A new approach to solvable KPZ models via a correspondence with free fermions at positive temperature

Randomness, Integrability and Universality

Galileo Galilei Institute for Theoretical Physics, Firenze
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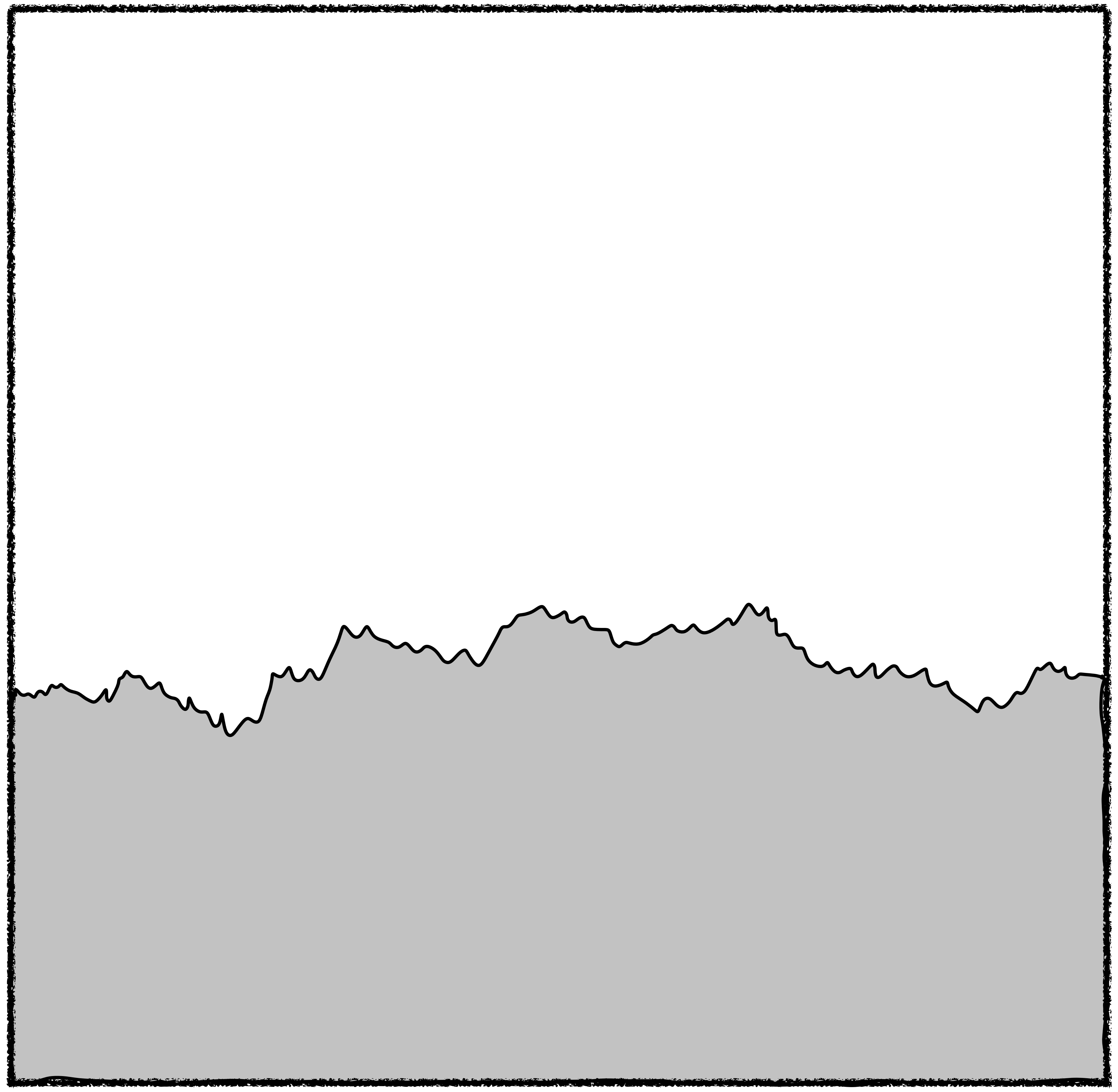
Matteo Mucciconi — based on a collaboration with Takashi Imamura and Tomohiro Sasamoto

arXiv:2106.11922[math.CO]
arXiv:2106.11913[math.CO]
arXiv:2204.08420[math.PR]

KPZ equation

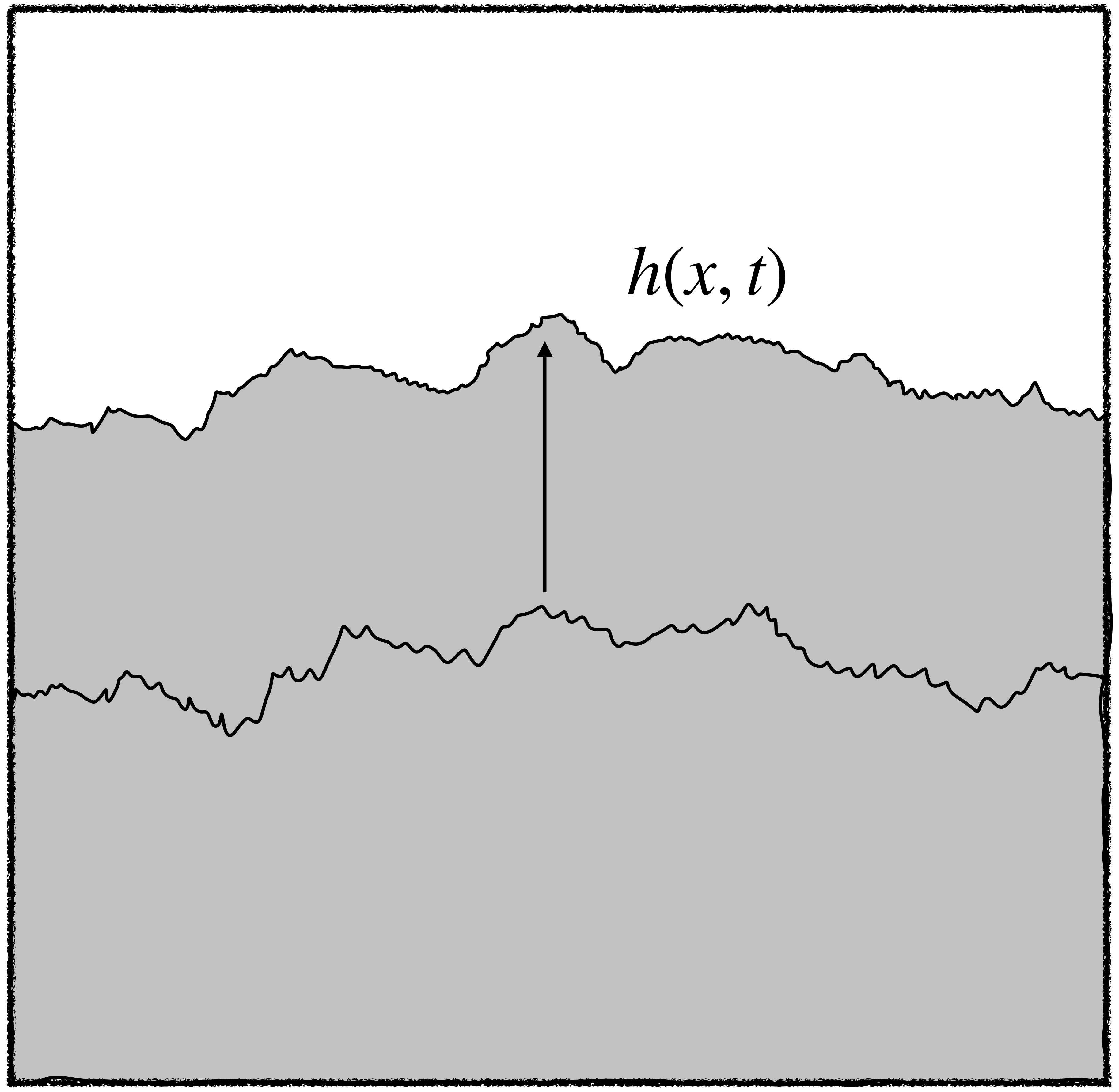
[Kardar-Parisi-Zhang '86]



$$\partial_t h = \frac{1}{2}\partial_x^2 h + \frac{1}{2}(\partial_x h)^2 + \eta$$

- h = random height function
  η = space-time white noise

KPZ equation [Kardar-Parisi-Zhang '86]



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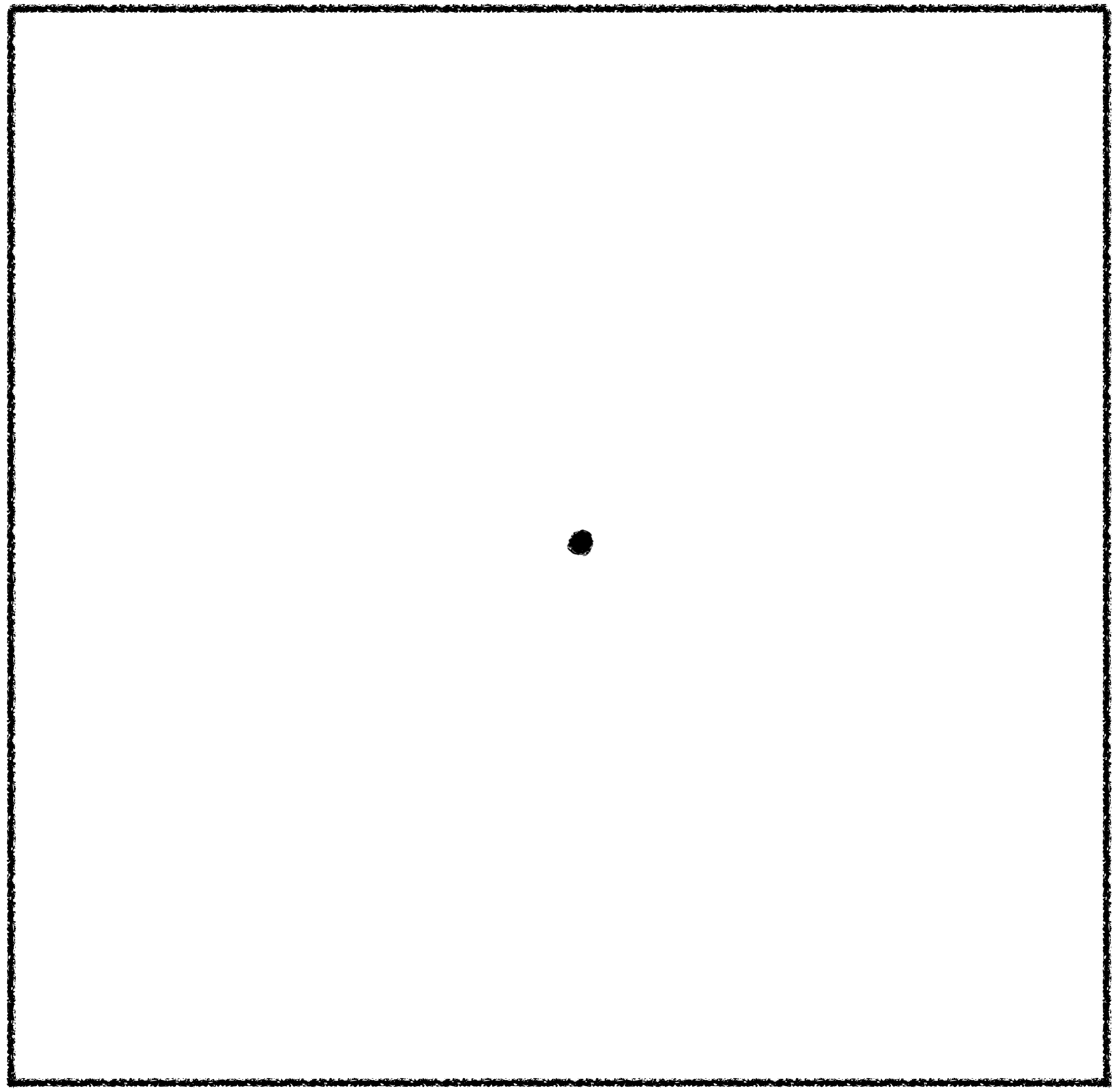
  $\eta =$ space-time white noise
- Well posedness: [Bertini-Giacomin '97], [Hairer '11], [Gubinelli-Perkowski-Imkeller '12]

KPZ equation : exact solutions



- Narrow wedge initial conditions

$$\begin{cases} \partial_t h = \frac{1}{2}\partial_x^2 h + \frac{1}{2}(\partial_x h)^2 + \eta \\ h(x,0) = \log(\delta_x) \end{cases}$$

