Ballistic macroscopic fluctuation theory

for

Integrable systems

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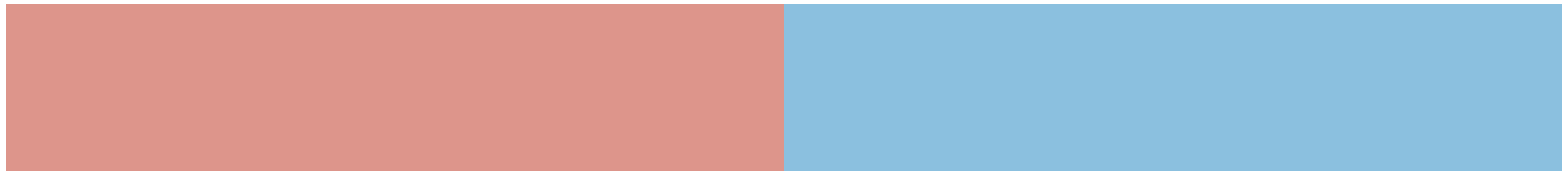
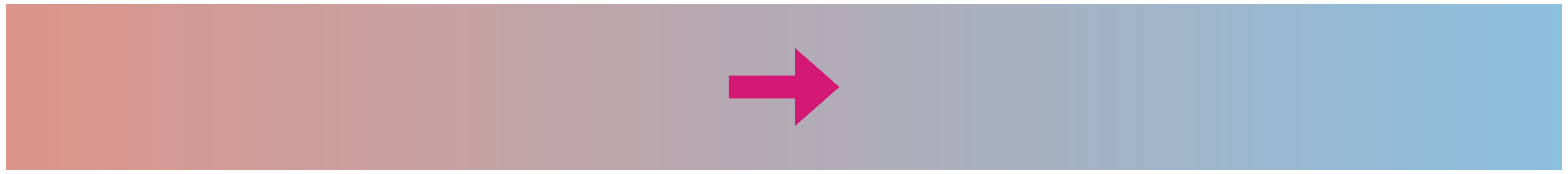
Inhomogeneous dynamics in quantum many-body systems

$t = 0 \qquad |\Psi(0)\rangle =$



$T_L \qquad\qquad T_R$

$t \to \infty \qquad |\Psi\rangle_{\mathrm{NESS}} =$



$\left(|\Psi\rangle_{\mathrm{NESS}} = \lim_{t\to\infty} |\Psi(t)\rangle\right)$

$J_{\mathrm{particle}}, J_{\mathrm{energy}}, \cdots$

How can we characterise the persistent (ballistic) currents?

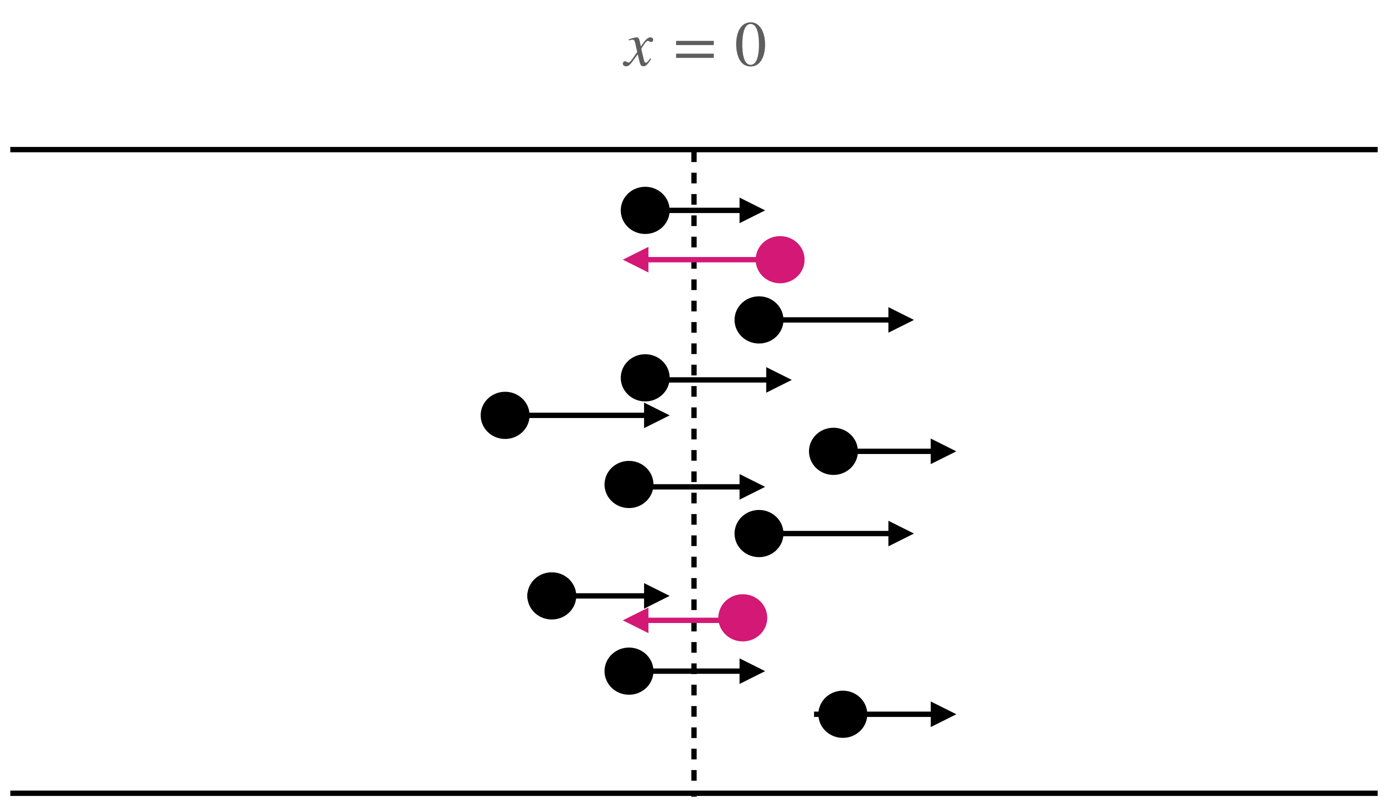
$$\hat{J}(T) := \int_0^T \mathrm{d}t\, \hat{j}(0,t) = \int_0^\infty \mathrm{d}x\, (\hat{q}(x,T) - \hat{q}(0,T))$$

- One of the most important quantities is the average of the currents $\lim_{T\to\infty} \frac{1}{T}\langle \Psi(0) | \hat{J}(T) | \Psi(0) \rangle$

- But of course that's not the only thing that describes the properties of currents! More intricate information is encoded in **correlations** pertaining to **rare** fluctuations

$$\lim_{T\to\infty} \frac{1}{T}\langle \Psi(0) | \hat{J}(T)\hat{J}(T) | \Psi(0) \rangle, \lim_{T\to\infty} \frac{1}{T}\langle \Psi(0) | \hat{J}(T)\hat{J}(T)\hat{J}(T) | \Psi(0) \rangle, \cdots$$

- They are known as (scaled) **cumulants**, and capture the **statistics** of currents, e.g. variance, skewness, etc.



- More formally

Scaled cumulant generating function $$F(\lambda) := \lim_{T\to\infty} \frac{1}{T} \log\langle e^{\lambda \hat{J}(T)}\rangle, \quad \langle \bullet \rangle := \langle \Psi(0) | \bullet | \Psi(0)\rangle$$

- The nontrivial $F(\lambda)$ exists when the probability $\mathbb{P}(\hat{J}(T) = Tj)$ follows the large deviation principle $\mathbb{P}(\hat{J}(T) = Tj) \asymp e^{-TI(j)}$ with a convex large deviation function $I(j)$.

- How would we compute $F(\lambda)$ for ballistic many-body systems?

- The ballistic fluctuation theory (BFT) was devised for this purpose but it is restricted to the homogeneous initial condition

[Doyon and Myer, 2020; Myers, Bhaseen, Harris, and Doyon, 2020]

- We propose a general framework to describe the physics pertaining to (rare) fluctuations. It can be seen as the ballistic version of the diffusive macroscopic fluctuation theory (MFT). Its general idea was also introduced in Benjamin's talk.

Hydrodynamics and local relaxation

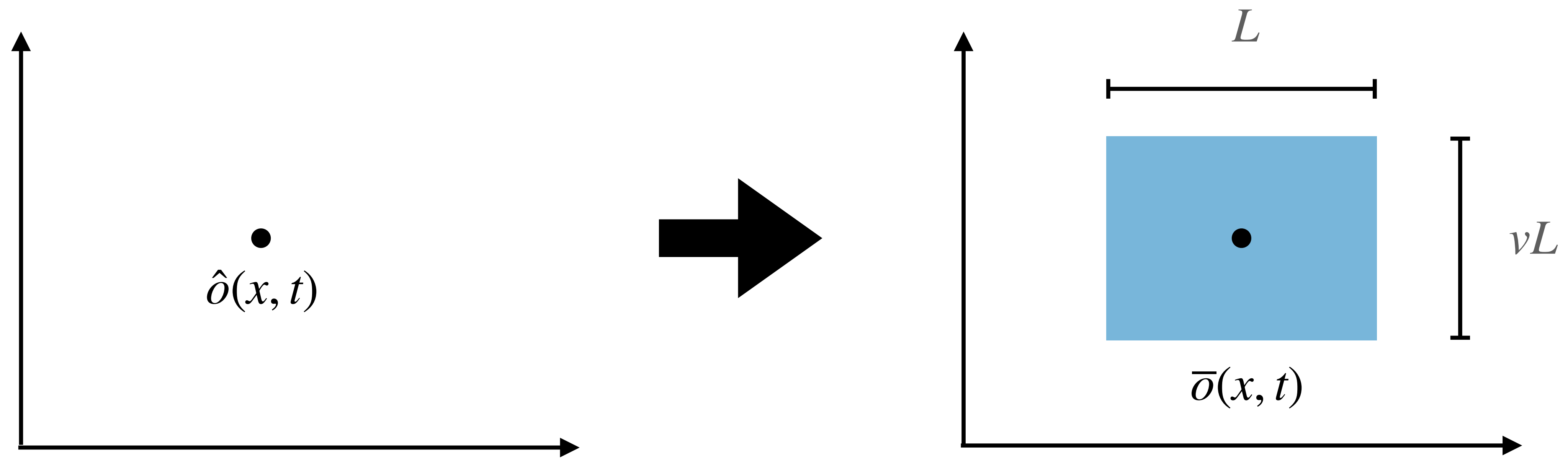
- We consider a many-body system with N conservation laws, which support ballistic transport. Let us start with recalling the principle of Euler hydrodynamics

