Topological Expansion and Phase Diagram in Ensembles of Random Matrices with Complex Potentials

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This is an ongoing project with Alfredo Deaño, Maxim Yattselev, Ahmad Barhoumi, Ken McLaughlin, and Roozbeh Gharakhloo

Topological Expansion

We begin with a brief review of the *topological expansion* in unitary ensembles of random matrices.

Unitary Ensemble of Random Matrices

Let us consider a real polynomial $V(x) = \sum_{j=1}^{d} t_j x^j$, where d is even and $t_j \in \mathbb{R},\ j = 1, \ldots, d$, with $t_d > 0$. The corresponding *unitary ensemble of random matrices* is the probability distribution

$$d\mu_N(M) = \frac{1}{Z_N} e^{-N \operatorname{Tr} V(M)} dM,$$

on the space of Hermitian $N \times N$ random matrices $M \in \mathcal{H}_N$, where

$$Z_N = \int_{\mathcal{H}_N} e^{-N \operatorname{Tr} V(M)} dM$$

is the *partition function*.

Quartic Ensemble

As an example, let us consider the quartic polynomial

$$V(x) = \frac{x^2}{2} + ux^4\,, \quad u = t_4 > 0.$$

The partition function of the quartic ensemble is equal to

$$Z_N(u) = \int_{\mathcal{H}_N} e^{-N\left(\frac{\mathrm{Tr}\, M^2}{2} + u\mathrm{Tr}\, M^4\right)} dM.$$

In particular, $Z_N(0)$ is the partition function of the Gaussian unitary ensemble (GUE),

$$Z_N(0) = \int_{\mathcal{H}_N} e^{-\frac{N\mathrm{Tr}\, M^2}{2}} dM = Z_N^{\mathrm{GUE}} = \left(\frac{\pi}{N}\right)^{\frac{N^2}{2}} \left(\frac{1}{2}\right)^{\frac{N(N-1)}{2}}.$$

Normalized Partition Function

The normalized partition function is the quotient,

$$\frac{Z_N(u)}{Z_N(0)} = \frac{\int_{\mathcal{H}_N} e^{-\frac{N\mathrm{Tr}\, M^2}{2} - Nu\mathrm{Tr}\, M^4}\, dM}{\int_{\mathcal{H}_N} e^{-\frac{N\mathrm{Tr}\, M^2}{2}}\, dM}.$$

It is convenient to make the change of variable, $M' = M\sqrt{N}$. Then denoting M' back by M, we obtain that

$$\frac{Z_N(u)}{Z_N(0)} = \frac{\int_{\mathcal{H}_N} e^{-\frac{\mathrm{Tr}\, M^2}{2} - \frac{u\mathrm{Tr}\, M^4}{N}}\, dM}{\int_{\mathcal{H}_N} e^{-\frac{\mathrm{Tr}\, M^2}{2}}\, dM}.$$

Normalized Partition Function

Thus,

$$\frac{Z_N(u)}{Z_N(0)} = \frac{\int_{\mathcal{H}_N} e^{-\frac{\operatorname{Tr} M^2}{2} - \frac{u \operatorname{Tr} M^4}{N}}\, dM}{\int_{\mathcal{H}_N} e^{-\frac{\operatorname{Tr} M^2}{2}}\, dM} = \left\langle e^{-\frac{u \operatorname{Tr} M^4}{N}} \right\rangle_{\mathrm{GUE}},$$

where

$$\langle f(M) \rangle_{\mathrm{GUE}} = \frac{\int_{\mathcal{H}_N} f(M) e^{-\frac{\operatorname{Tr} M^2}{2}}\, dM}{\int_{\mathcal{H}_N} e^{-\frac{\operatorname{Tr} M^2}{2}}\, dM}.$$

Evaluation of $\frac{Z_N(u)}{Z_N(0)}$

Expanding exp into the Taylor series, we obtain that

$$\frac{Z_N(u)}{Z_N(0)} = \left\langle \sum_{p=0}^{\infty} \frac{1}{p!} \left(-\frac{u}{N}\right)^p \left(\operatorname{Tr} M^4\right)^p \right\rangle_{\mathrm{GUE}}.$$

Observe that

$$\langle M_{ij} M_{kl} \rangle_{\mathrm{GUE}} = \delta_{il}\delta_{jk}.$$

Let us evaluate

$$E_p = \left\langle \left(\operatorname{Tr} M^4\right)^p \right\rangle_{\mathrm{GUE}}, \quad p = 1, 2, \ldots$$

Evaluation of E_p

Let us start with

$$E_1 = \langle \operatorname{Tr} M^4 \rangle_{\mathrm{GUE}} = \left\langle \sum_{i,j,k,l=1}^{N} M_{ij} M_{jk} M_{kl} M_{li} \right\rangle_{\mathrm{GUE}} .$$