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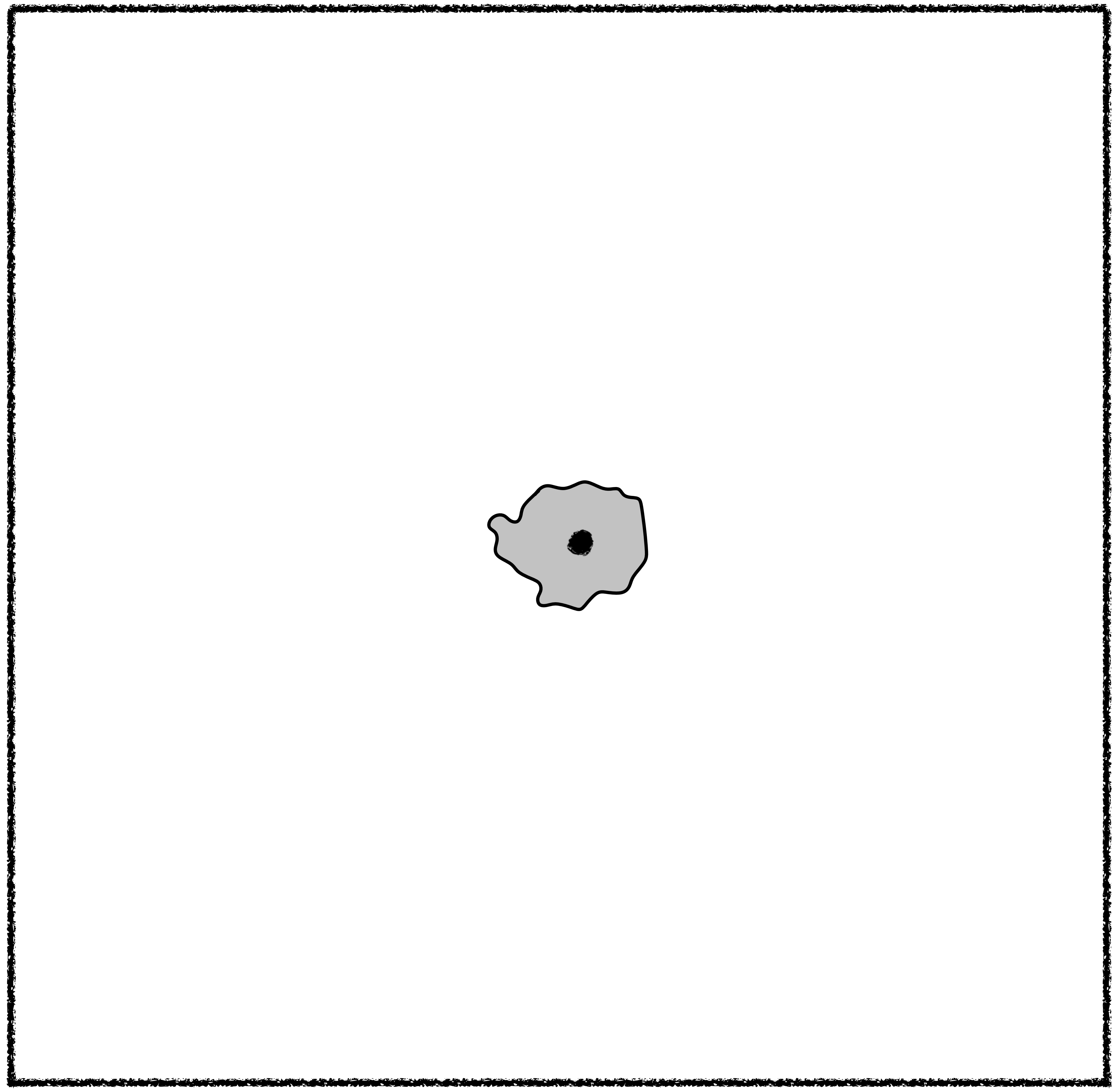

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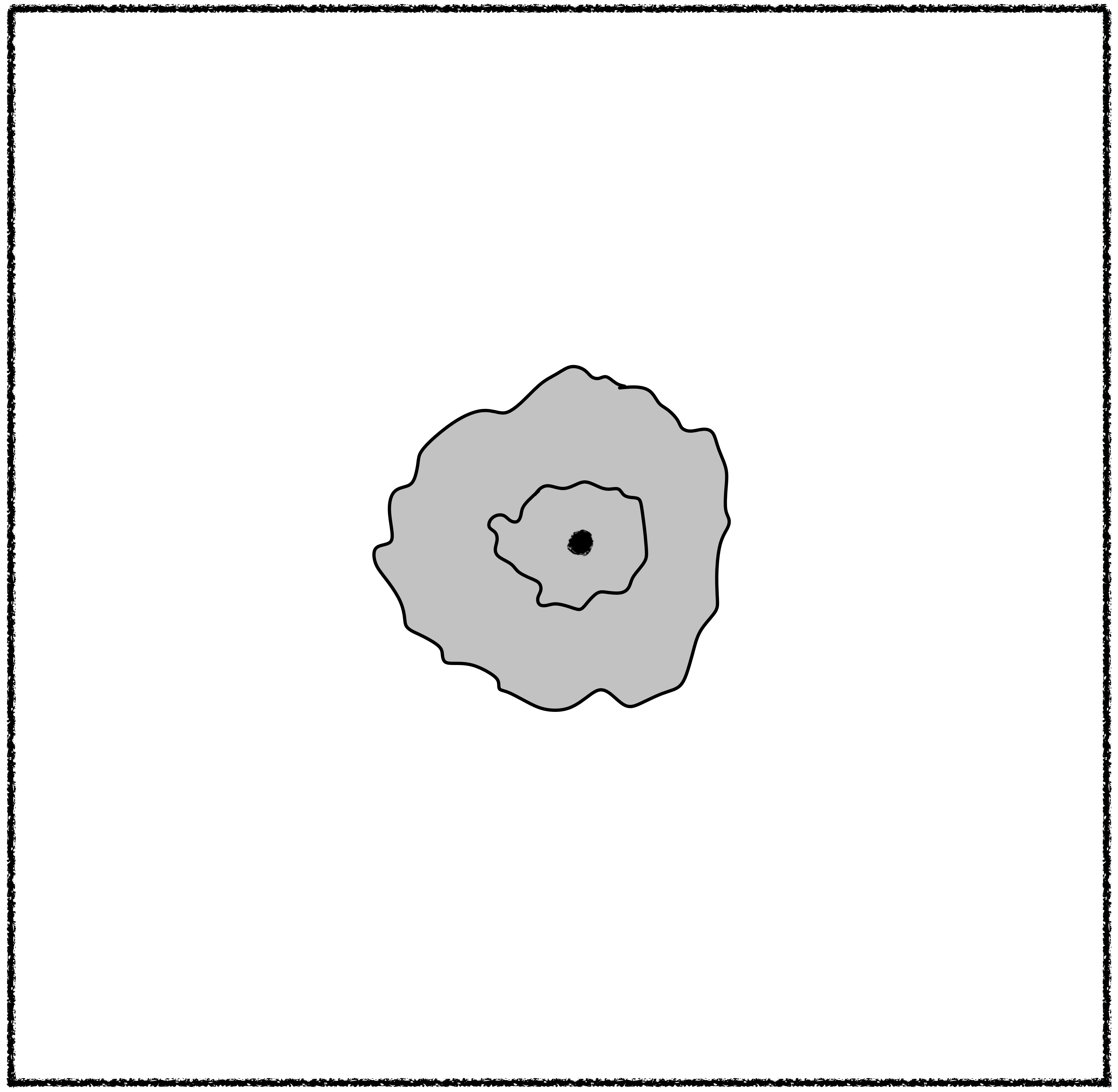

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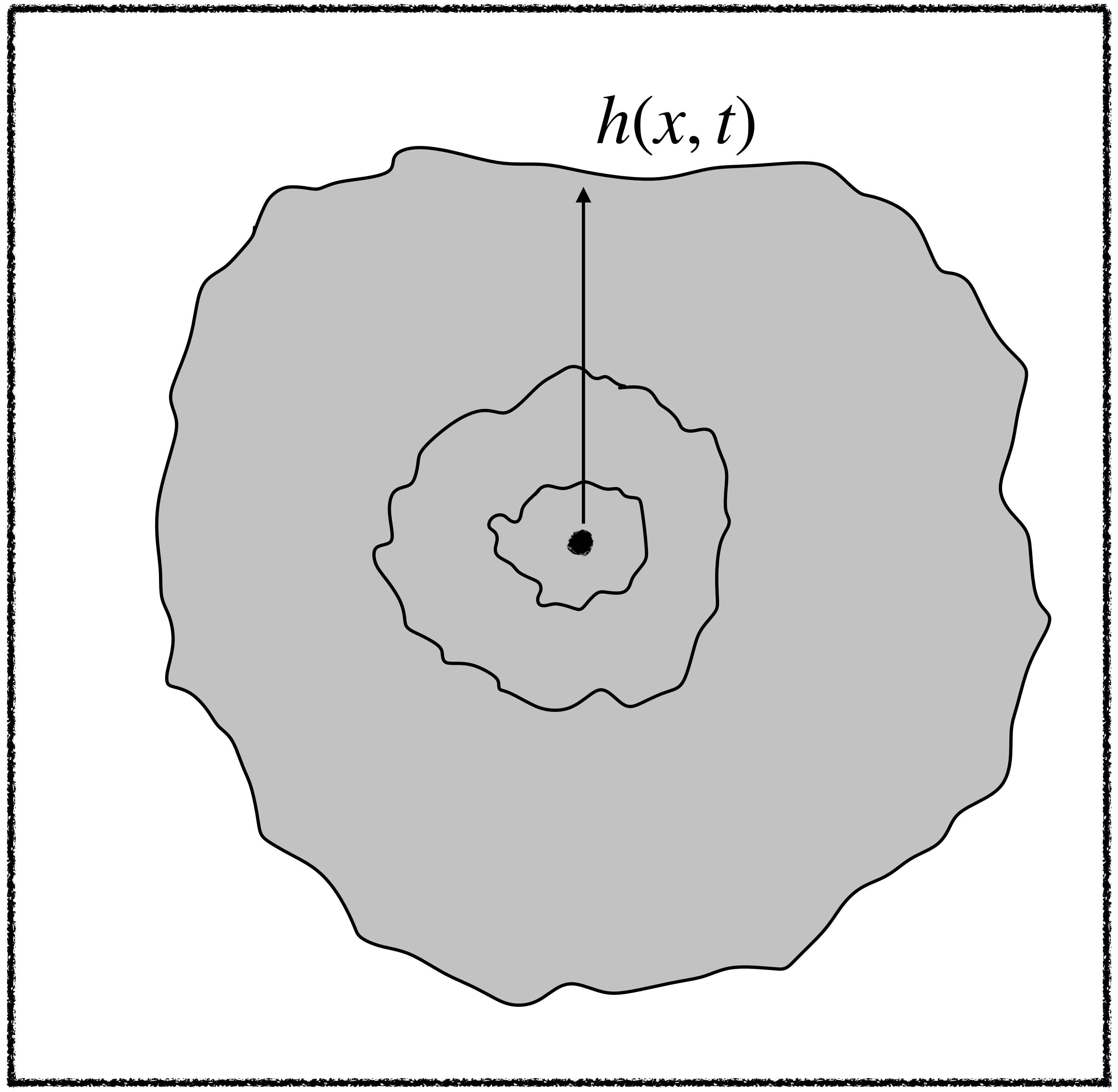