- Suppose the initial condition is given by some local (generalised) Gibbs ensemble with the statistical average

$$\langle \bullet \rangle_\ell = \frac{1}{Z}\mathrm{Tr}\left[\exp\left[-\sum_{i=0}^{N-1}\int_{\mathbb{R}} \mathrm{d}x\,\beta^i(x/\ell)\hat{q}_i(x,0)\right]\bullet\right]$$

- The primary objects in hydrodynamics are the space-time averaged mesoscopic observables

$$\overline{o}(x,t) = \frac{1}{vL^2}\int_{-L/2}^{L/2}\mathrm{d}y\int_{-vL/2}^{vL/2}\mathrm{d}s\,\hat{o}(x+y,t+s)$$



- Note that we eventually take $1 \ll L/\ell_{\mathrm{micro}} \ll \ell$

- For instance hydrodynamics predicts that

$$\lim_{\ell\to\infty} \langle \overline{o}(\ell x, \ell t)\rangle_\ell = \langle \hat{o}\rangle_{\underline{\beta}(x,t)}, \quad \langle \bullet \rangle_{\underline{\beta}} := \frac{1}{Z}\mathrm{Tr}\left[e^{-\sum_{i=0}^{N-1}\beta^i Q_i}\bullet\right]$$

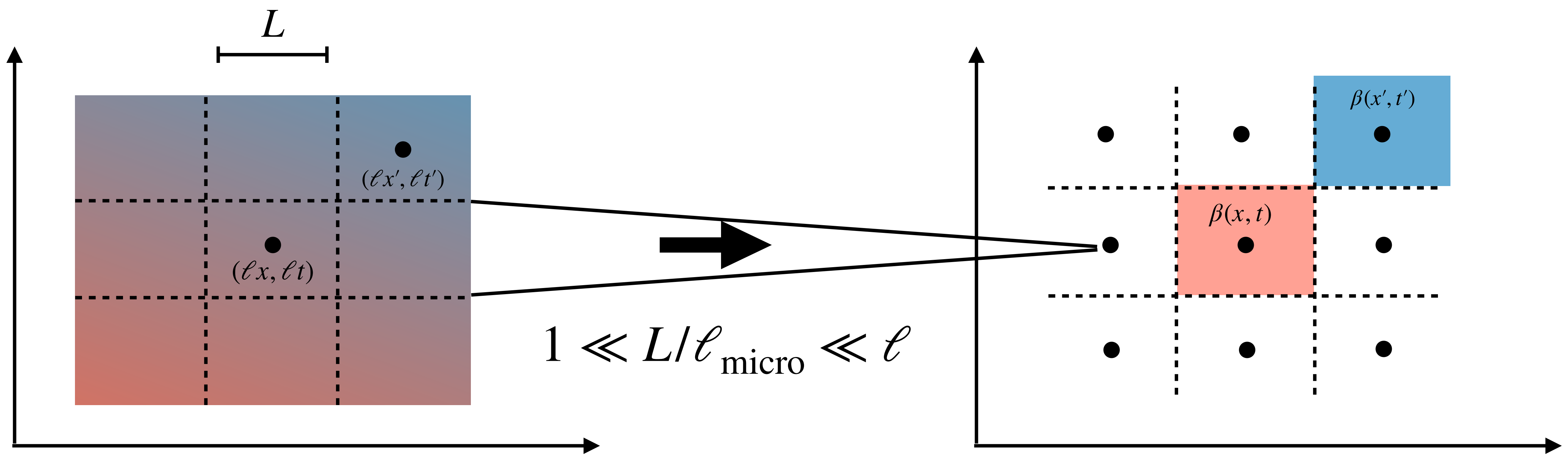
- Note that $\beta^i(x,t)$ characterise the fluid cell at the (macroscopic) space-time $(\ell x, \ell t)$, so their arguments (x,t) are **macroscopic**.

- The above equality implies that the **local equilibrium** average of the (mesoscopic) observable at the macroscopic point $(\ell x, \ell t)$ is given by its **equilibrium** average with the spatio-temporal Lagrange multipliers $\beta^i(x,t)$.

- On the Euler scale the average of any local operator can be thought of as a functional of $\underline{\mathrm{q}}(x,t) := \langle \hat{q}\rangle_{\underline{\beta}(x,t)}$, i.e.

$$\mathrm{o}(x,t) = \langle \hat{o}\rangle[\underline{\mathrm{q}}(x,t)]$$

Local relaxation of averages



- In terms of $\underline{\mathsf{q}}(x,t)$ and $\underline{\mathsf{j}}(x,t)$, the Euler hydrodynamic equation reads $\partial_t \mathsf{q}_i(x,t) + \partial_x \mathsf{j}_i(x,t) = 0$, or equivalently,

$$\partial_t \mathsf{q}_i(x,t) + \mathsf{A}_i^{\ j}[\underline{\mathsf{q}}(x,t)]\partial_x \mathsf{q}_j(x,t) = 0, \quad \mathsf{A}_i^{\ j}[\underline{\mathsf{q}}] := \frac{\partial \mathsf{j}_j[\underline{\mathsf{q}}]}{\partial \mathsf{q}_i} = \frac{\partial \mathsf{j}_j}{\partial \beta^k}\mathsf{C}^{ki}$$

- Here $\mathsf{C}^{ki} = (\mathsf{C}^{-1})_{ki}$ with the susceptibility matrix $\mathsf{C}_{ij} = -\dfrac{\partial \mathsf{q}_j}{\partial \beta^i} = \displaystyle\int_{\mathbb{R}} \mathrm{d}x \, \langle \hat{q}_i(x,0)\hat{q}_j(0,0)\rangle_{\underline{\beta}}$

- One can also write down a hydro equation for the Lagrange multipliers

$$\partial_t \beta^i(x,t) + \mathsf{A}_j^{\ i}[\underline{\beta}(x,t)]\partial_x \beta^j(x,t) = 0$$

Positivity of C implies a bijection $\underline{q} \leftrightarrow \underline{\beta}$

- Hydro is all about the dynamics of **averages** (and equilibrium correlation functions obtained out of them). In order to access the quantities that are strongly affected by the (rare) fluctuations, one also has to know how the **fluctuations** (regardless of their origins) propagate in time.

Initial fluctuations

- To consider the propagation of fluctuations, one has to first characterise the fluctuations of the initial condition

- We are interested in the fluctuations of the mesoscopic variables $q_i(x,0) \equiv \overline{q}_i(\ell x,0)$. Its correlations can be obtained from a measure

$\mathrm{d}\mathbb{P}_{\text{ini}}[\underline{q}(\cdot,0)] = \mathrm{d}\mu[\underline{q}(\cdot,0)]\, e^{-\ell\mathscr{F}[\underline{q}(\cdot,0)]}$, where $e^{-\ell\mathscr{F}[\underline{q}(\cdot,0)]}$ is the probability of observing the density profile $q_i(x,0)$ given the initial background density $\mathtt{q}_{\text{ini},i}(x)$

$$\mathscr{F}[\underline{q}(\cdot,0)] = \int_{\mathbb{R}} \mathrm{d}x \left(\beta^i_{\text{ini}}(x)(q_i(x,0) - \mathtt{q}_{\text{ini},i}(x)) + s[\underline{\mathtt{q}_{\text{ini}}}(x)] - s[\underline{q}(x,0)]\right).$$

- Defining $\underline{\beta}(x,0)$ via $q_i(x,0) = \mathtt{q}_i[\underline{\beta}(x,0)]$, the saddle point of $\mathrm{d}\mathbb{P}_{\text{ini}}[\underline{q}(\cdot,0)]$ gives the $\beta^i(x,0) = \beta^i_{\text{ini}}(x)$

- How do we generalise $\mathrm{d}\mathbb{P}_{\text{ini}}[\underline{q}(\cdot,0)]$ to $\mathrm{d}\mathbb{P}_{\text{ini}}[\underline{q}(\cdot,\cdot)]$?

Local relaxation of fluctuations

- The idea of **local relaxation of fluctuations** constitutes the cornerstone of the BMFT. This is the only **assumption** we make.

