda \hat{J}_\ell(T)} \hat{I}(\ell T) \rangle \hat{I}(\ell T). \]
  The MFT equation is

  \[
  \begin{align*}
  \lambda h^\theta \delta^\theta \Theta(x) - \beta^\theta(x,0) + \beta^\theta_{ini}(x) - H^\theta(x,0) &= 0 \\
  -\lambda h^\theta \delta^\theta \Theta(x) + H^\theta(x, T) &= 0 \\
  \partial_t \beta^\theta(x, t) + A_\phi^\theta(x, t) \partial_x \beta^\phi(x, t) &= 0 \\
  \partial_t H^\theta(x, t) + A_\phi^\theta(x, t) \partial_x H^\phi(x, t) &= 0
  \end{align*}
  \]

  \[
  \hat{Q}_i |\theta\rangle = h^\theta_i |\theta\rangle \\
  h^\theta := h^\theta_i
  \]

- We rewrite it in terms of normal modes. Recall that \( \beta^\theta \) and \( e_\theta \) are related by \( (R^{-1})^\phi \partial_{t,x} \beta^\phi = \partial_{t,x} e^\theta \). Motivated by this we define normal modes associated to \( H^\theta \):

  \[
  (R^{-1})^\theta \partial_{t,x} H^\phi =: \partial_{t,x} G^\theta
  \]

- The property \( \partial_t \partial_x G^\theta = \partial_x \partial_t G^\theta \) is instrumental
• The MFT equation for the auxiliary field becomes

\[ \lambda h^{\text{dr} \theta}(0,T) \Theta(x) - G^\theta(x, T) = 0 \]
\[ \partial_t G^\theta(x, t) + v^{\text{eff} \theta}(x, t) \partial_x G^\theta(x, t) = 0, \quad (a^{\text{dr} \theta} := (R^{-T})^\theta \Phi^{a \phi}) \]

• The method of characteristics allows us to solve the equation via \( G^\theta(x, t) = G^\theta(r^\theta(x, t), T) = h^{\text{dr} \theta}(0,T) \Theta(r^\theta(x, t)), \) where \( r^\theta(x, t) = \mathcal{U}^\theta(x, t; T). \)

• With this the MFT equation is now recast into the GHD equation with the \( \lambda \)-dependent initial condition

\[ \beta^\theta(x,0) = \beta_{\text{ini}}^\theta(x) + \lambda h^\theta \Theta(x) - \lambda R^\theta(0,T) \Theta(x - u^\phi(0,T)) h^{\text{dr} \phi}(0,T) \]
\[ \partial_t \beta^\theta(x, t) + A^\theta(x, t) \partial_x \beta^\phi(x, t) = 0 \]

• The evaluation of the current \( j^\theta(\lambda)(0,t) \) gives \( F(\lambda, T) \)
• In the homogeneous case one can readily compute the cumulants and get

\[ c_{2}^{\text{hom}} = \chi_{\theta} | v_{\theta}^{\text{eff}} | (h_{i_{s}}^{\text{dr};\theta})^{2} \]

\[ c_{3}^{\text{hom}} = \chi_{\phi} | v_{\phi}^{\text{eff}} | h_{i_{s}}^{\text{dr};\phi} \left( s_{\phi} f_{\phi} (h_{i_{s}}^{\text{dr};\phi})^{2} + 3 [s f (h_{i_{s}}^{\text{dr}})^{2}]_{\phi} \right). \]

• They coincide with the results obtained by the Ballistic fluctuation theory, which were also corroborated against Hard-rod simulations.

• A virtue of the BMFT is that the extension to inhomogeneous cases is straightforward. For instance we computed \( c_{2} \) for the partitioning protocol and obtained

\[ c_{2}^{\text{part}} = \chi_{\theta} (0) | v_{\theta}^{\text{eff}} (0) | (h_{i_{s}}^{\text{dr};\theta}(0))^2 \]

**Fully fixed by the NESS at \( \xi = 0! \)**
Comparison against hard-rod simulations

- A gas of hard-rods consists of rigid rods that scatter elastically, and hence is an integrable system

\[ H = \sum_{j=1}^{N} \frac{1}{2} p_j^2 + \sum_{j=1}^{N-1} V_{HR}(q_{j+1} - q_j), \quad V_{HR}(x) = \begin{cases} \infty & |x| < a \\ 0 & |x| \geq a \end{cases} \]

- The onset to the stationary value is controlled by the diffusive corrections
Euler dynamical correlation functions in integrable systems

- The MFT equation for the correlation function $S_{\hat{q}_1, \hat{q}_2}(x_1, t_1; x_2, t_2)$ is

$$
\begin{align*}
\beta^\theta(x,0) - \beta^\theta_{\text{ini}}(x) + H^\theta(x,0) &= 0 \\
H^\theta(x, T) &= 0 \\
\partial_t \beta^\theta + A^\theta \partial_x \beta^\phi &= 0 \\
\partial_t H^\theta + A^\theta \partial_x H^\phi + \lambda h_i^\theta \delta(x - x_1) \delta(t - t_1) &= 0
\end{align*}
$$

- In terms of the solution, the correlator is computed by

$$S_{\hat{q}_1, \hat{q}_2}(x_1, t_1; x_2, t_2) = \frac{d}{d\lambda} q_{i_2}(x_2, t_2) \bigg|_{\lambda=0} = -[\langle h_i \rangle^\theta \chi^\theta_{\partial_x} e^\theta](x_2, t_2) \bigg|_{\lambda=0}.$$
• As in the case of SCGF, the MFT equation is reduced to the following GHD equation

\[
\beta^\theta(x,0) = \beta^\theta_{\text{lin}}(x) - \lambda \partial \left( (R^T)^\theta_{\phi}(x_1, t_1) h^{\text{dr}}_{\phi}(x_1, t_1) \Theta(x - ut^\theta) \right) \\
\partial_t \beta^\theta(x, t) + A^\theta_{\phi}(x, t) \partial_x \beta^\phi(x, t) = 0
\]

• Let us take \( t_1 = t_2 = t \). The Euler dynamical correlator \( S_{\hat{q}_1, \hat{q}_2}(x_1, t; x_2, t) \) turns out to be given by

\[
S_{\hat{q}_1, \hat{q}_2}(x_1, t; x_2, t) = C_{i_1 i_2}(x_1, t) \delta(x_1 - x_2) + E_{i_1 i_2}(x_1, x_2; t), \quad C_{i_1 i_2}(x_1, t) := [(h^{{\Phi}_1})^{\text{dr}}, \chi^\theta(h^{{\Phi}_2})^{\text{dr}}, \theta](x_1, t)
\]
• Despite of the small numbers, the agreements are very satisfying!

• By increasing $\ell$, one can observe the convergence of the numbers
Conclusions and outlooks

• The BMFT is a new theory to study the fluctuation-induced physics, such as current fluctuations and large scale dynamical correlation functions, in ballistic many-body systems.

• The underlying idea of the BMFT is local relaxation of fluctuations.

• It works particularly well for integrable systems. The results also agree with hard-rods simulations very well.

• The BMFT should be applicable to other ballistic systems, e.g. the anharmonic chain. Will we indeed see the expected long-range correlations?

• It is highly desirable to derive our predictions microscopically using a simple model such as the AHR model.

• Obtaining the KPZ function from the BMFT+superdiffusive corrections?

• Quantum fluctuations?