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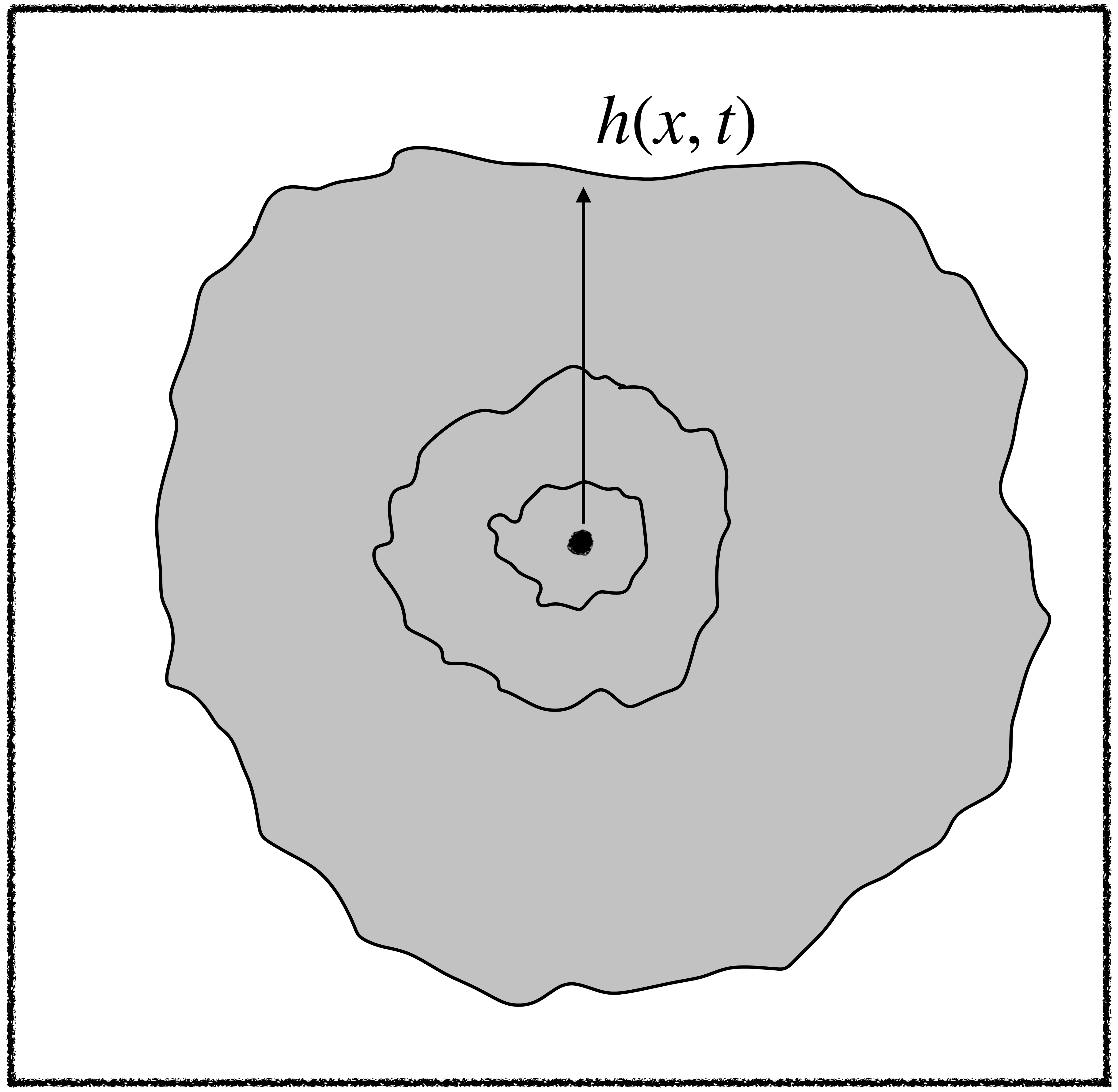

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- [Amir-Corwin-Quastel, Calabrese-Le Doussal, Dotsenko, Sasamoto-Spohn ’11]

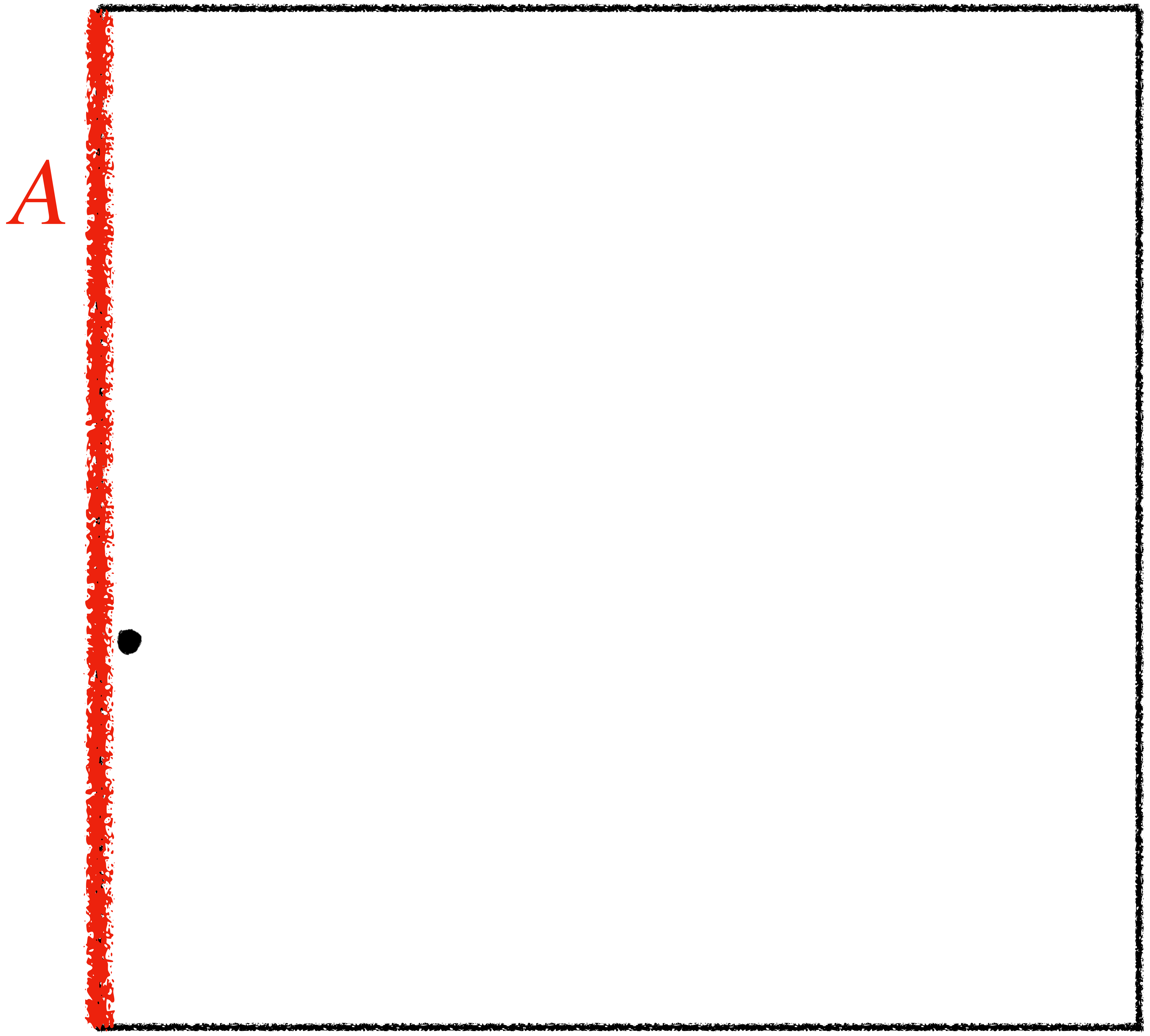
$$\mathbb{E}\left[\exp\left(-ze^{h(0,t)+t/24}\right)\right] = \det\left(1 - f K_{\mathrm{Airy}}\right)_{\mathscr{L}^2(\mathbb{R})}$$

$$K_{\mathrm{Airy}}(x,y) = \int_0^\infty \mathrm{Ai}(x+z)\mathrm{Ai}(y+z)dz \qquad f(x) = \frac{1}{1+e^{-xt^{1/3}}/z}$$

Airy Kernel

Fermi factor

KPZ equation : exact solutions



- Narrow wedge initial conditions in half space

$$\begin{cases} \partial_t h = \frac{1}{2}\partial_x^2 h + \frac{1}{2}(\partial_x h)^2 + \eta & x \in \mathbb{R}_+ \\ h(x,0) = \log(\delta_x), \qquad \partial_x h(x,t)\Big|_{x=0} = A \end{cases}$$

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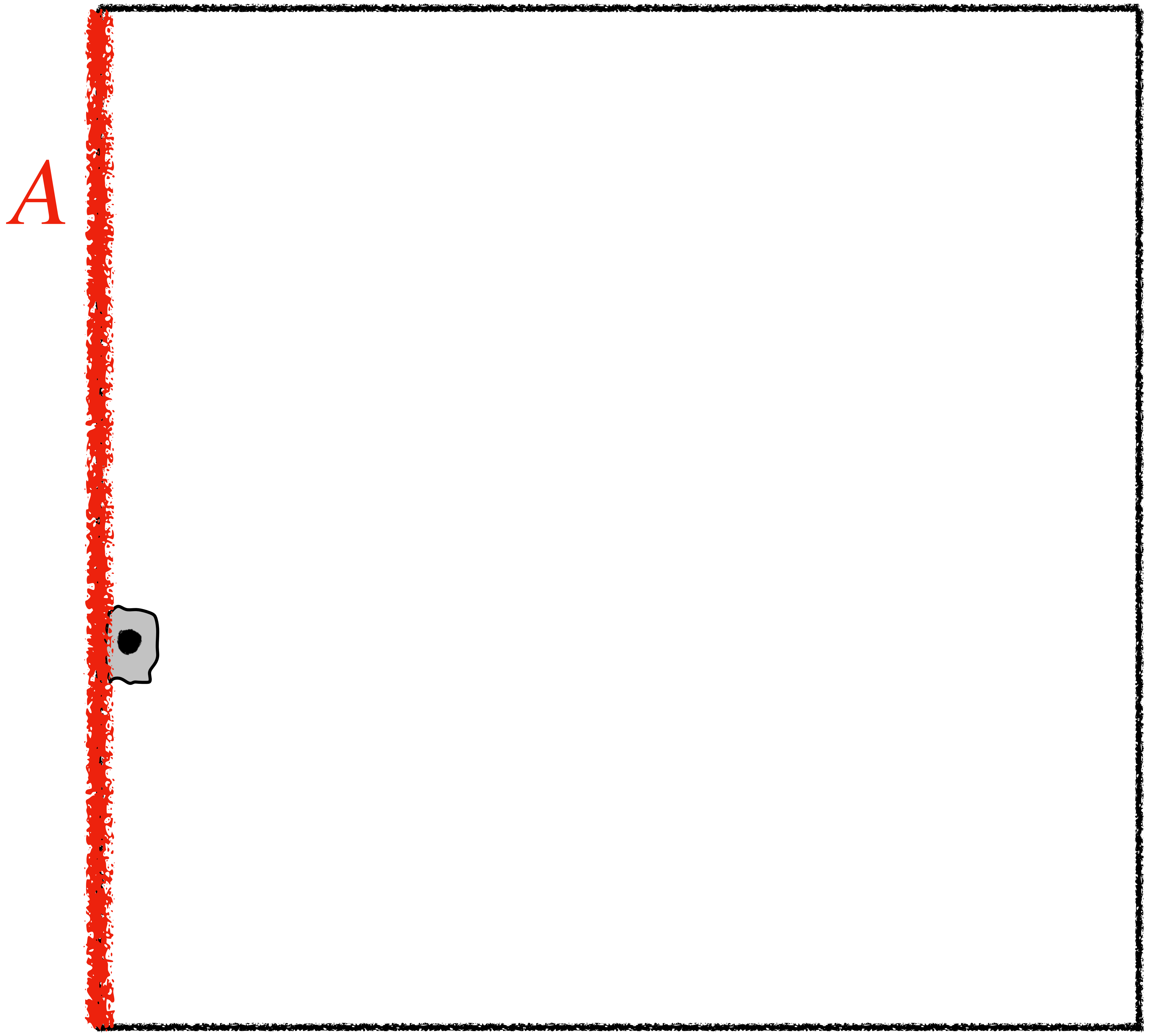