By the *Wick Theorem*,

$$\begin{aligned}
\langle M_{ij} M_{jk} M_{kl} M_{li} \rangle_{\mathrm{GUE}} &= \langle M_{ij} M_{jk} \rangle_{\mathrm{GUE}} \langle M_{kl} M_{li} \rangle_{\mathrm{GUE}} \\
&+ \langle M_{ij} M_{kl} \rangle_{\mathrm{GUE}} \langle M_{jk} M_{li} \rangle_{\mathrm{GUE}} \\
&+ \langle M_{ij} M_{li} \rangle_{\mathrm{GUE}} \langle M_{jk} M_{kl} \rangle_{\mathrm{GUE}} \\
&= \delta_{ik} + \delta_{il}\delta_{jk}\delta_{ij} + \delta_{kl},
\end{aligned}$$

hence

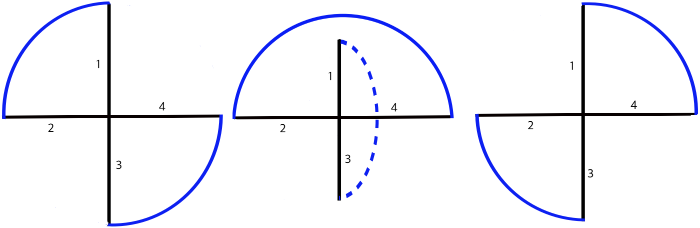
$$E_1 = \sum_{i,j,k,l=1}^{N} (\delta_{ik} + \delta_{il}\delta_{jk}\delta_{ij} + \delta_{kl}) = N^3 + N + N^3 = 2N^3 + N.$$

The three terms (pairings) in the Wick Theorem,

$$\begin{aligned}\langle M_{ij}M_{jk}M_{kl}M_{li}\rangle_{\mathrm{GUE}} &= \langle M_{ij}M_{jk}\rangle_{\mathrm{GUE}}\,\langle M_{kl}M_{li}\rangle_{\mathrm{GUE}} \\ &+ \langle M_{ij}M_{kl}\rangle_{\mathrm{GUE}}\,\langle M_{jk}M_{li}\rangle_{\mathrm{GUE}} \\ &+ \langle M_{ij}M_{li}\rangle_{\mathrm{GUE}}\,\langle M_{jk}M_{kl}\rangle_{\mathrm{GUE}}\,,\end{aligned}$$

can be represented by the three Feynman diagrams with one vertex. The first and third diagrams are planar, of the genus $g = 0$, and the second one is toroidal, of the genus $g = 1$.

Feynman's Diagrams



The Powers of N as the Number of Faces in the Feynman Diagram

We have

$$E_1 = \left\langle \mathrm{Tr}\, M^4 \right\rangle_{\mathrm{GUE}} = N^3 + N + N^3 = 2N^3 + N.$$

The N^3 terms correspond to the two Feynman diagrams (graphs) on the plane (or sphere), and the power 3 is the *number of faces* in these graphs. The N term corresponds to the Feynman diagram on the torus, and the number of faces in this graph on the torus is 1. Thus,

$$E_1 = \sum_{\pi} N^{f(\pi)}.$$

where $f(\pi)$ is the number of faces in the Feynman graph π realized without self-intersections on a Riemann surface of a minimal genus g.

Evaluation of E_2

Consider now

$$E_2 = \left\langle \left(\operatorname{Tr} M^4\right)^2 \right\rangle_{\mathrm{GUE}}$$
$$= \left\langle \sum_{i,j,k,l=1}^{N} \sum_{p,q,r,s=1}^{N} M_{ij} M_{jk} M_{kl} M_{li} M_{pq} M_{qr} M_{rs} M_{sp} \right\rangle_{\mathrm{GUE}} .$$

By the Wick Theorem,

$$\langle M_{ij} M_{jk} M_{kl} M_{li} M_{pq} M_{qr} M_{rs} M_{sp} \rangle_{\mathrm{GUE}}$$
$$= \sum_{\pi \in \Pi} \prod_{(\alpha\beta,\gamma\delta)\in\pi} \langle M_{\alpha\beta} M_{\gamma\delta} \rangle_{\mathrm{GUE}} ,$$

where $\Pi = \{\pi\}$ is the set of all partitions π of the set

$$V = \{(ij), (jk), \ldots, (sp)\}$$

into pairs.

Connected Part of E_2

The set $\Pi = \{\pi\}$ of all partitions π of the set

$$V = \{(ij), (jk), \dots, (sp)\}$$

into pairs is divided into two parts:

1. Π_0, of partitions π, with pairs in the set $\{(i,j),(j,k),(k,l),(l,i)\}$ and separately in the set $\{(p,q),(q,r),(r,s),(s,p)\}$, and
2. Π_c, of partitions π, such that at least one pair connects the sets $\{(i,j),(j,k),(k,l),(l,i)\}$ and $\{(p,q),(q,r),(r,s),(s,p)\}$.