Mesoscopic means of local observables (coarse-grained observables), $\overline{o}(\ell x, \ell t)$, do not fluctuate **independently** but are **fixed** functionals of conserved densities, i.e.
$$\overline{o}(\ell x, \ell t) = o[\underline{q}(x,t)]$$

- This is closely related to local relaxation of averages $\mathtt{o}(x,t) = \langle \hat{o} \rangle [\underline{\mathtt{q}}(x,t)]$. To fix $o[\,\bullet\,]$, we can determine it by taking $t = 0$ and invoke

$$\langle \hat{o} \rangle_{\underline{\beta}(x,0)} = \int_{(\mathbb{R})} \mathrm{d}\mathbb{P}_{\mathrm{ini}}[\underline{q}(\cdot,0)]\, o[\underline{q}(x,0)]$$
(because $o(x,0) = o[\underline{q}(x,0)]$ according to the ansatz)

- The saddle point of it yields $o[\underline{q}(x,0)] = \mathtt{o}[\underline{q}(x,0)]$, i.e. $\overline{o}(\ell x, \ell t) = \mathtt{o}[\underline{q}(x,t)]$.

$$\langle o(x,t) \rangle_\ell = \mathtt{o}[\underline{\mathtt{q}}(x,t)] \qquad o(x,t) = \mathtt{o}[\underline{q}(x,t)]$$

Local relaxation of averages **Local relaxation of fluctuations**

$\ell \to \infty$

- Having this assumption with $j_i(x,t) = \bar{j}_i(\ell x, \ell t) = \mathsf{j}_i[\underline{q}(x,t)]$, the measure $\mathrm{d}\mathbb{P}_{\mathrm{ini}}[\underline{q}(\,\cdot\,,\cdot\,)]$ we use for the BMFT is given by

$$\boxed{\mathrm{d}\mathbb{P}[\underline{q}(\,\cdot\,,\cdot\,)] = \underbrace{\mathrm{d}\mu[\underline{q}(\,\cdot\,,\cdot\,)]}_{\text{flat measure}}\,\underbrace{e^{-\ell\mathscr{F}[\underline{q}(\cdot,0)]}}_{\text{initial fluctuation}}\underbrace{\delta[\partial_t\underline{q} + \partial_x\underline{\mathsf{j}}[\underline{q}]]}_{\text{continuity equation} + \text{local relaxation of fluctuations}}}$$

- With this measure we write the BMFT average as

$$\langle\langle\,\bullet\,\rangle\rangle_\ell = \frac{1}{Z}\int_{(\mathbb{S})}\mathrm{d}\mu[\underline{q}(\,\cdot\,,\cdot\,)]\,e^{-\ell\mathscr{F}[\underline{q}(\cdot,0)]}\delta(\partial_t\underline{q} + \partial_x\underline{\mathsf{j}}[\underline{q}])\,\bullet\,, \qquad \mathbb{S} := \mathbb{R}\times[0,T]$$

which we identify with $\langle\,\bullet\,\rangle_\ell$.

- Closely related to the **conventional** MFT ansatz

$$\mathrm{d}\mathbb{P}[\underline{q}(\,\cdot\,,\cdot\,)] = \mathrm{d}\mu[\underline{q}(\,\cdot\,,\cdot\,)]\mathrm{d}\mu[\underline{\eta}(\,\cdot\,,\cdot\,)]\,e^{-\ell\mathscr{F}[\underline{q}(\cdot,0)]}\delta[\partial_t\underline{q} + \partial_x\underline{j}_{\mathrm{diff}}], \quad j_{\mathrm{diff},i} := -\mathfrak{D}_i^{\ j}\partial_x q_j + \eta_i$$

- Using the ansatz for the measure, one can compute any average of observables. In particular we're interested in two objects:

$$S_{\hat{o}_1,\ldots,\hat{o}_n}(x_1,t_1;\cdots;x_n,t_n) := \lim_{\ell\to\infty} \ell^{n-1}\langle \overline{o_1}(\ell x_1,\ell t_1)\cdots\overline{o_n}(\ell x_n,\ell t_n)\rangle_\ell^{\mathrm{c}}$$

Euler scale dynamical correlation functions

$$F(\lambda,T) := \lim_{\ell\to\infty}\frac{1}{\ell T}\log\langle e^{\lambda \hat{J}_{i*}(\ell T)}\rangle_\ell$$

Scaled cumulant generating function

- The BMFT allows us to evaluate these quantities via the path-integrals

$$S_{\hat{o}_1,\ldots,\hat{o}_n}(x_1,t_1;\cdots;x_n,t_n) = \lim_{\ell\to\infty} \ell^{n-1}\langle\langle \mathrm{o}_1[\underline{q}(x_1,t_1)]\cdots\mathrm{o}_n[\underline{q}(x_n,t_n)]\rangle\rangle_\ell^{\mathrm{c}}$$

$$F(\lambda,T) = \lim_{\ell\to\infty}\frac{1}{\ell}\log\langle\langle e^{\lambda T J_{i*}(T)}\rangle\rangle_\ell$$

- The power of the (B)MFT is that we don't even have to evaluate the path-integrals, as they turn out to be dominated by their saddle points when $\ell \to \infty$!

Current fluctuations from the BMFT

- Let us first look into the current fluctuations, i.e. the evaluation of $F(\lambda, T)$

$$\langle e^{\lambda \hat{J}_{i_*}(\ell T)} \rangle = \int_{(\mathbb{S})} \mathrm{d}\mu[\underline{q}(\,\cdot\,,\,\cdot\,)]\, e^{\ell \lambda J_{i_*}(T)} e^{-\ell \mathscr{F}[\underline{q}(\cdot,0)]} \delta(\partial_t \underline{q} + \partial_x \underline{\mathtt{j}}[\underline{q}])$$
$$=: \int_{(\mathbb{S})} \mathrm{d}\mu[\underline{q}(\,\cdot\,,\,\cdot\,)] \mathrm{d}\mu[\underline{H}(\,\cdot\,,\,\cdot\,)] e^{-\ell S_{\mathrm{curr}}[\underline{q},\underline{H}]}$$

with $S_{\mathrm{curr}}[\underline{q}, \underline{H}] := -\lambda J_{i_*}(T) + \mathscr{F}[\underline{q}(\cdot,0)] + \int_{\mathbb{S}} \mathrm{d}t \mathrm{d}x\, H^i(\partial_t q_i + \partial_x \mathtt{j}_i[\underline{q}])$

- Clearly the path-integral is dominated by $\underline{q}$ and $\underline{H}$ such that $\delta S_{\mathrm{curr}} = 0$, which can be obtained by solving the MFT equations

$$\boxed{\begin{aligned}
&\lambda \delta^i{}_{i_*} \Theta(x) - \beta^i(x,0) + \beta^i_{\mathrm{ini}}(x) - H^i(x,0) = 0,\\
&\lambda \delta^i{}_{i_*} \Theta(x) - H^i(x, T) = 0,\\
&\partial_t \beta^i(x,t) + \mathsf{A}_j{}^i[\underline{\beta}(x,t)] \partial_x \beta^j(x,t) = 0,\\
&\partial_t H^i(x,t) + \mathsf{A}_j{}^i[\underline{\beta}(x,t)] \partial_x H^j(x,t) = 0.
\end{aligned}}$$

- Once we have the solution we can compute the SCGF as $F(\lambda, T) = \int_0^\lambda \mathrm{d}\lambda' \int_0^T \mathrm{d}t\, \mathrm{j}_{i*}^{(\lambda')}(0,t)/T$

- In general it is hard to solve the equations, as the solution tends to be not unique due to the appearance of shocks (e.g. the TASEP). Some progress has been made for the TASEP by taking the totally asymmetric limit of the MFT for the WASEP [Bodineau and Derrida, 2006]

- Fortunately no shock appears in integrable systems, which turns out to be a great help in doing the BMFT for integrable systems.

- Surprisingly, the structure of the MFT equations implies the Gallavotti-Cohen fluctuation theorem (GCFT). Suppose the initial condition is a step-initial condition $\beta^i_{\mathrm{ini}}(x) = \delta^i{}_{i*}(\beta_L \Theta(-x) + \beta_R \Theta(x))$. Then the GCFT claims the symmetry

$$F(\lambda) = F(\beta_L - \beta_R - \lambda), \quad F(\lambda) := F(\lambda, 1)$$

- It turns out that the symmetry follows from the fact that $\mathrm{q}_i^{(\lambda)}(x, t)$ and its time-reversal counterpart with the counting field $\tilde{\lambda} = \beta_L - \beta_R - \lambda$ satisfy the same MFT equation

Dynamical correlation functions from the BMFT

- In a similar way one can also compute the Euler-scale dynamical correlation function

$$S_{\hat{q}_{i_1},\hat{q}_{i_2}}(x_1,t_1;x_2,t_2) = \frac{\mathrm{d}^2}{\ell \mathrm{d}\lambda_1 \mathrm{d}\lambda_2} \log\left[\int_{(\mathbb{S})} \mathrm{d}\mu[\underline{q}(\,\cdot\,,\,\cdot\,)]\exp[\ell(\lambda_1 q_{i_1}(x_1,t_1)+\lambda_2 q_{i_2}(x_2,t_2))]e^{\ell\mathscr{F}[\underline{q}(\cdot,0)]}\delta(\partial_t \underline{q}+\partial_x \underline{\mathrm{j}}[\underline{q}])\right]\Bigg|_{\lambda_1=\lambda_2=0}$$

$$=: \frac{\mathrm{d}^2}{\ell \mathrm{d}\lambda_1 \mathrm{d}\lambda_2} \log\left[\int_{(\mathbb{S})} \mathrm{d}\mu[\underline{q}(\,\cdot\,,\,\cdot\,)]\mathrm{d}\mu[\underline{H}(\,\cdot\,,\,\cdot\,)]e^{-\ell S_{\mathrm{corr}}[\underline{q},\underline{H}]}\right],$$

with $S_{\mathrm{corr}}[\underline{q},\underline{H}] := -(\lambda_1 q_{i_1}(x_1,t_1)+\lambda_2 q_{i_2}(x_2,t_2)) + \mathscr{F}[\underline{q}(\cdot,0)] + \int_{\mathbb{S}} \mathrm{d}t\mathrm{d}x\, H^i(x,t)(\partial_t q_i(x,t)+\partial_x \mathrm{j}_i[\underline{q}(x,t)])$.

- As in the case of the SCGF, one can compute $S_{\hat{q}_{i_1},\hat{q}_{i_2}}(x_1,t_1;x_2,t_2)$ out of the solution of the MFT equation involving λ_1 and λ_2.

$$S_{\hat{q}_{i_1},\hat{q}_{i_2}}(x_1,t_1;x_2,t_2) = -\frac{\mathrm{d}^2}{\mathrm{d}\lambda_1\mathrm{d}\lambda_2} S^{\mathrm{sp}}_{\mathrm{corr}}\Bigg|_{\lambda_1=\lambda_2=0}.$$

- Since $S^{\mathrm{sp}}_{\mathrm{corr}}$ is the saddle point action, we have $\partial_{\lambda_2} S^{\mathrm{sp}}_{\mathrm{corr}} = -q_{i_2}(x_2,t_2)$. Hence

$$S_{\hat{q}_{i_1},\hat{q}_{i_2}}(x_1,t_1;x_2,t_2) = \frac{\mathrm{d}}{\mathrm{d}\lambda}\mathrm{q}_{i_2}(x_2,t_2)\Bigg|_{\lambda=0}, \quad \lambda := \lambda_1$$

- The associated MFT equation is then

$$\boxed{\begin{aligned} &\beta^i(x,0) - \beta^i_{\mathrm{ini}}(x) + H^i(x,0) = 0, \\ &H^i(x,T) = 0, \\ &\partial_t \beta^i + \mathsf{A}_j{}^i \partial_x \beta^j = 0, \\ &\partial_t H^i + \mathsf{A}_j{}^i \partial_x H^j + \lambda \delta^i{}_{i_1} \delta(x - x_1)\delta(t - t_1) = 0. \end{aligned}}$$

- The solution of the MFT equation actually predicts the existence of the long-range correlation amongst the fluid cells on the same time slice $t_1 = t_2 = t$, i.e.