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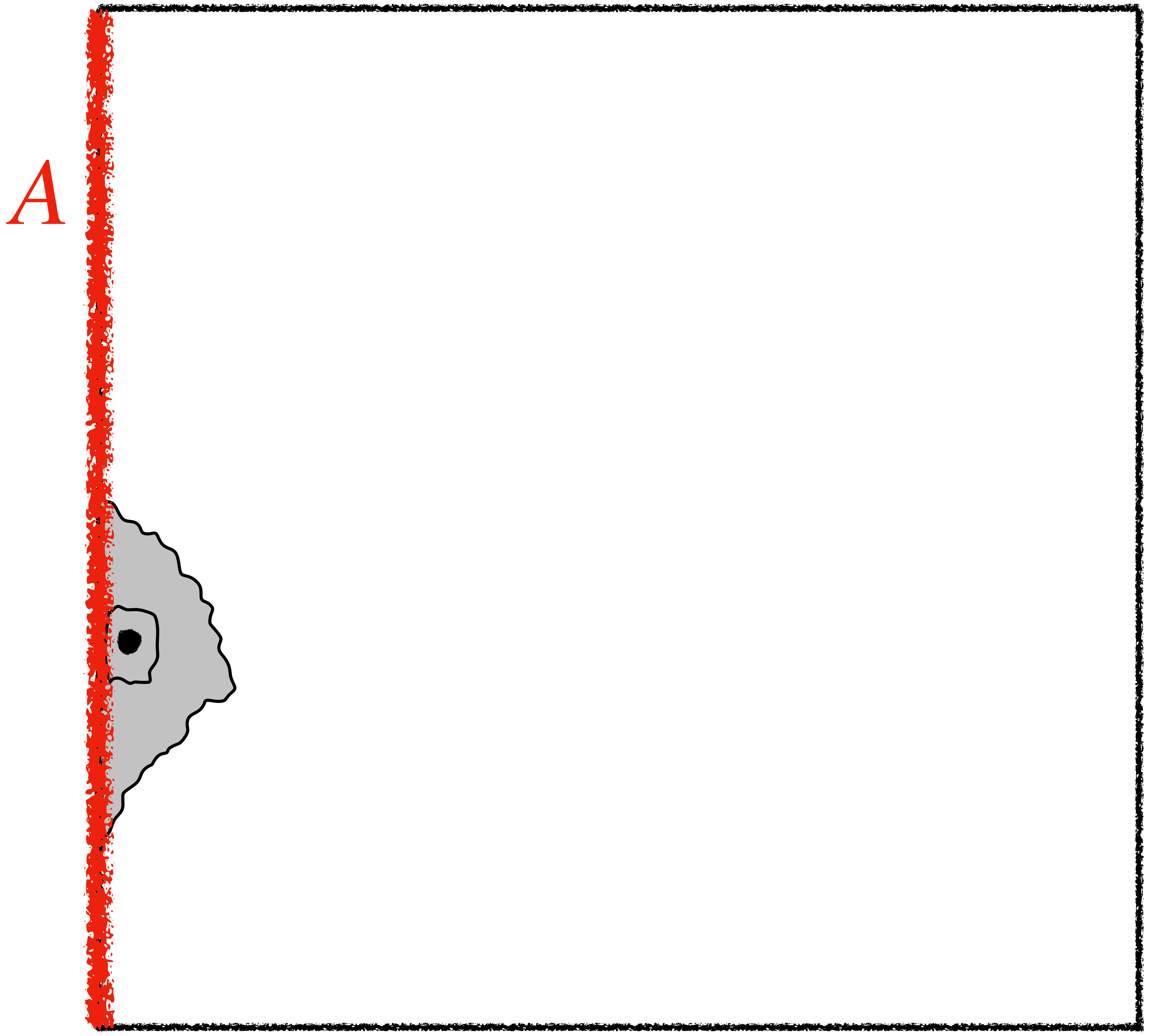

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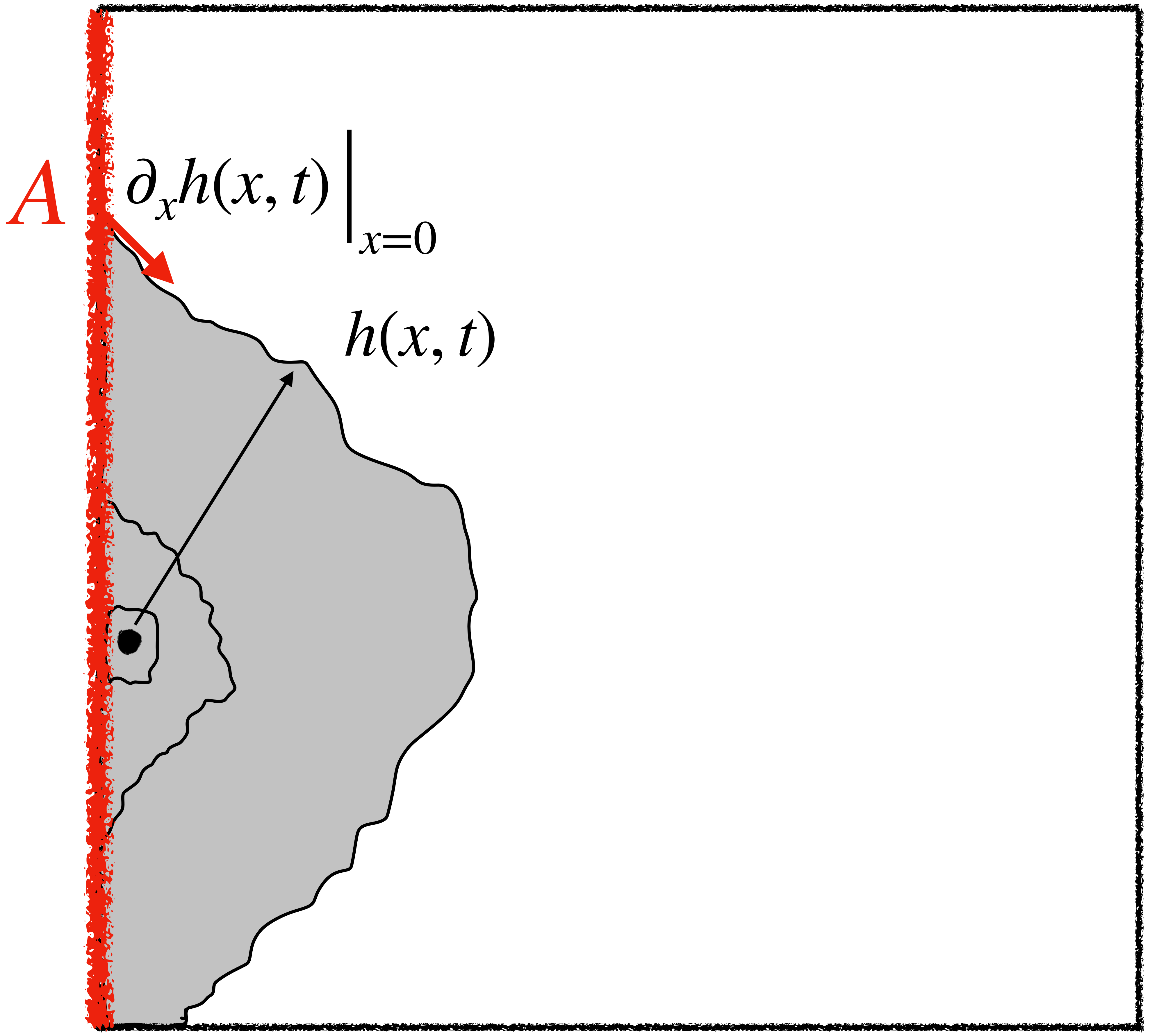

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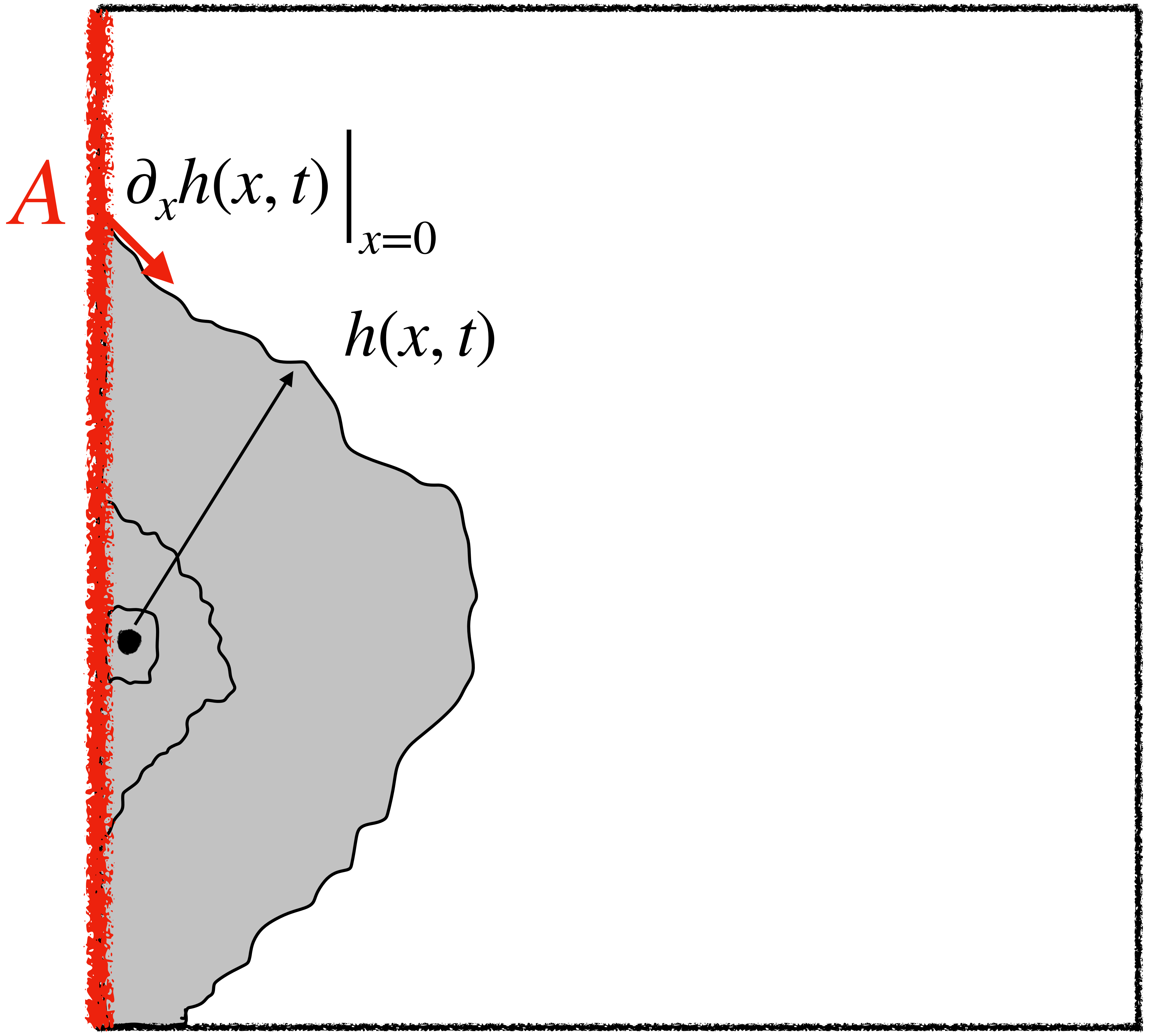

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- [Gueudre-Le Doussal'12, Borodin-Bufetov-Corwin'16, Barraquand-Borodin-Corwin-Wheeler'17 (*A*=-1/2), Krajenbrink-Le Doussal'19, De Nardis-Krajenbrink-Le Doussal-Thiery'20]

$$\mathbb{E}_{\mathrm{hs}}\left[\exp\left(-z e^{h(0,t)+t/24}\right)\right] = \mathrm{Pf}\left(J - fK\right)_{\mathscr{L}^2(\mathbb{R})}$$

$K(x,y)$

2x2 matrix kernel (Airy-like)

$$f(x) = \frac{1}{1+e^{-xt^{1/3}}/z}$$

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KPZ equation : exact solutions

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- Apparent only from the solutions
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- Solutions are obtained through Bethe Ansatz (BA)
- BA is very powerful but requires difficult calculations and (often) non-rigorous arguments
- Can we create an elementary theory to solve the KPZ eq.?

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    - Solvable polymer models (e.g. Log-Gamma polymers)
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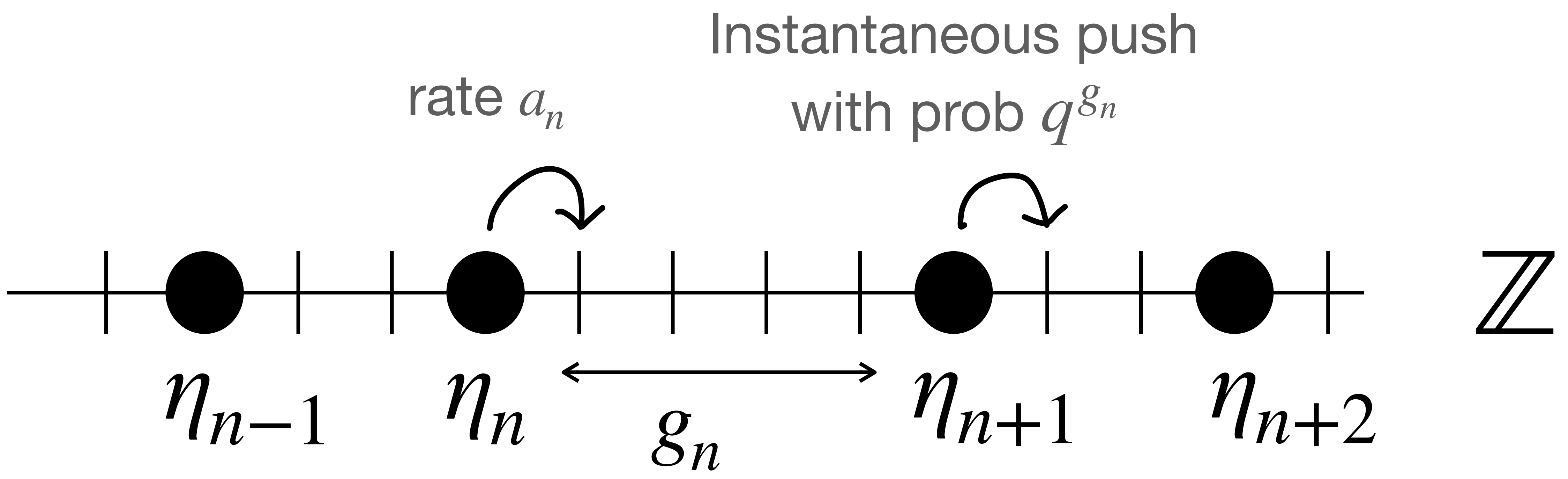
- …how to solve discrete models?