Connected Part of E_2

Then

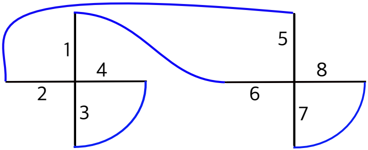
$$E_2 = (E_1)^2 + E_2^c,$$

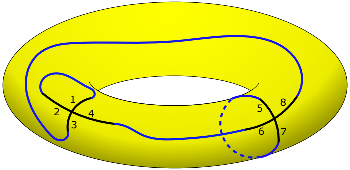
where

$$E_2^c = \sum_{\pi \in \Pi_c} \prod_{(\alpha\beta,\gamma\delta)\in\pi} \langle M_{\alpha\beta} M_{\gamma\delta} \rangle_{\mathrm{GUE}},$$

with the sum over connected regular Feynman diagrams of degree 4 with two vertices.

Examples of Regular Connected Feynman's Diagrams of Degree 4 with 2 Vertices





Connected Part of E_p

Thus,

$$E_2^c = \left\langle \left(\operatorname{Tr} M^4\right)^2 \right\rangle_{\mathrm{GUE}}^c = \sum_{\pi\in\Pi_c} \prod_{(\alpha\beta,\gamma\delta)\in\pi} \langle M_{\alpha\beta} M_{\gamma\delta}\rangle_{\mathrm{GUE}} = \sum_{\pi\in\Pi_c} N^{f(\pi)},$$

where $f(\pi)$ is the number of faces of the Feynman diagram π realized on the Riemann surface of a smallest possible genus. This can be extended to the connected parts of the subsequent moments,

$$E_p^c = \left\langle \left(\operatorname{Tr} M^4\right)^p \right\rangle_{\mathrm{GUE}}^c = \sum_{\pi\in\Pi_c^p} N^{f(\pi)},$$

where Π_c^p is the set of connected regular Feynman diagrams of degree 4 with p vertices.

The Second Wick Theorem

The (normalized) free energy is defined as

$$F_N(u) = \frac{1}{N^2} \ln \frac{Z_N(u)}{Z_N(0)}\,.$$

By the *second Wick theorem*,

$$F_N(u) = \frac{1}{N^2} \sum_{p=0}^{\infty} \frac{1}{p!} (-N^{-1}u)^p \left\langle \left(\operatorname{Tr} M^4\right)^p \right\rangle_{\mathrm{GUE}}^c\,.$$

Since

$$\left\langle \left(\operatorname{Tr} M^4\right)^p \right\rangle_{\mathrm{GUE}}^c = \sum_{\pi \in \Pi_c^p} N^{f(\pi)},$$

we obtain that

$$F_N(u) = \frac{1}{N^2} \sum_{p=0}^{\infty} \frac{1}{p!} (-u)^p \sum_{\pi \in \Pi_c^p} N^{f(\pi)-p}.$$

The Euler Formula

In the formula

$$F_N(u) = \frac{1}{N^2} \sum_{p=0}^{\infty} \frac{(-u)^p}{p!} \sum_{\pi \in \Pi_c^p} N^{f(\pi)-p}.$$

the Feynman diagram $\pi \in \Pi_c^p$ is a regular connected graph of degree 4 with p vertices. The number of edges in π is equal to $l = \frac{4p}{2} = 2p$. By the Euler characteristic formula,

$$v - l + f = 2 - 2g \implies p - 2p + f = 2 - 2g \implies f - p = 2 - 2g.$$

hence

$$F_N(u) = \sum_{p=0}^{\infty} \frac{(-u)^p}{p!} \sum_{\pi \in \Pi_c^p} N^{-2g}.$$

$\frac{1}{N^2}$-Expansion of the Free Energy

Interchanging the order of summation, we obtain that

$$\begin{aligned} F_N(u) &= \sum_{p=0}^{\infty} \frac{(-u)^p}{p!} \sum_{\pi \in \Pi_c^p} N^{-2g} \\ &\sim \sum_{g=0}^{\infty} \frac{1}{N^{2g}} \sum_{p=0}^{\infty} \frac{(-u)^p}{p!} \mathcal{A}_g(p), \end{aligned}$$

where $\mathcal{A}_g(p)$ is the number of connected regular Feynman diagrams of degree 4 with p vertices on a closed oriented Riemannian surface of genus g.

The Topological Expansion of the Free Energy

The $\frac{1}{N^2}$-expansion of the free energy,

$$F_N(u) \sim \sum_{g=0}^{\infty} \frac{1}{N^{2g}} \mathcal{F}_g(u),$$

is called the *topological expansion*, and its coefficients $\mathcal{F}_g(u)$ are the generating functions for the numbers $\mathcal{A}_g(p)$ of connected regular Feynman diagrams of degree 4 with p vertices on a closed oriented Riemannian surface of genus g,

$$\mathcal{F}_g(u) = \sum_{p=0}^{\infty} \frac{(-u)^p}{p!} \mathcal{A}_g(p).$$

Works on Topological Expansion in Random Matrix Models

Physics

t'Hooft, 1974; Brézin, Itzykson, Parisi, Zuber, 1978; Bessis, Itzykson, Zuber, 1980; Boulatov, Kazakov, Kostov, Migdal, 1986; Brézin, Kazakov, 1990; Douglas, Shenker, 1990; Gross, Migdal, 1990; David, 1990; Witten, 1990; Di Francesco, Ginsparg, J. Zinn-Justin, 1993; Di Francesco, 2004; Eynard, 2005, and others.