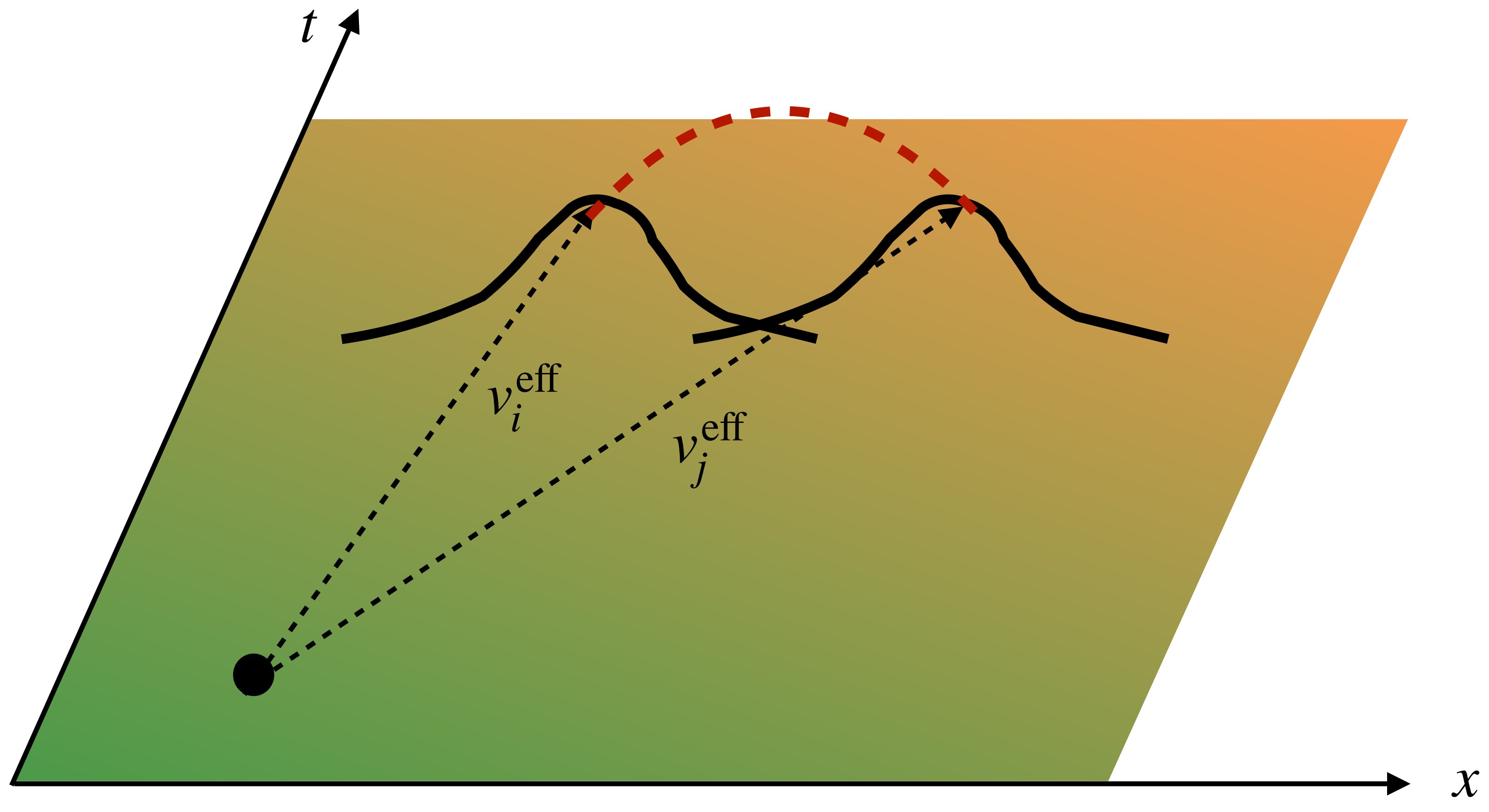
$$\lim_{\ell \to \infty} \langle \bar{q}_{i_1}(\ell x_1, \ell t)\bar{q}_{i_2}(\ell x_2, \ell t)\rangle_\ell = S_{\hat{q}_{i_1},\hat{q}_{i_2}}(x_1, t; x_2, t) = C_{i_1 i_2}(x_1, t)\delta(x_1 - x_2) + E_{i_1 i_2}(x_1, x_2; t)$$

$$\Rightarrow \quad \langle \bullet \rangle_\ell \neq \frac{1}{Z}\mathrm{Tr}\left[\exp\left[-\sum_{i=0}^{N-1}\int_{\mathbb{R}} \mathrm{d}x\, \beta^i(x/\ell, t/\ell)\hat{q}_i(x,0)\right]\bullet\right]$$

for higher point functions

- Even if there is no long-range correlation initially, it could build up by the coupling between normal modes on an inhomogeneous background

- Note that the long-range correlations observed here is of purely **hydrodynamic** nature



multiple conservation laws ✔
interaction ✔
initial inhomogeneity ✔

[Ortiz de Zárate and Sengers, 2004]

- For instance it is readily seen that TASEP cannot develop long-range correlations based on this mechanism

- Long-range correlations have also been observed in NESS in the partitioning protocol for different models e.g. free 1d Klein-Gordon model, Lieb-Liniger model where $\langle \hat{q}_0(x,0)\hat{q}(0,0)\rangle^{\mathrm{c}}_{\mathrm{NESS}}$ decays as $1/x^2$ [Doyon, Lucas, Schalm, and Bhaseen, 2015; De Nardis and Panfil, 2018]. But this has a different origin, which is a non-locality of the NESS density operator. A similar observation was also made for the symmetric simple exclusion process [Derrida, 2007]

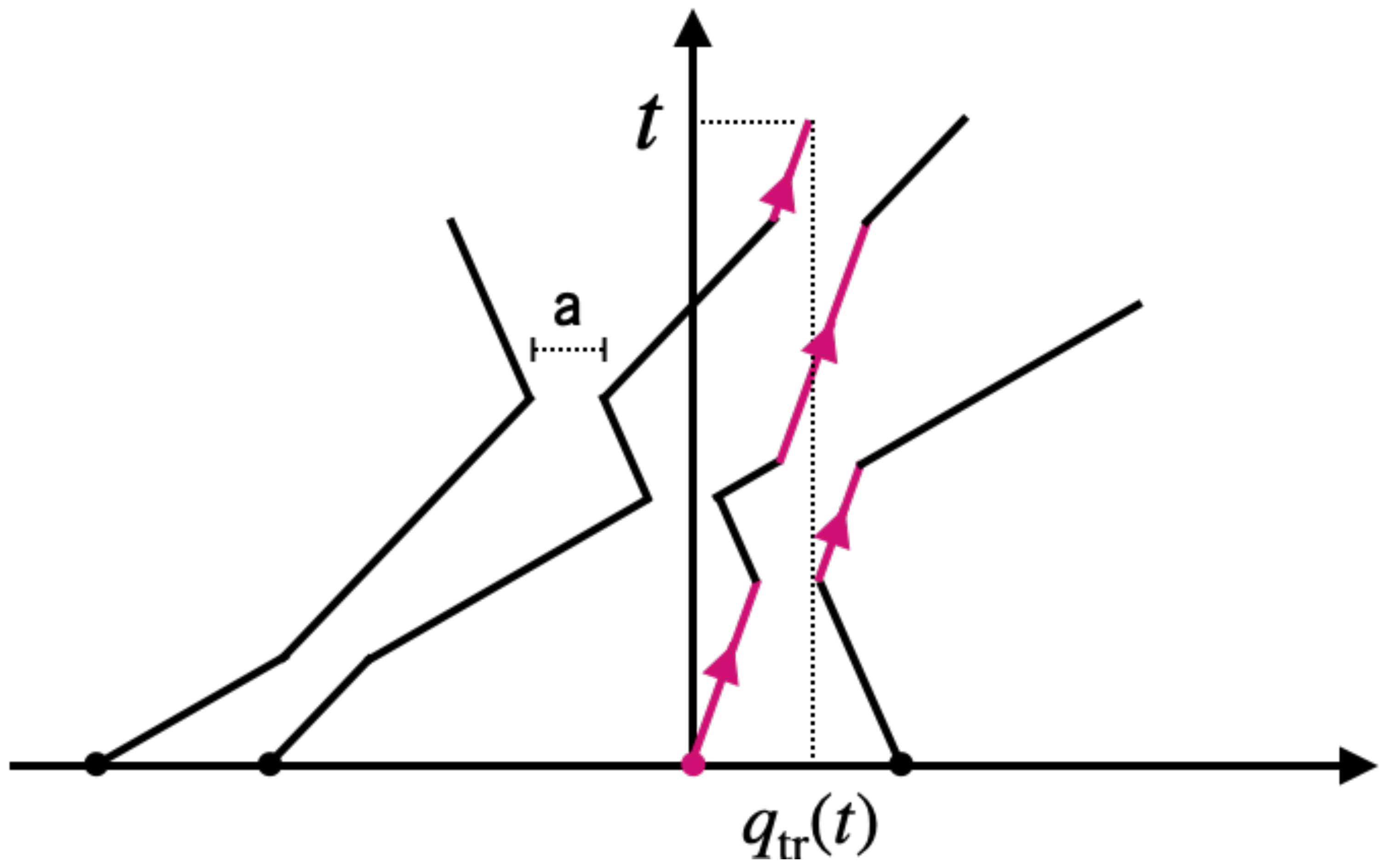
- We are going to compute these quantities explicitly for **integrable systems**

Generalised hydrodynamics

- In integrable systems thermodynamics as well as the dynamics of the system are dictated by the scattering data: particle species, dispersion relation, and the two-body S-matrix.