Solvable KPZ models and symmetric functions

- Typical model (FULL SPACE): *q-Push TASEP* [Borodin-Petrov '12]



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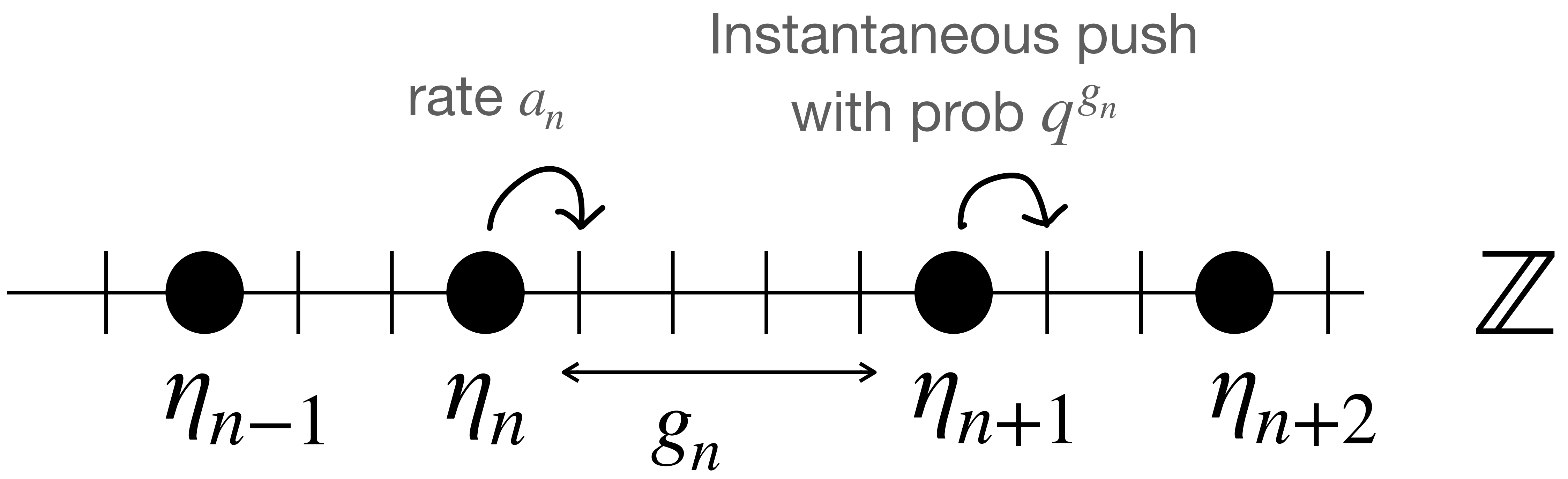

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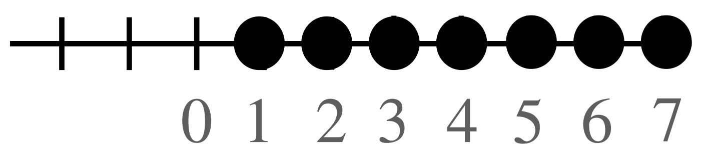

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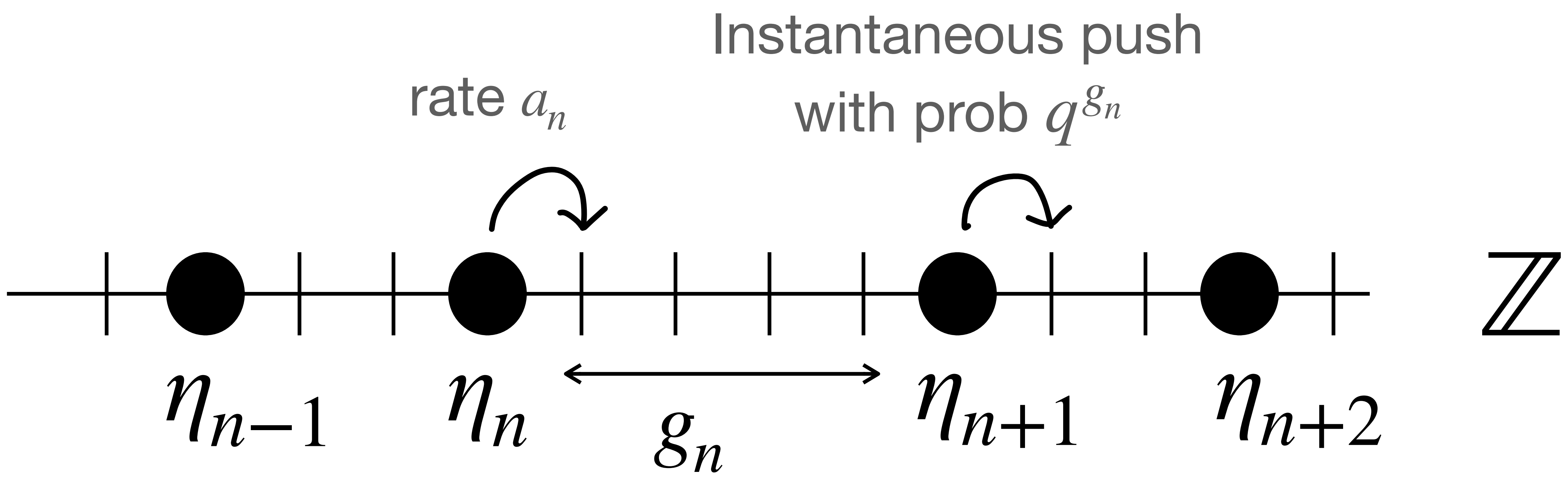

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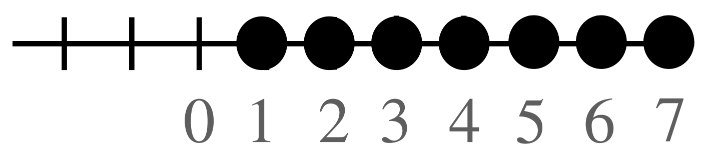

$$\mathbb{P}(\eta_n(t) - n = k) = \sum_{\mu_1 = k} \frac{b_\mu \mathscr{P}_\mu(a)\mathscr{P}_\mu(b_t)}{Z^q_{a,b_t}}$$

q-Whittaker measure
[Borodin-Corwin '11]

$\mu = (\mu_1 \geq \mu_2 \geq \cdots \geq 0)$

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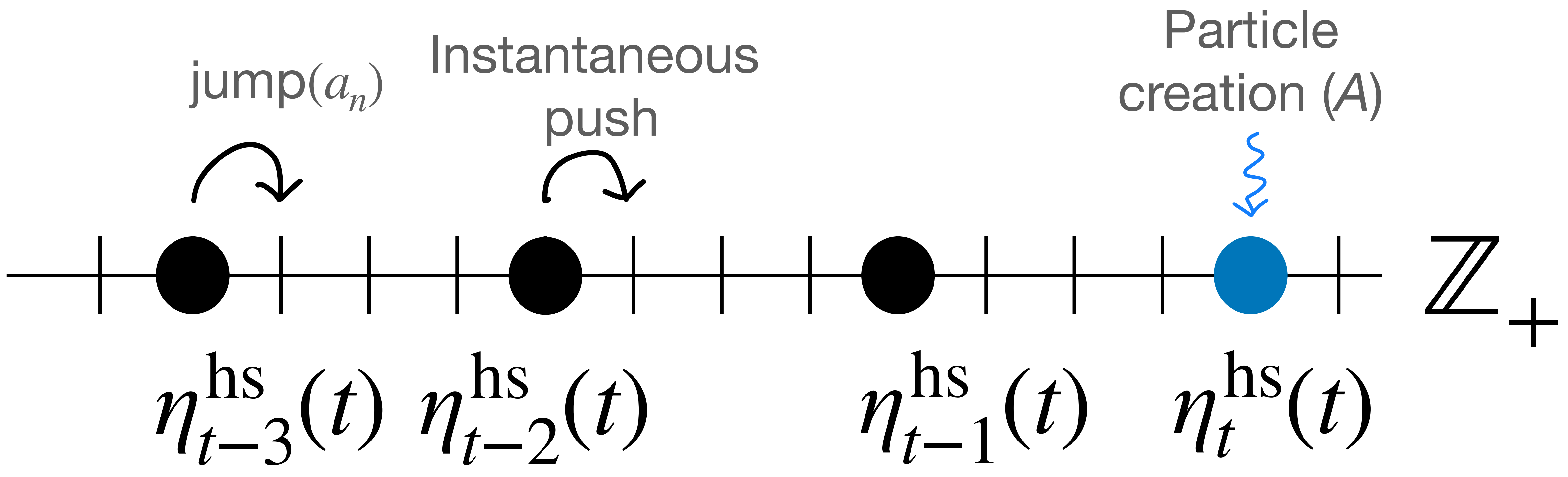

$$h_q^{\text{hs}}(x,t) = \eta_x^{\text{hs}}(t)$$

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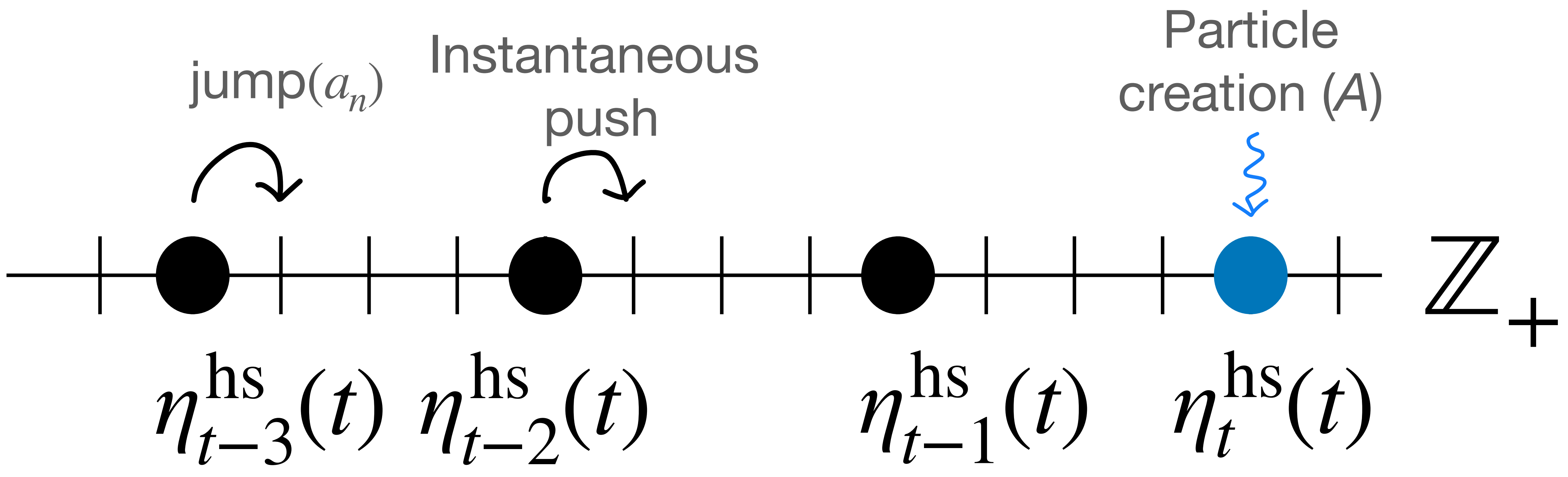

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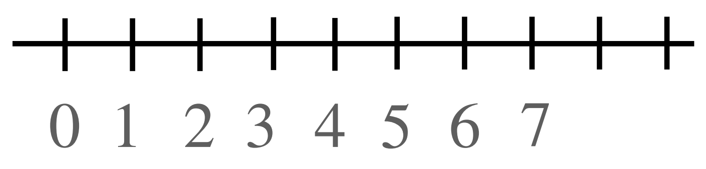