Mathematics

Mulase, 1997; Zvonkin, 1997; Ercolani, McLaughlin, 2003; Bleher, Its, 2005; Ercolani, McLaughlin, Pierce, 2008; Borot, Guionnet, 2011; Bleher, Deaño, 2013; Bleher, Gharakhloo, McLaughlin 2021, and others.

The Riemann–Hilbert Approach to the Topological Expansion

The Riemann–Hilbert approach to an evaluation of the topological expansion is based on the eigenvalue representation of the free energy,

$$F_N(u) = \frac{1}{N^2} \ln \frac{\mathcal{Z}_N(u)}{\mathcal{Z}_N(0)} ,$$

where

$$\mathcal{Z}_N(u) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \prod_{1\le j<k\le n} (x_j - x_k)^2 \times \prod_{j=1}^{n} \exp\left[-N\left(\frac{x_j^2}{2} + \frac{ux_j^4}{4}\right)\right] dx_1 \cdots dx_N$$

is the *partition function of the ensemble of eigenvalues*. (We replace u by $\frac{u}{4}$ to simplify some formulae.)

Reduction to Orthogonal Polynomials

The partition function of the ensemble of eigenvalues $\mathcal{Z}_N(u)$ can be expressed in terms of associated orthogonal polynomials.

Associated Orthogonal Polynomials

Consider monic orthogonal polynomials $P_n(x) = x^n + \dots$ such that

$$\int_{-\infty}^{\infty} P_m(x) P_n(x) \exp\left[-N\left(\frac{x^2}{2} + \frac{ux^4}{4}\right)\right] dx = h_n \delta_{mn} \,.$$

The polynomials $P_n(x) = P_n(x; u, N)$ satisfy the *three term recurrence relation*

$$xP_n(x) = P_{n+1}(x) + R_n P_{n-1}(x), \quad R_n = \frac{h_n}{h_{n-1}}, \quad n = 1, 2, \dots,$$

and the recurrence coefficients $R_n = R_n(u, N)$ satisfy the *string equation*,

$$R_n(1 + uR_{n-1} + uR_n + uR_{n+1}) = \frac{n}{N} \,.$$

$\mathcal{Z}_N$ in Terms of Associated Orthogonal Polynomials

The partition function of the ensemble of eigenvalues $\mathcal{Z}_N$ is expressed in terms of the normalizing constants h_n of the associated orthogonal polynomials as

$$\mathcal{Z}_N = N! \prod_{n=0}^{N-1} h_n.$$

This formula is not convenient for the topological $\frac{1}{N^2}$ expansion because it contains h_n with small n. We use a different approach based on the recurrence coefficients and deformation equations for the free energy.

Topological Expansion of Recurrence Coefficients

By using the *Riemann–Hilbert approach* (nonlinear steepest descent method) to orthogonal polynomials, we obtain a uniform topological expansion of the recurrence coefficient $R_n = R_n(u, N)$, such that for some constants $0 < C_1 < 1 < C_2 < \infty$,

$$R_n(u) \sim \sum_{g=0}^{\infty} \frac{r_g(\eta, u)}{N^{2g}}, \quad \eta = \frac{n}{N}, \quad C_1 \leq \eta \leq C_2,$$

where the coefficients $r_g(\eta, u)$ are analytic functions of η and u at the point $\eta = 1$, $u = 0$.

Leading Order of the Recurrence Coefficients

Substituting the topological expansion of the recurrence coefficients into the string equation, we obtain recursively the coefficients $r_g = r_g(\eta, u)$ of the topological expansion of the recurrence coefficients. In particular, the zeroth order equation is

$$r_0(1 + 3ur_0) = \eta,$$

whose solution is

$$r_0 = \frac{-1 + \sqrt{1 + 12\eta u}}{6u}.$$

Higher Orders of the Recurrence Coefficients

Higher order coefficients r_g, $g = 1, 2, \ldots$, are calculated recursively by the formula

$$r_g = -\frac{1}{1+3ur_0}\sum_{\ell=1}^{g} r_{g-\ell}\left[3ur_\ell + 2u\sum_{k=0}^{\ell-1}\frac{1}{(2\ell-2k)!}\frac{\partial^{2\ell-2k} r_k}{\partial\eta^{2\ell-2k}}\right]$$

In particular,

$$r_1 = \frac{u\left(-1+\sqrt{1+12\eta u}\right)}{(1+12\eta u)^2},$$

$$r_2 = \frac{63u^3\left(-3-8\eta u+3\sqrt{1+12\eta u}\right)}{(1+12\eta u)^{9/2}},$$

and so on.