- We shall consider a diagonally-scattering integrable model with a single species defined on a line. Generalisations to more complicated models are straightforward.

- A kinetic intuition behind GHD is that on a hydrodynamic scale, quasi-particles in integrable systems behave pretty much like tracer particles of hard-rods. [Boldrighini, Dobrushin, and Sukhov, 1983; Spohn, 1991; Doyon and Spohn, 2017; Doyon, TY, and Caux, 2018]



[Cortés Cubero, TY, and Spohn, 2021]

- This underlying similarity of kinetics among integrable systems amounts to universal structures of hydrodynamic equations.

- An exact expression of the current average turns out to be instrumental in GHD. [See reviews: Borsi, Pozsgay, and Pristyák, 2021; Cortés Cubero, TY, and Spohn, 2021]

- On the Euler scale, quasi-particles in integrable systems are transported according to the GHD equation

[Castro-Alvaredo, Doyon, and TY, 2016; Bertini, Collura, De Nardis, Fagotti, 2016]

$$\partial_t \underline{\mathrm{q}}(x,t) + \partial_x \underline{\mathrm{j}}(x,t) = 0 \quad \longleftrightarrow \quad \partial_t \rho_\theta(x,t) + \partial_x (v_\theta^{\mathrm{eff}}(x,t)\rho_\theta(x,t)) = 0$$

- In MFT it is in fact more convenient to work with the Lagrange multipliers β^θ (we are considering a GGE $\varrho \sim e^{-\beta^\theta \hat{Q}_\theta}$)

$$\partial_t \beta^\theta(x,t) + A_\phi^\theta[\rho_\cdot(x,t)]\partial_x \beta^\phi(x,t) = 0$$

Note $C_{ij} = -\frac{\partial \rho_i}{\partial \beta^j}$ and $AC = CA^{\mathrm{T}}$ with $A_\theta^\phi := \frac{\partial(\rho_\theta v_\theta^{\mathrm{eff}})}{\partial \rho_\phi}$

- To solve initial value problems we shall use the GHD equation in terms of the normal mode

$$\partial_t \epsilon_\theta(x,t) + v_\theta^{\mathrm{eff}}(x,t)\partial_x \epsilon_\theta(x,t) = 0$$

- To go from the equation for β^θ to that for ϵ_θ, we used $(R^{-1})_\phi^\theta \partial_{t,x}\beta^\phi = \partial_{t,x}\epsilon^\theta$ where $R = 1 - nT$ diagonalises the flux matrix: $RAR^{-1} = \mathrm{diag}\, v^{\mathrm{eff}}$.

- One of the crucial properties of the GHD equation is that its solutions always involve neither shocks nor rarefaction waves but only contact discontinuities (CD). CDs can be thought of as shocks without entropy production.

Initial value problems in GHD

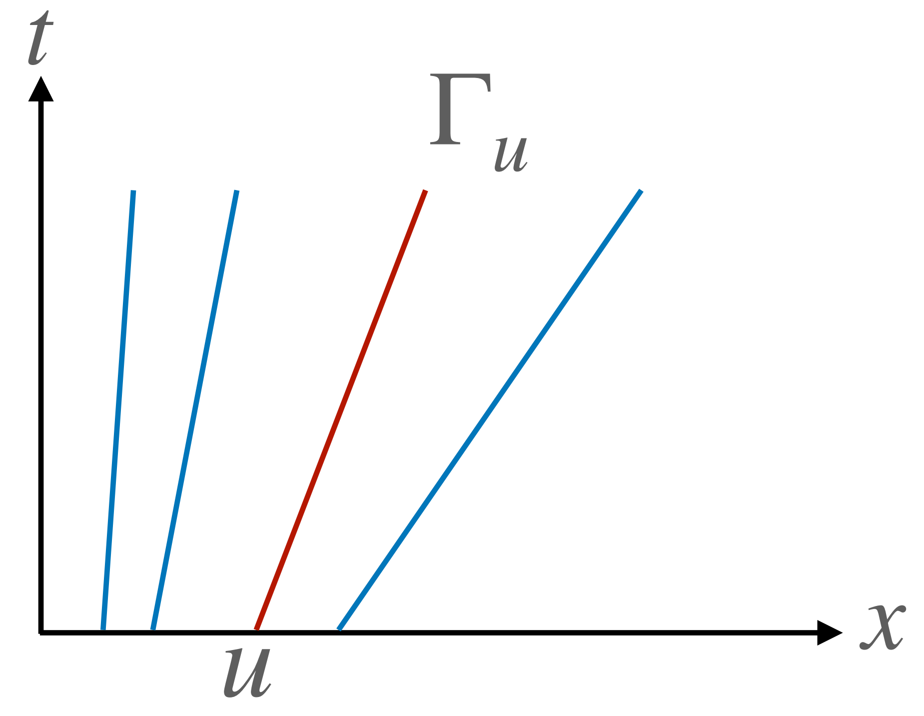
[Doyon, Spohn, and TY, 2017; Doyon 2020]

- Let us start with recalling how the method of characteristics works in a simple case: $\partial_t \rho + v(\rho)\partial_x \rho = 0$ with $\rho(x,0) = \rho_0(x)$.
- For each $x = u$ at $t = 0$, we have the characteristic curve Γ_u along which $\rho(x,t)$ is constant: $\frac{\mathrm{d}x(u,t)}{\mathrm{d}t} = v(\rho(x(u,t),t))$. This implies

$$\frac{\mathrm{d}}{\mathrm{d}t}\rho(x,t) = \frac{\partial}{\partial t}\rho(x,t) + \frac{\mathrm{d}x}{\mathrm{d}t}\frac{\partial}{\partial x}\rho(x,t) = \frac{\partial}{\partial t}\rho(x,t) + v(\rho)\frac{\partial}{\partial x}\rho(x,t) = 0$$

- Furthermore the characteristic curve is straight because clearly $\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} = 0$. The equation of characteristics can be solved as $\frac{\mathrm{d}x}{\mathrm{d}t} = v(\rho(x,t)) = v(\rho(u,0)) = v(\rho_0(u))$, i.e.

$$x = v(\rho_0(u))t + u$$



- Having $u(x,t)$ by solving the equation, we obtain $\rho(x,t) = \rho(u(x,t),0) = \rho_0(u(x,t))$.

- We want to do the same for GHD.

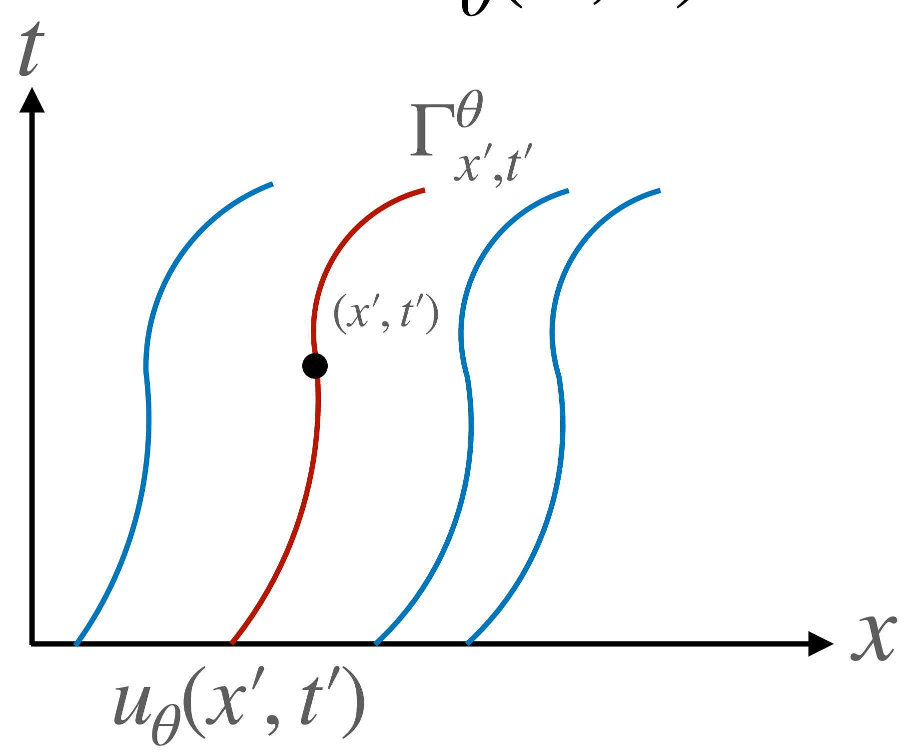
- The characteristic curve in GHD is defined by $\frac{\mathrm{d}x_\theta(u,t)}{\mathrm{d}t} = v^{\mathrm{eff}}_\theta[\epsilon_\cdot(x_\theta(u,t),t)]$, which immediately implies $\frac{\mathrm{d}\epsilon_\theta(x_\theta(u,t),t)}{\mathrm{d}t} = 0$ with $x_\theta(u,0) = u$.

- The characteristic curve is **not** straight, i.e. $\frac{\mathrm{d}^2 x_\theta(u,t)}{\mathrm{d}t^2} \neq 0$. One gets $\epsilon_\theta(x_\theta(u,t),t) = \epsilon_\theta(x_\theta(u,0),0) = \epsilon_\theta(u,0)$.

- In fact it is more convenient to fix the space time (x',t') and then construct a characteristic curve that passes $x = u_\theta(x',t')$ at $t = 0$.

- We thus redefine $u = u_\theta(x,t), x_\theta(u_\theta(x,t),t) = x$ with which we have

$$\epsilon_\theta(x,t) = \epsilon_\theta(u_\theta(x,t),0)$$



- How do we determine $u_\theta(x,t)$? Clearly the characteristic curves being not straight isn't helpful.