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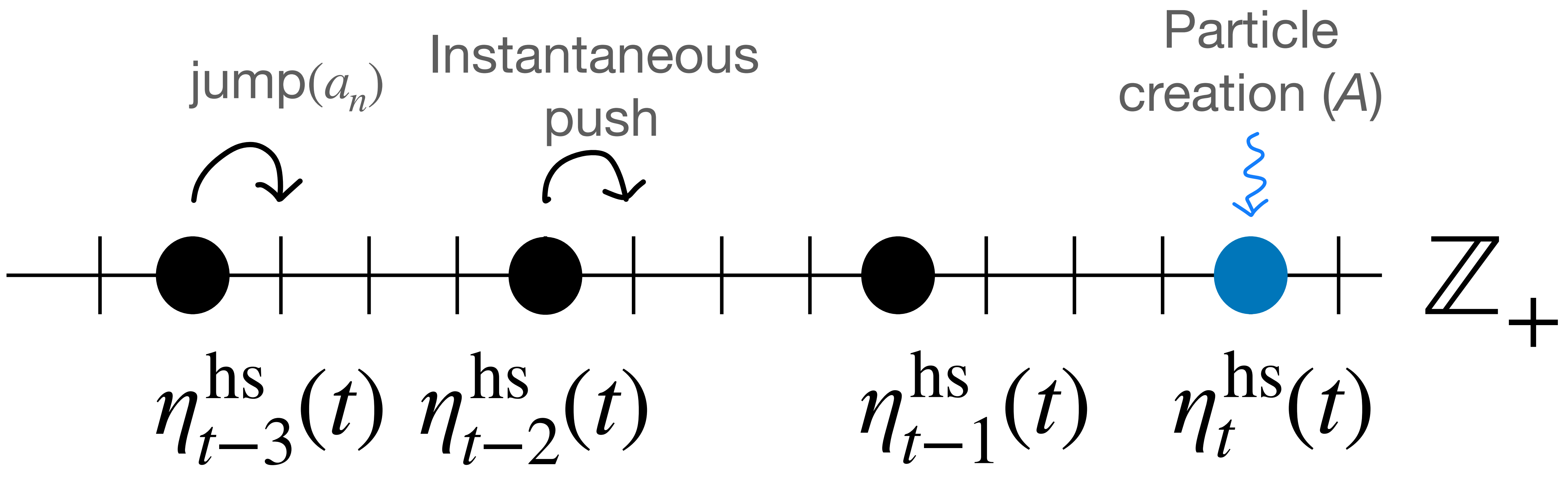

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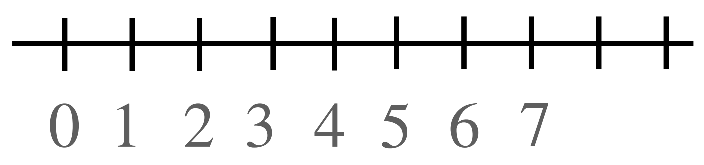

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Half space *q*-Whittaker measure [BBC '18]

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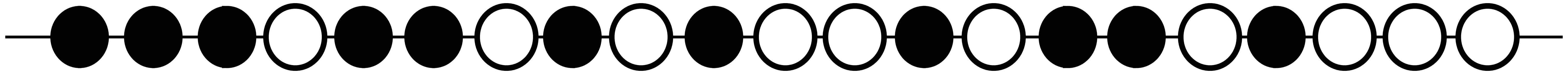
- **GOAL** : relate μ_1 with natural statistics of a determinantal/pfaffian point process

Positive temperature Free Fermions



Creation

$$\psi_k \,|\, \{n\} \rangle = \delta_{n_k} (-1)^{\sum_{j>k} n_j} \,|\, \{n + \mathbf{e}_k\} \rangle$$

Annihilation

$$\psi_k^\dagger \,|\, \{n\} \rangle = \delta_{n_k - 1} (-1)^{\sum_{j>k} n_j} \,|\, \{n - \mathbf{e}_k\} \rangle$$

Fermion number

$$\mathscr{N} = \sum_k : \psi_k \psi_k^\dagger :$$

Potential energy

$$\mathscr{H} = \sum_k k : \psi_k \psi_k^\dagger :$$

Hopping term

$$\Gamma_\pm(a) = \exp \left\{ \sum_n \frac{p_n(a)}{n} \sum_k \psi_{k \mp n} \psi_k^\dagger \right\}$$

Fermi-Dirac density

$$\varrho = \exp \left\{ \beta(\mathscr{H} - \nu \mathscr{N}) \right\}$$

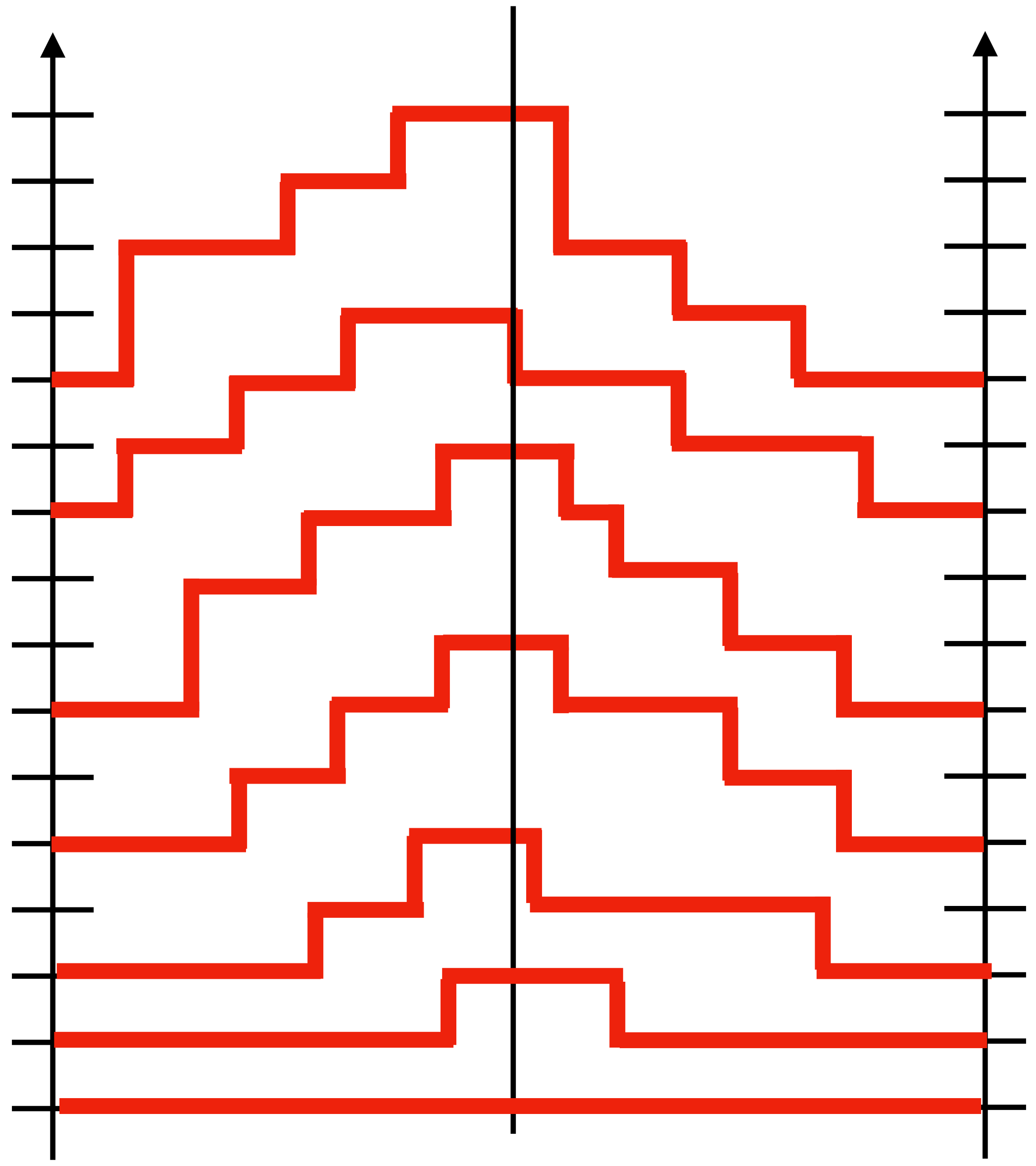
$$\langle O \rangle_{\beta,\nu}^{a,b} \propto \operatorname{tr} \left\{ \varrho \Gamma_-(b) O \Gamma_+(a) \right\}$$

Positive temperature Free Fermions

- Periodic Schur measure [Borodin '06]

$$\mathbb{P}(\lambda) = \frac{1}{\tilde{Z}^q_{a,b}} \sum_{\rho} q^{|\rho|} s_{\lambda/\rho}(a) s_{\lambda/\rho}(b)$$

$$s_{\lambda/\rho}(a) = \langle \rho, c \,|\, \Gamma_+(a) \,|\, \lambda, c \rangle$$ Schur polynomials



- $\mathbb{P}(S = k) \propto q^{k^2/2} t^k$ for $k \in \mathbb{Z}$ independent of λ
- $(\lambda_1 + S, \lambda_2 + S, \lambda_3 + S, \ldots)$ is a determinantal point process