The Deformation Equation for the Free Energy

To obtain the topological expansion of the free energy,

$$F_N(u) = \frac{1}{N^2} \ln \frac{\mathcal{Z}_N(u)}{\mathcal{Z}_N(0)}\,,$$

we derive and use the deformation equation

$$F_N'(u) = R_N(u)\left[\frac{1}{u} + R_{N-1}(u)R_{N+1}(u)\right] - \frac{1}{4u}\,.$$

This gives a topological expansion first for $F_N'(u)$ and then for $F_N(u)$.

Theorem 1. *We have the topological expansion of the free energy,*

$$F_N(u) \sim \sum_{g=0}^{\infty} \frac{1}{N^{2g}} \mathcal{F}_g(u),$$

where

$$\mathcal{F}_0(u) = \sum_{j=1}^{\infty} \frac{(-1)^j 12^j (2j-1)!}{(j)!(j+2)!} u^j\,,$$

$$\mathcal{F}_1(u) = \frac{1}{24} \sum_{j=1}^{\infty} \frac{(-1)^j 12^j}{j} \left[1 - \frac{(2j)!}{4^j (j!)^2}\right] u^j,$$

$$\mathcal{F}_2(u) = \frac{1}{2304} \sum_{j=3}^{\infty} \frac{(-1)^j 12^j}{j} \left[\frac{8(2j)!(28j+9)}{15 \cdot 4^j (j-2)! j!} - 13j(j-1)\right] u^j,$$

and so on.

The Number of Regular Feynman Graphs of Degree 4 on the Riemannian Surface

Comparing formulae of Theorem 1 to the Feynman diagram expansion,

$$F_N(u) \sim \sum_{g=0}^{\infty} \frac{1}{N^{2g}} \mathcal{F}_g(u),$$

where

$$\mathcal{F}_g(u) = \sum_{p=0}^{\infty} \frac{(-u)^p}{4^p p!} \mathcal{A}_g(p),$$

and $\mathcal{A}_g(p)$ is the number of 4 valent Feynman diagrams with p vertices on a closed oriented Riemannian surface of genus g, we obtain formulae for $\mathcal{A}_g(p)$.

Large p Asymptotics of the Coefficients of the Topological Expansion

For g larger than 2 the formulae for the coefficients of the topological expansion of the free energy $\mathcal{F}_g(u)$ become rather complicated. We can nevertheless find the asymptotic behavior of their Taylor coefficients,

$$\mathcal{F}_g(u) = \sum_{p=0}^{\infty} \frac{(-u)^p}{4^p p!} \mathcal{A}_g(p),$$

as $p \to \infty$. Namely, we have

Theorem 2. *As* $p \to \infty$,

$$\mathcal{A}_g(p) = \frac{\mathcal{C}_g p^{5g-7}}{u_c^p} \left(1 + \mathcal{O}\left(\frac{1}{p}\right)\right),$$

with $u_c = -\frac{1}{12}$ *and some constants* $\mathcal{C}_g > 0$.

Painlevé I Equation

The constants $\mathcal{C}_g > 0$ in Theorem 2 are related to the *Boutroux tritronquée solution of the Painlevé I equation*. Namely, consider the Painlevé I equation,

$$u''(x) = 6u^2(x) + x.$$

Let $u(x)$ be the Boutroux tritronquée solution of this equation and $y(x)$ a scaling of $u(x)$ such that

$$y(x) = -2^{\frac{8}{5}} 3^{\frac{2}{5}} u\left(-2^{\frac{9}{5}} 3^{\frac{6}{5}} x\right).$$

Then as $x \to \infty$ the function $y(x)$ admits the asymptotic expansion

$$y(x) = \sum_{g=0}^{\infty} \mathcal{Y}_g x^{\frac{1-5g}{2}}.$$

Theorem 2'. *The constants* $\mathcal{C}_g > 0$ *in Theorem 2 coincide with the coefficients* $\mathcal{Y}_g > 0$ *of the asymptotic expansion of the rescaled Boutroux tritronquée solution* $y(t)$ *at infinity, so that*

$$\mathcal{C}_g = \mathcal{Y}_g.$$

Analytic Continuation of the Quartic Partition Function to the Complex u-Plane

To better understand the asymptotics of the coefficients of the topological expansion and the appearance of the Painlevé I equation, we consider an analytic continuation of the partition function

$$\mathcal{Z}_N(u) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \prod_{1 \le j < k \le N} (z_j - z_k)^2 \prod_{j=1}^{N} e^{-N\left(\frac{z_j^2}{2} + \frac{u z_j^4}{4}\right)} dz_1 \cdots dz_N$$

to the *complex u-plane*. The integral is well-defined for $\Re u > 0$, but it diverges for $\Re u < 0$. To define a regularization of the integral, we can either rotate the real axis of integration or make a change of variables. We will use a change of variables.