- The following observation makes things simpler: by changing the state-dependent coordinate change we get

$$\partial_t \hat{\epsilon}_\theta(q,t) + v^-_\theta \partial_x \hat{\epsilon}_\theta(q,t) = 0, \quad v^-_\theta := v^{\mathrm{eff}}_\theta \mathscr{K}_\theta[n^-]$$

[Doyon, Spohn, and TY, 2017]

- Here $\hat{\epsilon}_\theta(q,t)$ is defined by $\hat{\epsilon}_\theta(q_\theta(x,t),t) = \epsilon_\theta(x,t)$ with

$$q_\theta(x,t) := \int_{x_0}^{x} \mathrm{d}y\, \mathscr{K}_\theta[\epsilon_\cdot(y,t)], \quad \mathscr{K}_\theta[\epsilon_\cdot] := \frac{(p')^{\mathrm{dr}}_\theta[\epsilon_\cdot]}{p'_\theta}, \quad h^{\mathrm{dr}}_\theta := (R^{-\mathrm{T}})_\theta^{\ \phi} h_\phi$$

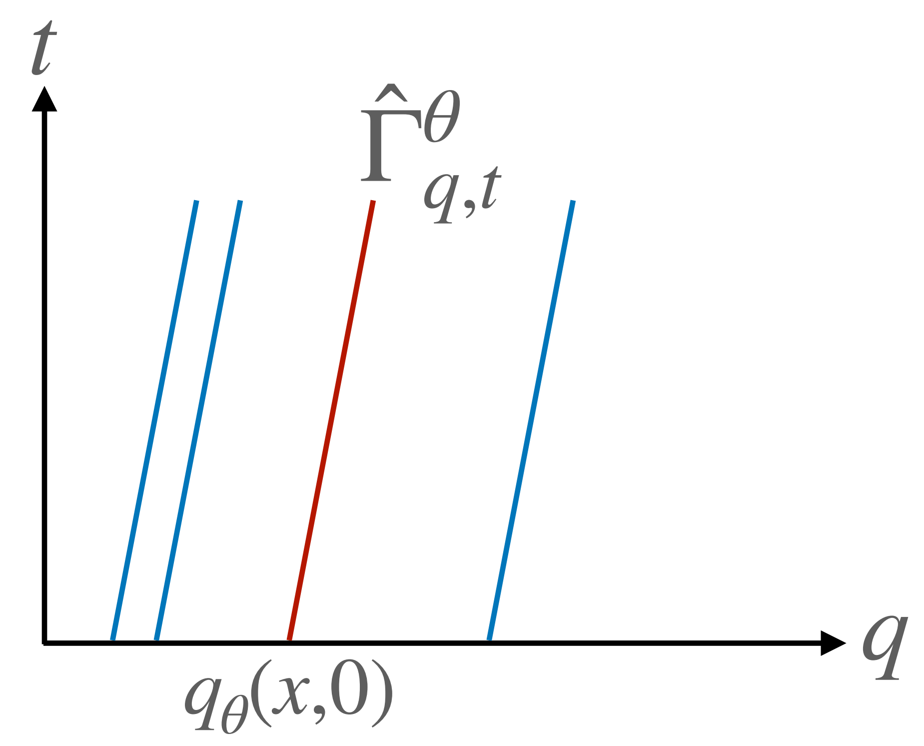
- Quasi-particles are now transported freely according to the above equation but on the state-dependent phase space $\mathrm{d}q\mathrm{d}\theta = \mathcal{K}_\theta[\epsilon(x)]\mathrm{d}x\mathrm{d}\theta$. The asymptotic coordinate x_0 is chosen so that $\rho_\theta(x,t) = \rho_\theta^-$ for all $x_0 \leq x$ at any time $t \in [0,T]$.

- The equation is trivially solved by $\hat{\epsilon}_\theta(q,t) = \hat{\epsilon}(q - v_\theta^- t, 0)$. Using the definition of $u_\theta(x,t)$ i.e. $\epsilon_\theta(x,t) = \epsilon_\theta(u_\theta(x,t),0)$ and $\hat{\epsilon}_\theta(q_\theta(x,t),t) = \epsilon_\theta(x,t)$, it immediately follows that

$$\hat{\epsilon}_\theta(q_\theta(u_\theta(x,t),0),0) = \hat{\epsilon}(q_\theta(x,t) - v_\theta^- t, 0)$$

- We thus get the solution of the characteristics in GHD, which also determines $u_\theta(x,t)$

$$\boxed{q_\theta(x,t) = v_\theta^- t + q_\theta(u_\theta(x,t),0)}$$



- In the q-coordinate space the characteristic lines are all straight. Also importantly they share the same velocity v_θ^-, which is in accordance with the fact that there is no shock in GHD.

- Alternatively the above solution can also be written as

$$\int_{x_0}^{u_\theta(x,t)} \mathrm{d}y\, \mathcal{K}_\theta[\epsilon_\cdot^0(y)] + v_\theta^- t = \int_{x_0}^{x} \mathrm{d}y\, \mathcal{K}_\theta[\epsilon_\cdot(y,t)]$$

Current fluctuations in integrable systems

- We simply adopt the BMFT we formulated to (quantum) integrable systems. We want to compute $F(\lambda, T) = \lim_{\ell \to \infty} \log \langle e^{\lambda \hat{J}_{i*}(\ell T)} \rangle_{\ell} / (\ell T)$. The MFT equation is

$$\boxed{\begin{aligned} \lambda h^{\theta} \delta^{\theta}_{\theta_*} \Theta(x) - \beta^{\theta}(x,0) + \beta^{\theta}_{\text{ini}}(x) - H^{\theta}(x,0) &= 0 \\ -\lambda h^{\theta} \delta^{\theta}_{\theta_*} \Theta(x) + H^{\theta}(x,T) &= 0 \\ \partial_t \beta^{\theta}(x,t) + A^{\theta}_{\phi}(x,t) \partial_x \beta^{\phi}(x,t) &= 0 \\ \partial_t H^{\theta}(x,t) + A^{\theta}_{\phi}(x,t) \partial_x H^{\phi}(x,t) &= 0 \end{aligned}}$$

$$\hat{Q}_i | \theta \rangle = h_i^{\theta} | \theta \rangle$$
$$h^{\theta} := h^{\theta}_{i*}$$

- We rewrite it in terms of normal modes. Recall that β^{θ} and ϵ_{θ} are related by $(R^{-1})^{\theta}_{\phi} \partial_{t,x} \beta^{\phi} = \partial_{t,x} \epsilon^{\theta}$. Motivated by this we define normal modes associated to H^{θ}:

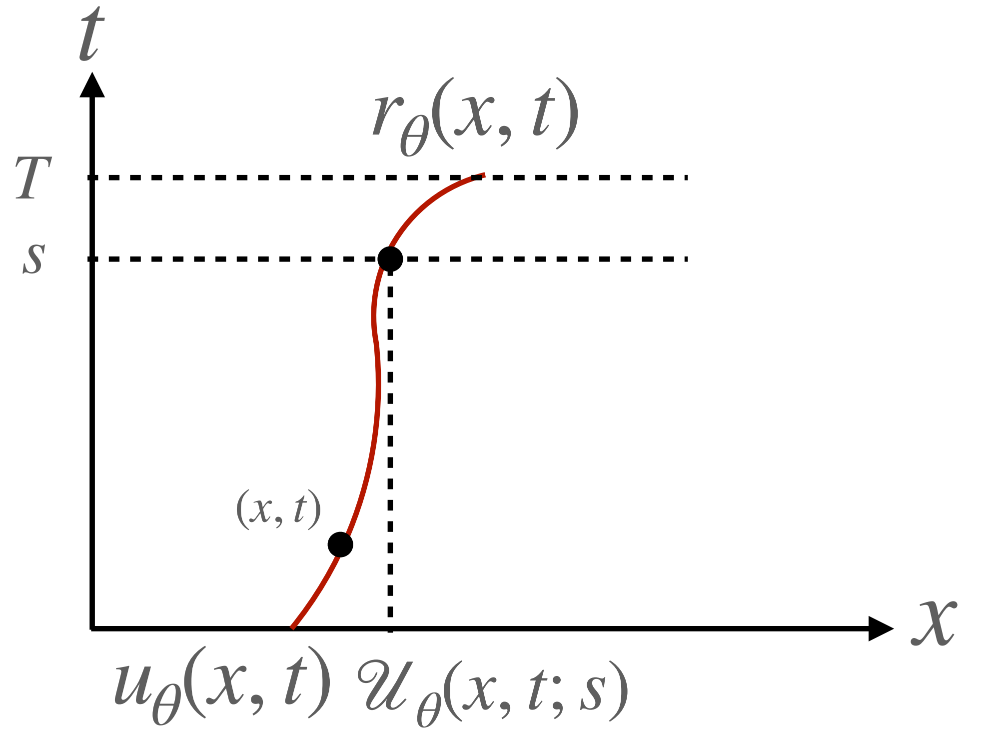
$$(R^{-1})^{\theta}_{\phi} \partial_{t,x} H^{\phi} =: \partial_{t,x} G^{\theta}$$

- The property $\partial_t \partial_x G^{\theta} = \partial_x \partial_t G^{\theta}$ is instrumental

- The MFT equation for the auxiliary field becomes

$$\begin{aligned} \lambda h^{\mathrm{dr};\theta}(0,T)\Theta(x) - G^{\theta}(x,T) = 0 \\ \partial_t G^{\theta}(x,t) + v^{\mathrm{eff};\theta}(x,t)\partial_x G^{\theta}(x,t) = 0 \end{aligned}, \quad (a^{\mathrm{dr},\theta} := (R^{-\mathrm{T}})^{\theta}{}_{\phi} a^{\phi})$$

- The method of characteristics allows us to solve the equation via $G^{\theta}(x,t) = G^{\theta}(r^{\theta}(x,t),T) = h^{\mathrm{dr};\theta}(0,T)\Theta(r^{\theta}(x,t))$, where $r^{\theta}(x,t) = \mathcal{U}^{\theta}(x,t;T)$.