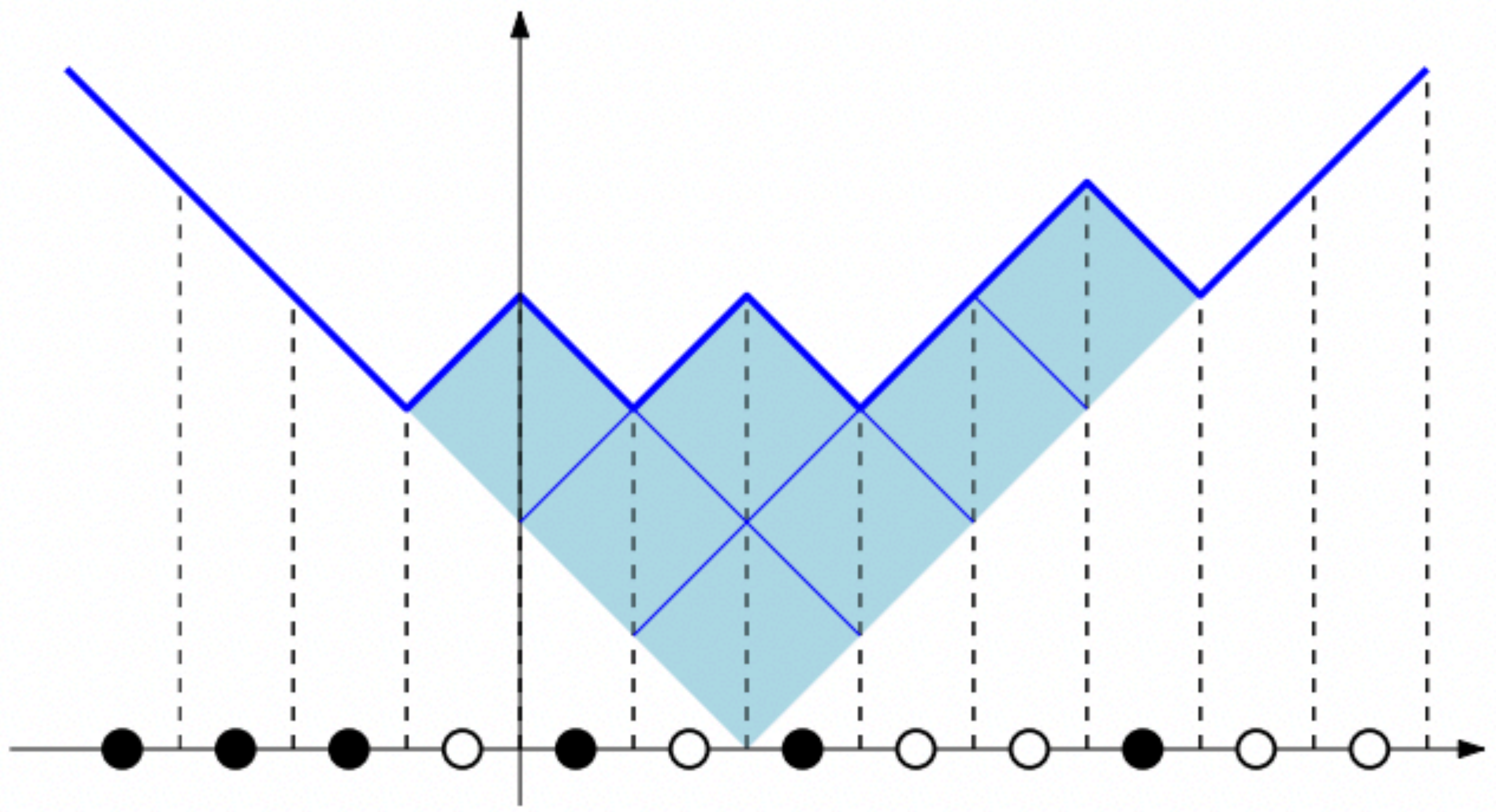
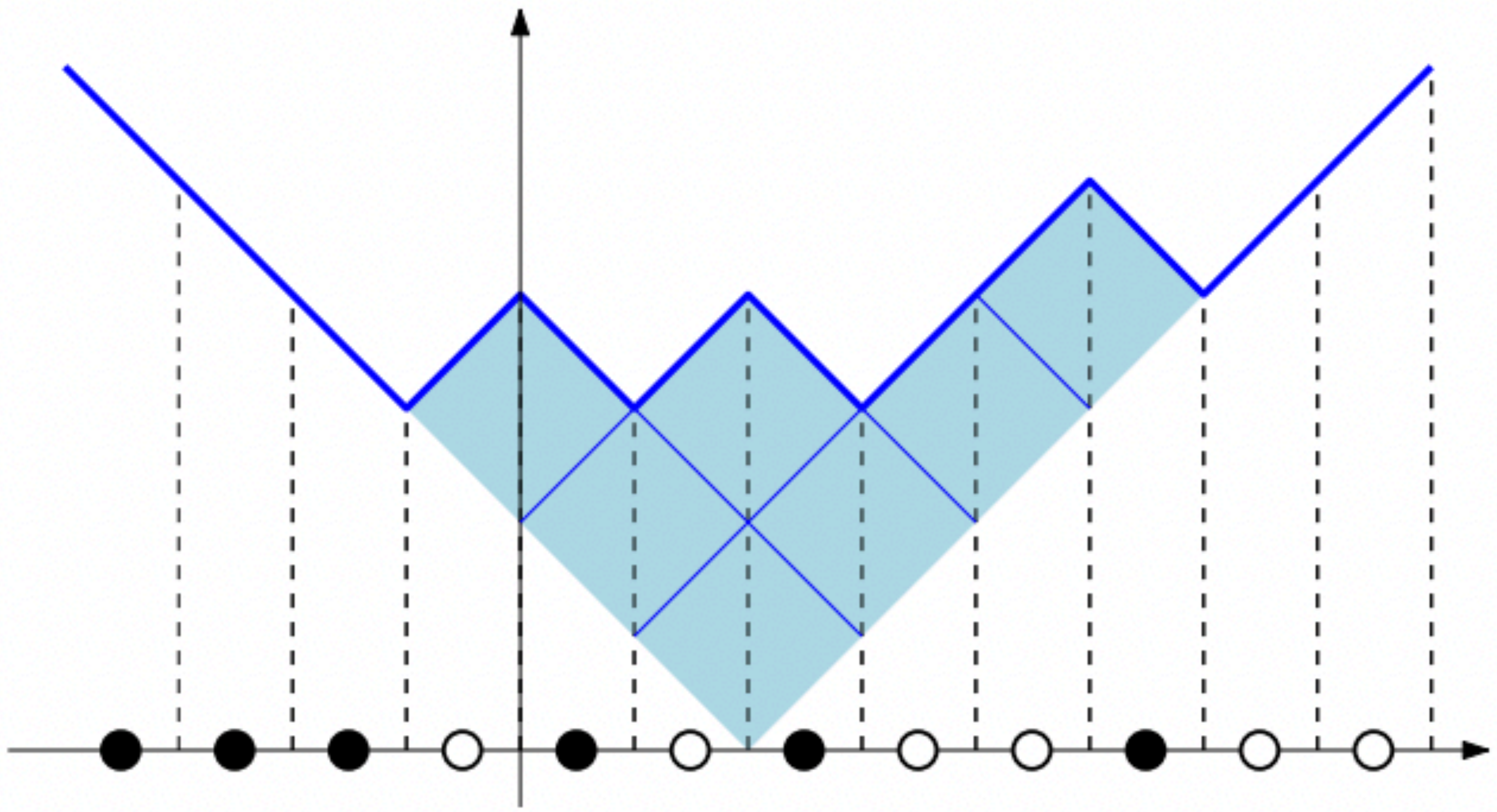


Figure from [Betea-Bouttier]

Positive temperature Free Fermions



$$\mathbb{P}(\lambda_1 + S \leq r) = \det(1 - K)_{\ell^2\{r+1, r+2, \ldots\}}$$

- $(\lambda_1 + S, \lambda_2 + S, \lambda_3 + S, \ldots)$ is a determinantal point process with correlation kernel

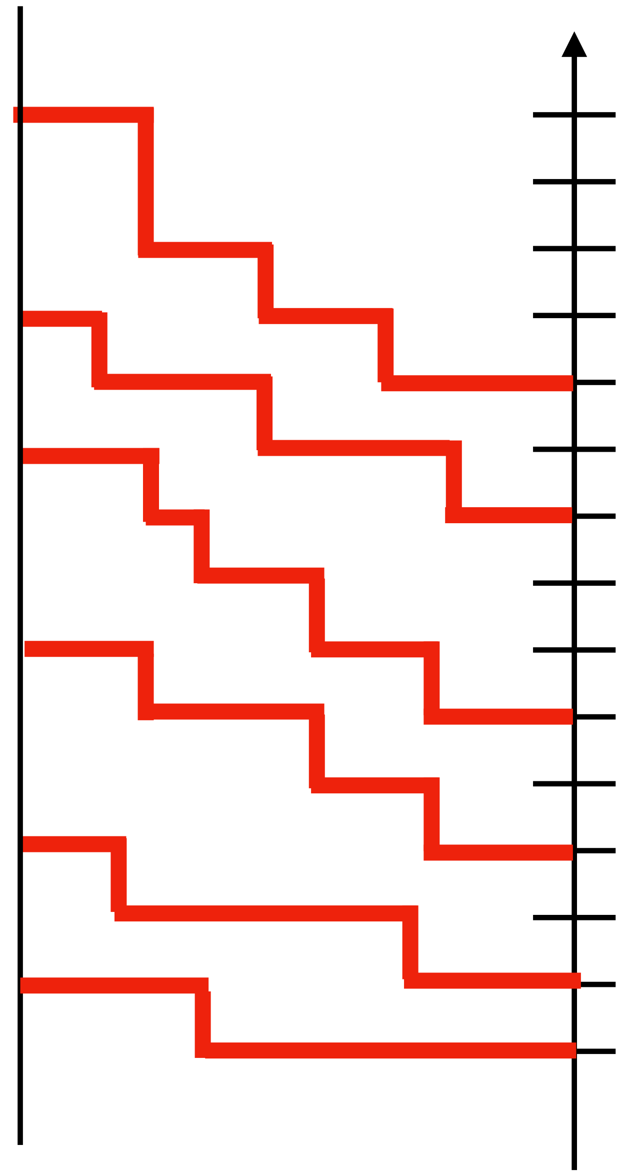
$$K(x, y) = \frac{1}{(2\pi \mathrm{i})^2} \oint_{|z|=r} \frac{dz}{z^{x+1}} \oint_{|w|=r'} \frac{dw}{w^{-y+1}} \frac{F(z)}{F(w)} \kappa(z, w),$$

$$F(z) = \prod_{i \geq 1} \frac{(b_i/z; q)_\infty}{(a_i z; q)_\infty} \qquad \kappa(z, w) = \sqrt{\frac{w}{z}} \frac{(q; q)_\infty^2}{(z/w, qw/z; q)_\infty} \frac{\vartheta_3(\zeta z/w; q)}{\vartheta_3(\zeta; q)} \qquad (z; q)_\infty = \prod_{\ell \geq 0} (1 - q^\ell z)$$

Free boundary Schur measure

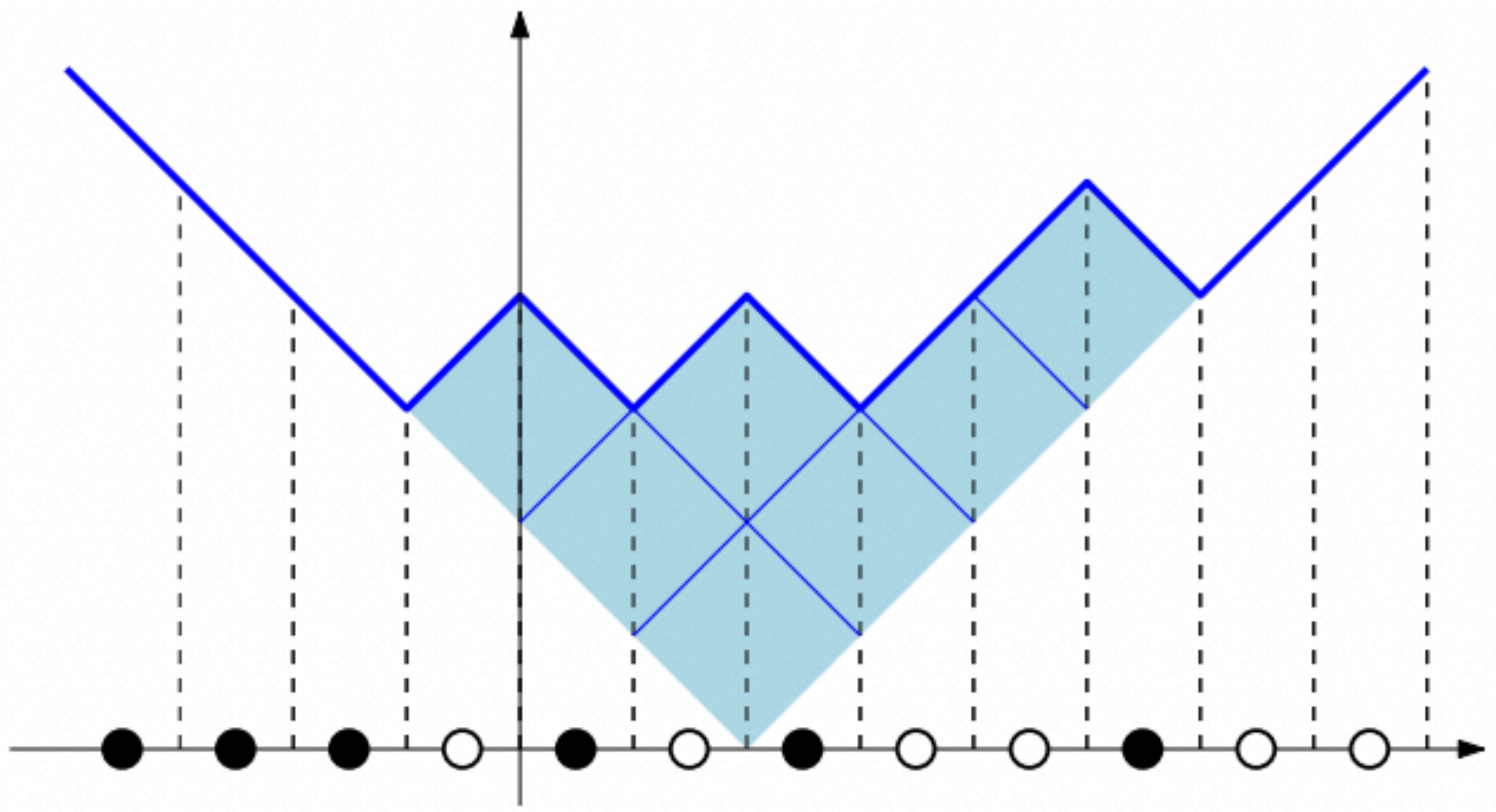
- Free boundary Schur measure [Betea-Bouttier-Nejjar-Vuletic ’17]

$$\mathbb{P}(\lambda) = \frac{\mathbf{1}_{\lambda' \text{even}}}{\tilde{Z}^q_{a;A}} \sum_{\rho' \text{even}} q^{|\rho|/2} s_{\lambda/\rho}(a; A)$$



- $\mathbb{P}(S = k) \propto q^{2k^2} t^{2k}$ for $k \in \mathbb{Z}$ independent of λ

- $(\lambda_1 + 2S, \lambda_2 + 2S, \lambda_3 + 2S, \dots)$ is a pfaffian point process