Change of Variables in $\mathcal{Z}_N(u)$

We define

$$\zeta = u^{1/4} z, \quad \sigma = u^{-1/2},$$

and

$$V(\zeta;\sigma) = \frac{\sigma\zeta^2}{2} + \frac{\zeta^4}{4}.$$

Then the corresponding partition function of eigenvalues is given by

$$Z_N(\sigma) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \prod_{1\le j<k\le N} (\zeta_j - \zeta_k)^2 \prod_{j=1}^{N} e^{-N\left(\frac{\sigma\zeta_j^2}{2} + \frac{\zeta_j^4}{4}\right)} d\zeta_1 \cdots d\zeta_n,$$

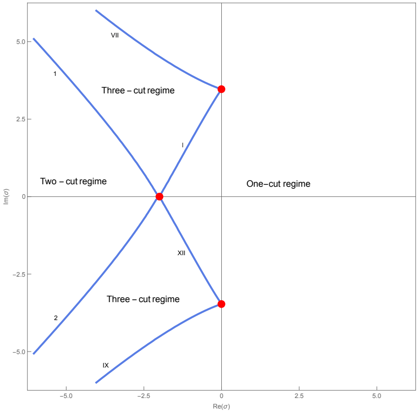
which converges for all $\sigma \in \mathbb{C}$. Note that

$$\mathcal{Z}_N(u) = \sigma^{\frac{N^2}{2}} Z_N(\sigma).$$

Asymptotics of $Z_N(\sigma)$

In what follows I present results of my joint work with Ken McLaughlin and Roozbeh Gharakhloo.

The Phase Diagram of the Quartic Model on the Complex σ-Plane



The critical points: $\sigma = -2$ (PII), $\sigma = \pm\sqrt{12}\,i$ (PI) (F. David).

The Equilibrium Measure

The phase regions are described in terms of max-min equilibrium measures on the complex plane. We use here the works of Kuijlaars and Silva, 2015. A contour Γ on the complex plane is called *admissible* if it goes from $(-\infty)$ to $(+\infty)$, and it is a finite union of analytic arcs. Let $\mathcal{P}(\Gamma)$ the space of probability measures ν on Γ. Consider the following real-valued functional on $\mathcal{P}(\Gamma)$:

$$I_\Gamma(\nu) := \iint_{\Gamma\times\Gamma} \log\frac{1}{|z-s|}\, d\nu(z)d\nu(s) + \int_\Gamma \Re V(s)\, d\nu(s).$$

By results of the potential theory, there exists a unique *minimizer* ν_Γ of the functional $I_\Gamma(\nu)$, so that

$$\min_{\nu\in\mathcal{P}(\Gamma)} I_\Gamma(\nu) = I_\Gamma(\nu_\Gamma).$$

The Euler–Lagrange Variational Conditions

The support of the minimizer ν_Γ is a compact set $J_\Gamma \subset \Gamma$. An important fact is that the equilibrium measure is uniquely determined by the *Euler–Lagrange variational conditions*. Namely, ν_Γ is the unique probability measure ν on Γ such that there exists a constant l, a Lagrange multiplier, such that

$$U^\nu(z) + \frac{1}{2}\Re V(z) = l, \quad z \in \operatorname{supp}\nu,$$
$$U^\nu(z) + \frac{1}{2}\Re V(z) \geq l, \quad z \in \Gamma \setminus \operatorname{supp}\nu,$$

where

$$U^\nu(z) = \int_\Gamma \log\frac{1}{|z-s|}\,d\nu(s)$$

is the *logarithmic potential* of the measure ν.

The Max-Min Equilibrium Measures

Now we *maximize* $I(\nu_\Gamma)$ over the set of admissible contours, $\Gamma \in \mathcal{T}$. Kuijlaars and Silva prove that

1. The maximizing contour $\Gamma_0 \in \mathcal{T}$ *exists*.
2. The equilibrium measure $\nu_0 = \nu_{\Gamma_0}$ is supported by a set $J \subset \Gamma_0$ which is a finite union of *analytic arcs (cuts)* $\Gamma_0[a_k, b_k]$,

$$J = \bigcup_{k=1}^{q} \Gamma_0[a_k, b_k], \quad a_1 < b_1 \le a_2 < b_2 \le \ldots \le a_q < b_q,$$

   that are *critical trajectories* of a quadratic differential $(-R(z))\,dz^2$, where $R(z)$ is a polynomial.
3. The set $J = \bigcup_{k=1}^{q} \Gamma_0[a_k, b_k]$ is *unique*.