- With this the MFT equation is now recast into the GHD equation with the λ-dependent initial condition

$$\boxed{\begin{aligned} &\beta^{\theta}(x,0) = \beta^{\theta}_{\mathrm{ini}}(x) + \lambda h^{\theta}\Theta(x) - \lambda R_{\phi}^{\theta}(0,T)\Theta(x - u^{\phi}(0,T))h^{\mathrm{dr};\phi}(0,T) \\ &\partial_t \beta^{\theta}(x,t) + \mathsf{A}_{\phi}^{\theta}(x,t)\partial_x \beta^{\phi}(x,t) = 0 \end{aligned}}$$

- The evaluation of the current $j_{\theta}^{(\lambda)}(0,t)$ gives $F(\lambda,T)$

- In the homogeneous case one can readily compute the cumulants and get

$$
\begin{aligned}
c_2^{\text{hom}} &= \chi_\theta \,|\, v_\theta^{\text{eff}} \,|\, (h_{i_*}^{\text{dr};\theta})^2 \\
c_3^{\text{hom}} &= \chi_\phi \,|\, v_\phi^{\text{eff}} \,|\, h_{i_*}^{\text{dr};\phi} \left(s_\phi \tilde{f}_\phi (h_{i_*}^{\text{dr};\phi})^2 + 3[sf(h_{i_*}^{\text{dr}})^2]_\phi^{\text{dr}} \right) .
\end{aligned}
$$

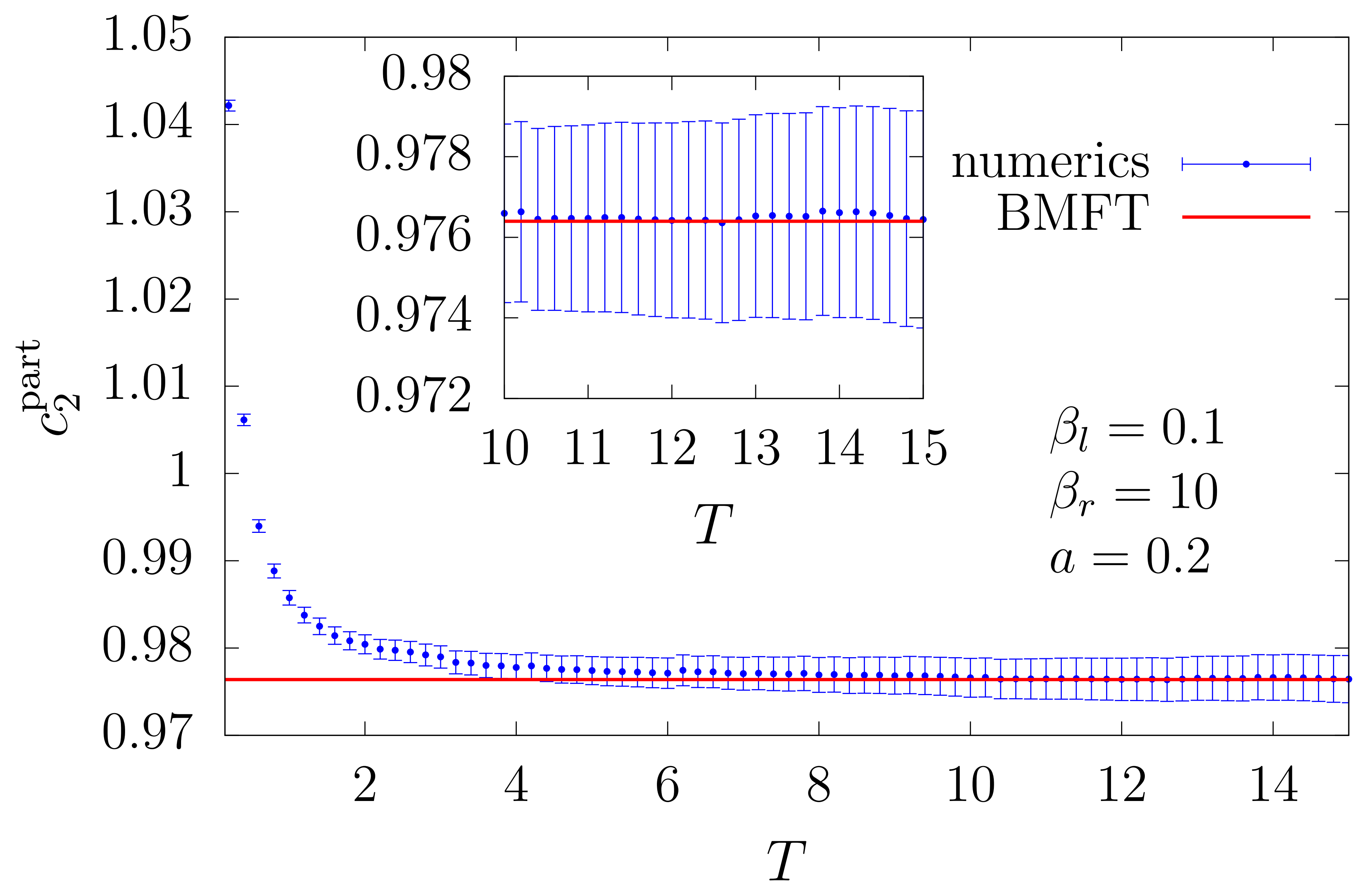
- They coincide with the results obtained by the Ballistic fluctuation theory, which were also corroborated against Hard-rod simulations.

- A virtue of the BMFT is that the extension to inhomogeneous cases is straightforward. For instance we computed c_2 for the partitioning protocol and obtained

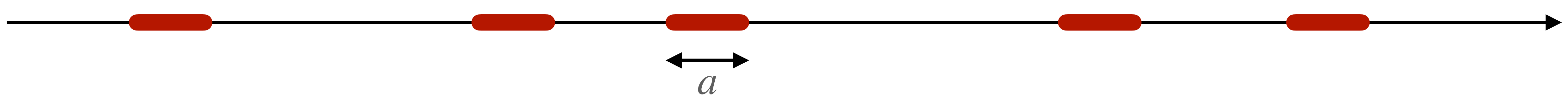
$$
c_2^{\text{part}} = \chi_\theta(0) \,|\, v_\theta^{\text{eff}}(0) \,|\, (h_{i_*}^{\text{dr};\theta}(0))^2
$$

Fully fixed by the NESS at $\xi = 0$!

Comparison against hard-rod simulations



- A gas of hard-rods consists of rigid rods that scatter elastically, and hence is an integrable system



$$H = \sum_{j=1}^{N} \frac{1}{2} p_j^2 + \sum_{j=1}^{N-1} V_{\text{HR}}(q_{j+1} - q_j), \quad V_{\text{HR}}(x) = \begin{cases} \infty & |x| < a \\ 0 & |x| \geq a \end{cases}$$

- The onset to the stationary value is controlled by the diffusive corrections

Euler dynamical correlation functions in integrable systems

- The MFT equation for the correlation function $S_{\hat{q}_{i_1},\hat{q}_{i_2}}(x_1, t_1; x_2, t_2)$ is

$$\boxed{\begin{aligned} &\beta^\theta(x,0) - \beta^\theta_{\mathrm{ini}}(x) + H^\theta(x,0) = 0 \\ &H^\theta(x,T) = 0 \\ &\partial_t \beta^\theta + \mathsf{A}_\phi^\theta \partial_x \beta^\phi = 0 \\ &\partial_t H^\theta + \mathsf{A}_\phi^\theta \partial_x H^\phi + \lambda h^\theta_{i_1} \delta(x - x_1)\delta(t - t_1) = 0 \end{aligned} \quad , \quad t_1, t_2 \neq 0,T}$$

- In terms of the solution, the correlator is computed by

$$S_{\hat{q}_{i_1},\hat{q}_{i_2}}(x_1, t_1; x_2, t_2) = \frac{\mathrm{d}}{\mathrm{d}\lambda} \mathsf{q}_{i_2}(x_2, t_2)\Big|_{\lambda=0} = -\left[(h_{i_2})^{\mathrm{dr};\theta} \chi_\theta \partial_\lambda \epsilon^\theta\right](x_2, t_2)\Big|_{\lambda=0}.$$

- As in the case of SCGF, the MFT equation is reduced to the following GHD equation

$$\boxed{\begin{aligned}&\beta^\theta(x,0) = \beta^\theta_{\mathrm{ini}}(x) - \lambda\partial\left((R^{\mathrm{T}})^\theta{}_\phi(x_1,t_1)h^{\mathrm{dr};\phi}(x_1,t_1)\Theta(x-u^\phi)\right)\\ &\partial_t\beta^\theta(x,t) + \mathsf{A}^\theta_\phi(x,t)\partial_x\beta^\phi(x,t) = 0\end{aligned}}$$

- Let us take $t_1 = t_2 = t$. The Euler dynamical correlator $S_{\hat{q}_{i_1},\hat{q}_{i_2}}(x_1,t;x_2,t)$ turns out to be given by

$$S_{\hat{q}_{i_1},\hat{q}_{i_2}}(x_1,t;x_2,t) = C_{i_1i_2}(x_1,t)\delta(x_1-x_2) + E_{i_1i_2}(x_1,x_2;t), \quad C_{i_1i_2}(x_1,t) := [(h_{i_1})^{\mathrm{dr};\theta}\chi_\theta(h_{i_2})^{\mathrm{dr};\theta}](x_1,t)$$

$$E_{i_1i_2}(x_1,x_2;t) := -[h^{\mathrm{dr};\theta}_{i_2}\chi_\theta\mathscr{E}^\theta](x_2,t)$$

$$\mathscr{E}^\theta(x,t) = \mathscr{E}^\theta_0(x,t) + w^\theta(x,t)\int_{-\infty}^x \mathrm{d}y[\chi\mathscr{E}]^{\mathrm{dr};\theta}(y,t)$$

$$w^\theta(x,t) := \frac{\partial\epsilon^\theta_{\mathrm{ini}}(u^\theta(x,t))}{\rho^{\mathrm{tot};\theta}(u^\theta(x,t),0)}$$