Support of the Equilibrium Measure

The fact that the arcs $\Gamma_0[a_k, b_k],\ k = 1, \ldots, q,$ are critical trajectories of the quadratic differential $(-R(z))\, dz^2$ means that

1. $$R(a_1) = R(b_1) = \ldots = R(a_q) = R(b_q) = 0,$$

   and

2. $$-R(z)\, dz^2 > 0, \quad \forall\, z \in \bigcup_{k=1}^{q} (a_k, b_k).$$

A Formula for the Polynomial $R(z)$

Furthermore, Kuijlaars and Silva prove that the polynomial $R(z)$ is equal to

$$R(z) = \left(-\omega(z) + \frac{V'(z)}{2}\right)^2,$$

where

$$\omega(z) = \int_J \frac{d\nu_0(s)}{z-s} = \frac{1}{z} + \frac{m_1}{z^2} + \dots, \quad m_k = \int_J s^k d\nu_0(s),$$

is the *resolvent* of the measure ν_0.

The Density of the Equilibrium Measure

In addition, the equilibrium measure ν_0 is *absolutely continuous* with respect to the arc length, and

$$d\nu_0(s) = \frac{1}{\pi i} R_+(s)^{1/2} ds,$$

where $R_+(s)^{1/2}$ is the limiting value of the function

$$R(z)^{1/2} = -\int_J \frac{d\nu_0(s)}{z-s} + \frac{V'(z)}{2},$$

as $z \to s \in J$ from the left-hand side of J with respect to the orientation of the contour Γ_0 from $(-\infty)$ to ∞.

Regular and Singular Equilibrium Measures

An equilibrium measure ν_0 is called *regular* if the following three conditions hold:

1. The arcs $\Gamma_0[a_k, b_k],\ k = 1, \ldots, q,$ of the support of ν_0 (the cuts) are disjoint.
2. The end-points $\{a_k, b_k,\ k = 1, \ldots, q\}$ are simple zeros of the polynomial $R(s)$.
3. There is a contour Γ_0 containing the support J of ν_0 such that

$$U^{\nu}(z) + \frac{1}{2}\Re V(z) > l, \quad \forall\, z \in \Gamma_0 \setminus \operatorname{supp} \nu_0,$$

An equilibrium measure ν_0 is called *singular* (or *critical*) if it is not regular.

Equilibrium Measures of the Quartic Polynomial $V(z;\sigma)$

For the quartic polynomial in hand, $V(\zeta;\sigma) = \frac{\sigma\zeta^2}{2} + \frac{\zeta^4}{4}$, we have that

$$R(z) = \left(-\omega(z) + \frac{z^3+\sigma z}{2}\right)^2, \quad \omega(z) = \int_J \frac{d\nu_0(s)}{z-s}.$$

Since the polynomial $V(z)$ is even, the uniqueness of the equilibrium measure ν_0 implies that

1. ν_0 is even as well, $\nu_0(-s) = \nu_0(s)$,
2. The resolvent $\omega(z)$ is odd, $\omega(-z) = -\omega(z)$, and
3. The polynomial $R(z)$ is even, $R(-z) = R(z)$.

One-Cut Equilibrium Measure

When $q = 1$ (a one-cut equilibrium measure), we have that

$$R(z) = \frac{1}{4}(z^2 - z_0^2)^2(z^2 - b_1^2),$$

where $\pm b_1$ are the end-points of the equilibrium measure and $\pm z_0$ are double zeroes. Equating this expression to

$$R(z) = \left(-\omega(z) + \frac{V'(z)}{2}\right)^2,$$

we obtain that

$$(z^2 - z_0^2)^2(z^2 - b_1^2) = z^6 + 2\sigma z^4 + (\sigma^2 - 4)z^2 - 4(\sigma + m_2).$$

Comparing the coefficients at z^4 and z^2, we obtain the system of equations,

$$\begin{cases} b_1^2 + 2z_0^2 = -2\sigma, \\ 2b_1^2 z_0^2 + z_0^4 = \sigma^2 - 4. \end{cases}$$

Zeros of $R(z)$

Solving the above system of equations we obtain that

$$b_1^2 = \frac{2}{3}\left(-\sigma + \sqrt{12+\sigma^2}\right),$$
$$z_0^2 = \frac{1}{3}\left(-2\sigma - \sqrt{12+\sigma^2}\right).$$

An Example of the Equilibrium Measure with One-Cut Support

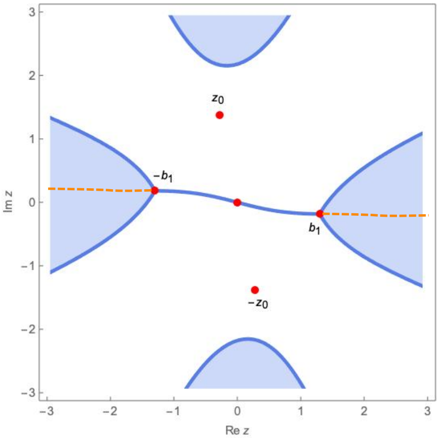


Figure: $\sigma = 1 + i$

Two-Cut Equilibrium Measure

When $q = 2$ (a two-cut equilibrium measure), we have that

$$R(z) = \frac{1}{4} z^2 (z^2 - a_1^2)(z^2 - b_1^2),$$

where $\pm a_1,\ \pm b_1$ are the end-points of the equilibrium measure. Equating this expression to