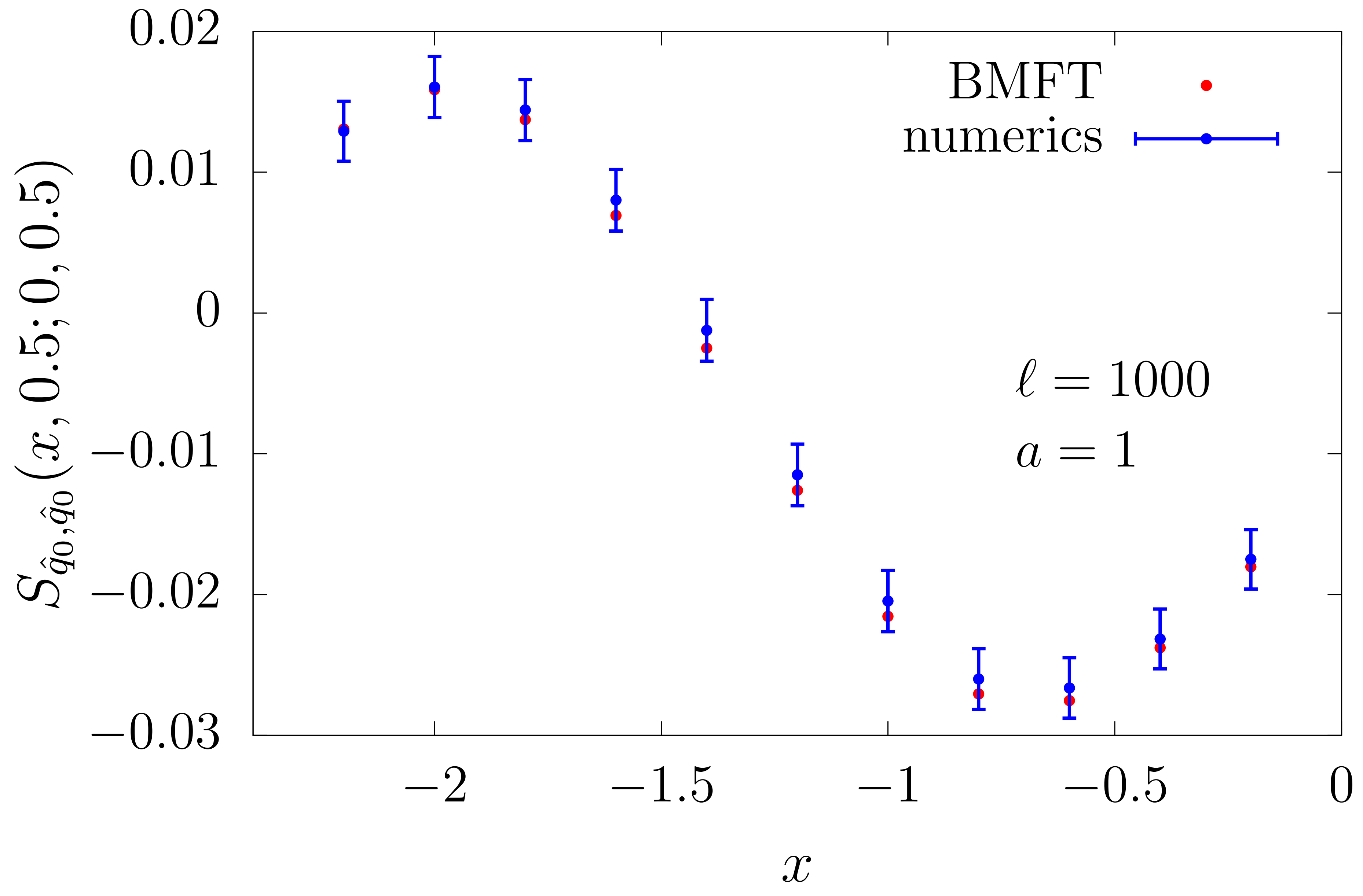
$$\mathscr{E}^\theta_0(x,t) = \mathscr{D}^\theta_1(x,t) + \mathscr{D}^\theta_2(x,t)$$

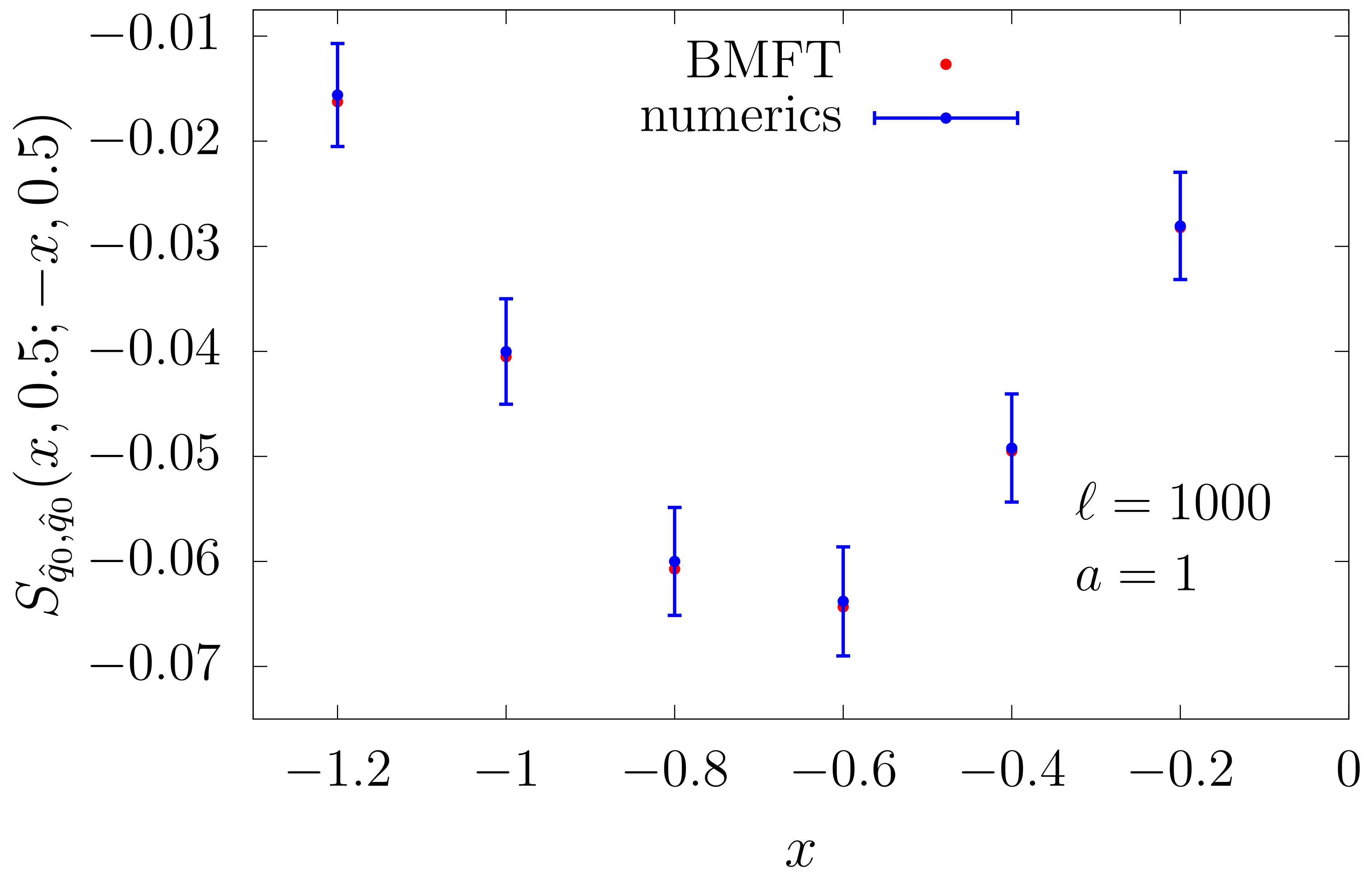
$$\mathscr{D}^\theta_1(x,t) := (R^{-\mathrm{T}})^\theta{}_\phi(u^\theta(x,t),0)\partial\left[(R^{\mathrm{T}})^\phi{}_\alpha h^{\mathrm{dr};\alpha}\right](x_1,t)\Theta(u^\theta(x,t)-u^\alpha(x_1,t)) - w^\theta(x,t)[\chi h^{\mathrm{dr}}]^{\mathrm{dr};\theta}(x_1,t)\Theta(x_2-x_1)$$

$$\mathscr{D}^\theta_2(x,t) := -w^\theta(x,t)\int_{-\infty}^{u^\theta(x,t)}\mathrm{d}y[\chi\mathscr{D}_3]^{\mathrm{dr};\theta}(y,0)$$

$$\mathscr{D}^\theta_3(x,0) := -h^{\mathrm{dr};\theta}(x_1,t)\frac{\delta(x-u^\theta(x_1,t))}{\partial\mathscr{U}^\theta(u^\theta(x_1,t),0;t)} + (R^{-\mathrm{T}})^\theta{}_\phi(x,0)\partial\left[(R^{\mathrm{T}})^\phi{}_\alpha h^{\mathrm{dr};\alpha}\right](x_1,t)\Theta(x-u^\alpha(x_1,t))$$

Comparison against hard-rod simulations (bump-release protocol)





- Despite of the small numbers, the agreements are very satisfying!

- By increasing ℓ, one can observe the convergence of the numbers

Conclusions and outlooks

- The BMFT is a new theory to study the fluctuation-induced physics, such as current fluctuations and large scale dynamical correlation functions, in ballistic many-body systems
- The underlying idea of the BMFT is local relaxation of fluctuations
- It works particularly well for integrable systems. The results also agree with hard-rods simulations very well.
- The BMFT should be applicable to other ballistic systems, e.g. the anharmonic chain. Will we indeed see the expected long-range correlations?
- It is highly desirable to derive our predictions microscopically using a simple model such as the AHR model.
- Obtaining the KPZ function from the BMFT+superdiffusive corrections?
- Quantum fluctuations?