$$R(z) = \left(-\omega(z) + \frac{V'(z)}{2} \right)^2,$$

we obtain the system of equations

$$\begin{aligned} a_2^2 + b_2^2 + 2\sigma &= 0, \\ a_2^4 - 2a_2^2 b_2^2 + b_2^4 &= 16, \end{aligned}$$

which yields

$$a_2^2 = -2 - \sigma, \quad b_2^2 = 2 - \sigma.$$

An Example of the Equilibrium Measure with Two-Cut Support

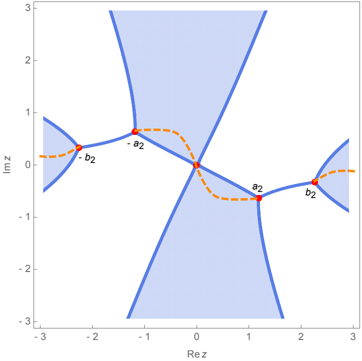


Figure: $\sigma = -3 + i$

The three-cut regime

In the three-cut regime the support of the equilibrium measure consists of three cuts,

$$J = [-c_3, -b_3] \cup [-a_3, a_3] \cup [b_3, c_3].$$

The algebraic end-point equations are

$$\begin{aligned} &a_3^2 + b_3^2 + c_3^2 + 2\sigma = 0, \\ &a_3^4 + b_3^4 + c_3^4 - 2a_3^2 b_3^2 - 2a_3^2 c_3^2 - 2b_3^2 c_3^2 = 16. \end{aligned}$$

In addition, we have the two real equations,

$$\Re\left(\int_{a_3}^{b_3} \sqrt{R(s)}ds\right) = 0, \quad \Re\left(\int_{c_3}^{b_3} \sqrt{R(s)}ds\right) = 0.$$

An Example of the Equilibrium Measure with Three-Cut Support

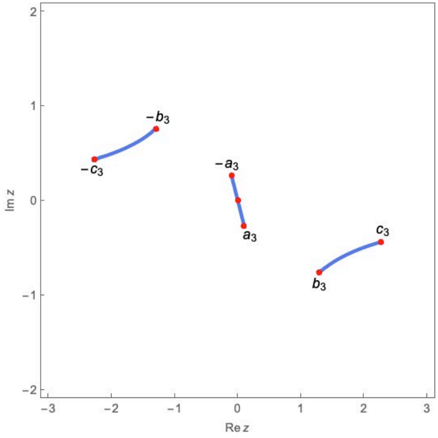


Figure: $\sigma = -3 + 2i$

Analytical Dependence of the Equilibrium Measure on σ

Theorem 3.

1. *In the one-cut and two-cut regions the equilibrium measure $\nu_{\mathrm{eq}}(\sigma)$ depends analytically on the parameter σ.*
2. *In the three-cut region the equilibrium measure $\nu_{\mathrm{eq}}(\sigma)$ depends analytically on $\Re\sigma$ and $\Im\sigma$, but not on σ (so that the Cauchy–Riemann equations fail).*

Critical Points

Theorem 4. (Bleher–Eynard) *At the critical point* $\sigma = -2$ *the free energy exhibits a third order phase transition on the real line.*
Theorem 5. (Bleher–Its) *At the critical point* $\sigma = -2$ *the double scaling limit is PII.*
Theorem 6. (Duits–Kuijlaars) *At the critical point* $\sigma = 12i$ *the double scaling limit is PI.*

An Equation of the Critical Curve on the Complex β-Plane

Theorem 7. *Let*

$$\sigma = -\frac{3}{4}\beta + \frac{4}{\beta},$$

Then the set of the critical parameters σ *is mapped onto critical trajectories of the quadratic differential*

$$S(\beta)d\beta^2 = \frac{(16-\beta^2)(16+3\beta^2)^3}{1024\beta^6}\, d\beta^2.$$

Theorem 4 (Bleher, Gharakhloo, and McLaughlin). *We have that*

1. *The critical curves of the complex quartic model are determined by the quadratic differentials and they do separate one-, two-, and three cut phases.*
2. *The associated orthogonal polynomials admit* $\frac{1}{N^2}$*-expansion in the one-cut and two-cut phase regions.*
3. *The free energy admits a topological* $\frac{1}{N^2}$*-expansion in the one-cut region.*

Topological Expansion for the Cubic Ensemble

The cubic ensemble with

$$V(x) = \frac{x^2}{2} + ux^3, \quad u > 0.$$

is very interesting because in this case the topological expansion gives generating functions for the number of regular Feynman graphs of degree 3 on Riemannian surfaces, or equivalently the number of triangulations of the surface. The phase diagram of the cubic ensemble was described in the physical work of David, and it was rigorously investigated in the works of Barhoumi, Bleher, Deaño, and Yattselev.

The End

Thank you!