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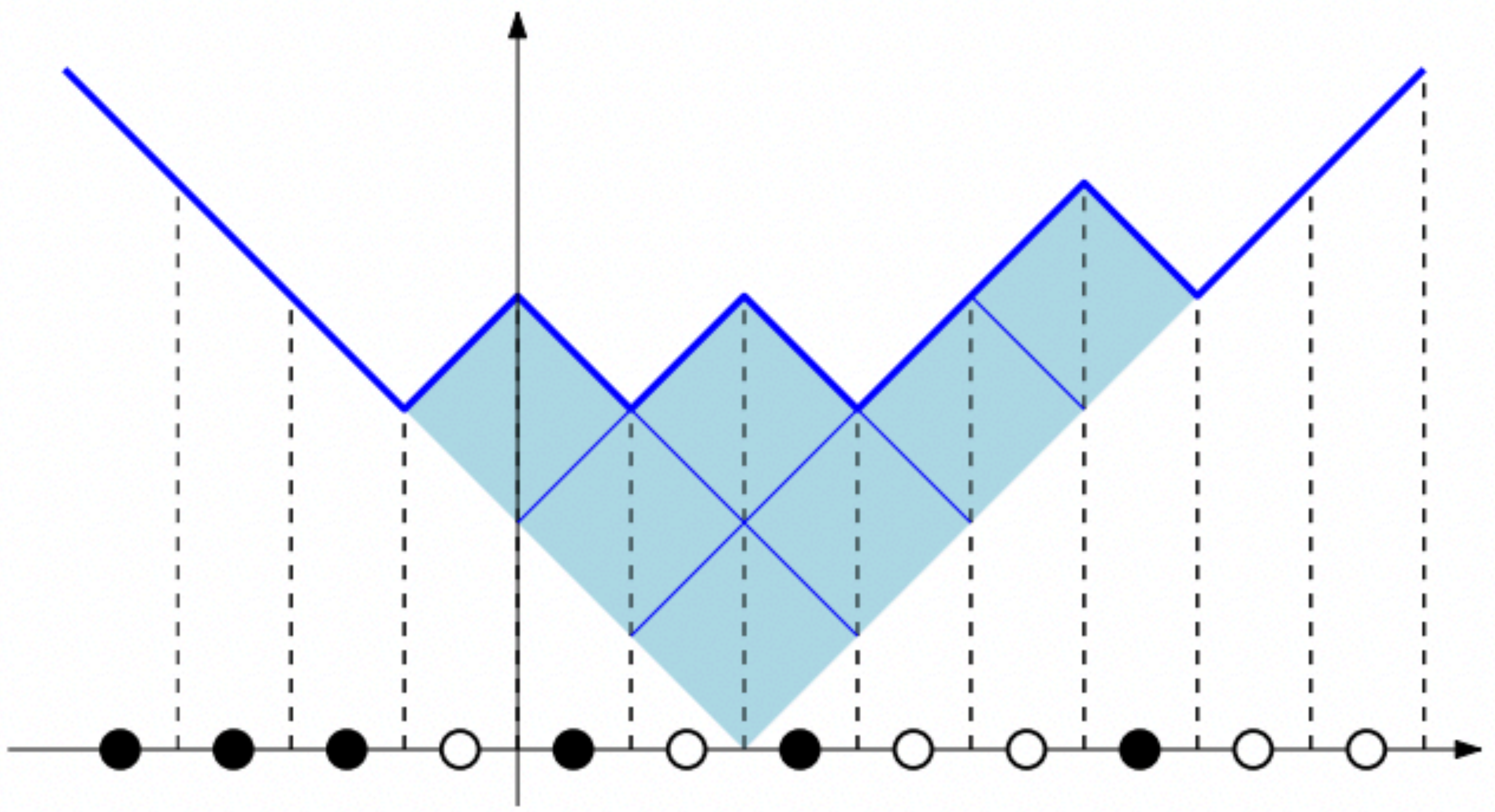
Free boundary Schur measure



$$\mathbb{P}(\lambda_1 + S \leq r) = \mathrm{Pf}(1 - L)_{\ell^2\{r+1, r+2, \dots\}}$$

$$L(x,y) = \begin{pmatrix} k(x,y) & -\nabla_y k(x,y) \\ -\nabla_x k(x,y) & \nabla_x \nabla_y k(x,y) \end{pmatrix}$$

$$k(x,y) = \frac{1}{(2\pi \mathrm{i})^2} \oint_{|z|=r} \frac{dz}{z^{x+3/2}} \oint_{|w|=r} \frac{dw}{w^{y+5/2}} F(z) F(w) \kappa^{\mathrm{hs}}(z,w)$$

$$\nabla g(x) = \frac{1}{2}\left[g(x+1) - g(x-1)\right]$$

$$F(z) = \frac{(A/z;q)_\infty}{(Az;q)_\infty} \prod_{i \geq 1} \frac{(a_i/z;q)_\infty}{(a_i z;q)_\infty}$$

$$\kappa^{\mathrm{hs}}(z,w) = \frac{(q, q, w/z, qz/w; q)_\infty}{(1/z^2, 1/w^2, 1/zw, qwz; q)_\infty} \frac{\vartheta_3(\zeta^2 z^2 w^2; q^2)}{\vartheta_3(\zeta^2; q^2)}$$

KPZ solvable models

Determinantal/Pfaffian point processes

Full space

$$\mu \sim \frac{b_\mu \mathscr{P}_\mu(x)\mathscr{P}_\mu(y)}{Z^q_{x,y}}$$

$$\lambda \sim \frac{1}{\tilde{Z}^q_{x,y}} \sum_\rho q^{|\rho|} s_{\lambda/\rho}(x) s_{\lambda/\rho}(y)$$

Half space

$$\mu^{\mathrm{h}s} \sim \frac{b^{\mathrm{el}}_\mu \mathscr{P}_\mu(a;A)}{Z^q_{a;A}}$$

$$\lambda^{\mathrm{h}s} \sim \frac{\mathbf{1}_{\lambda'\mathrm{even}}}{\tilde{Z}^q_{a;A}} \sum_{\rho'\mathrm{even}} q^{|\rho|/2} s_{\lambda/\rho}(a;A)$$

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THEOREM (Imamura-M.-Sasamoto '21)

$$\mu_1 + \chi \overset{\mathscr{D}}{=} \lambda_1 \qquad \mu_1^{\mathrm{hs}} + \chi \overset{\mathscr{D}}{=} \lambda_1^{\mathrm{hs}} \qquad (\star)$$

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Comments on ($\star$):

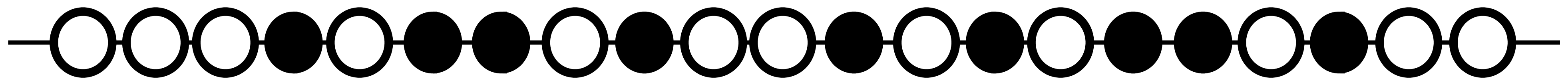
- Reveals the origin of determinantal formulas for KPZ solvable models at positive temperature
- Nice symmetrical relation between full and half space
- Suggests a new paradigm to solve models
- Earlier results relating KPZ models and free fermions:
    - [Dean-Le Doussal-Majumdar-Schehr'15]
    - [Borodin'16],[Borodin-Gorin'16],[Borodin-Ohlshanki'16],[Borodin-Corwin-Barraquand-Wheeler'17]

Formulas for KPZ models (full space)

- We use the correspondence between q-Whittaker measure and free fermions to get formulas

$$\mathbb{P}\left(\mu_1 + \chi + S \leq r\right) = \det\left(1 - K\right)_{\ell^2(r+1, r+2, \ldots)}$$

$$K(x, y) = \frac{1}{(2\pi \mathrm{i})^2} \oint_{|z|=r} \frac{dz}{z^{x+1}} \oint_{|w|=r'} \frac{dw}{w^{-y+1}} \frac{F(z)}{F(w)} \kappa(z, w),$$



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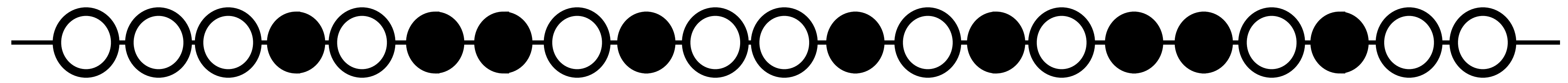
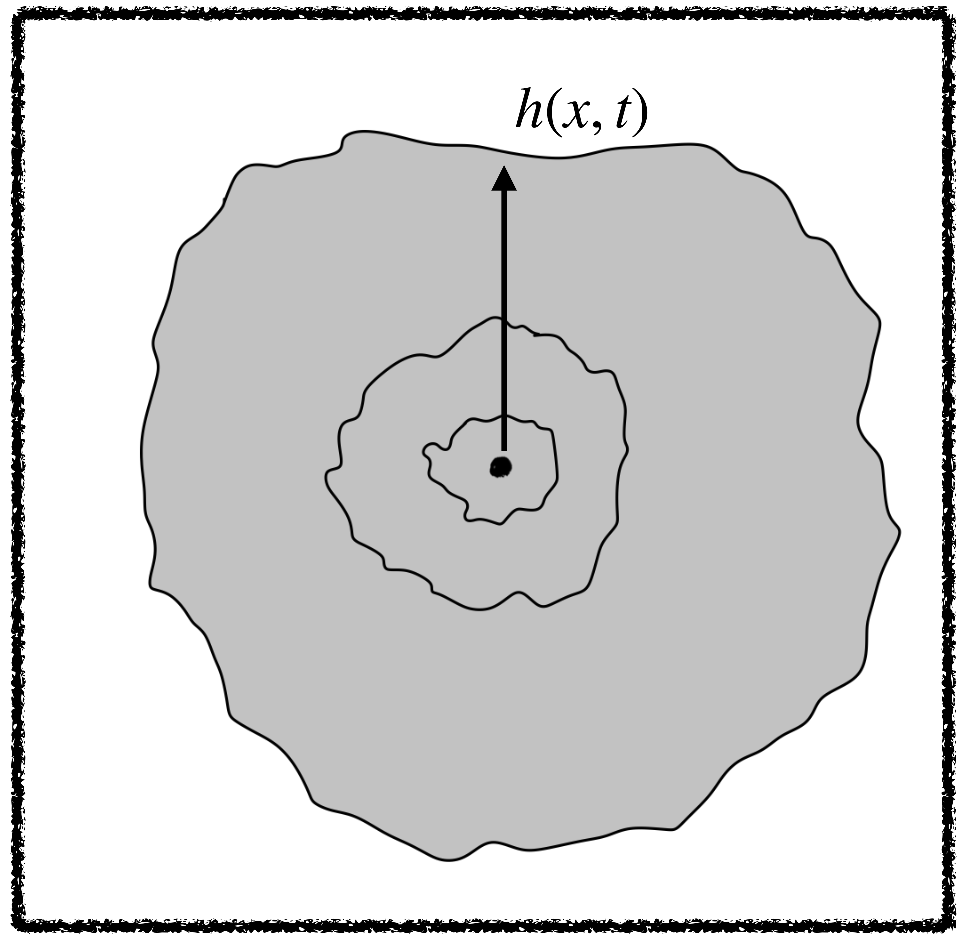


$$\mathbb{P}\left(h(0,t)+\frac{t}{24}+\mathcal{G}\le r\right)=\det\left(1-\mathcal{K}\right)_{\mathbb{L}^2(r,+\infty)}$$

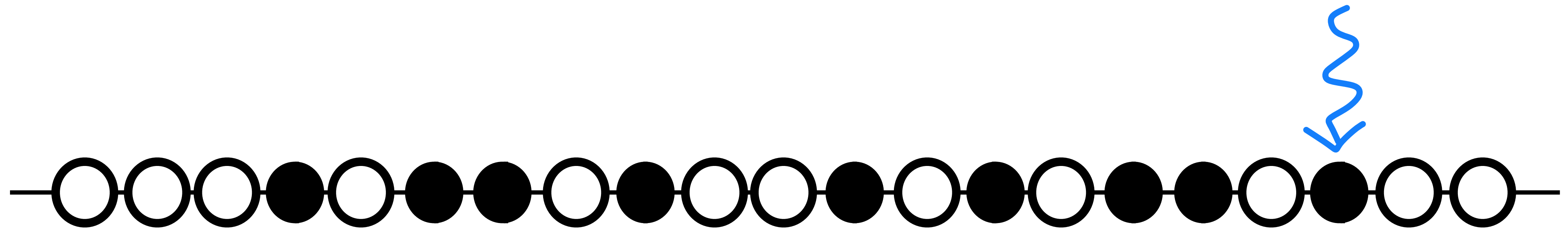
$\mathcal{G}=$ indep . Gumbel r . v .

$$\mathcal{K}(x,y)=\int_{\mathrm{i}\mathbb{R}-d}\frac{\mathrm{d}Z}{2\pi\mathrm{i}}\int_{\mathrm{i}\mathbb{R}+d}\frac{\mathrm{d}W}{2\pi\mathrm{i}}e^{-\frac{t}{2}\left(\frac{Z^3}{3}-\frac{W^3}{3}\right)+Zx-Wy}\frac{\pi}{\sin[\pi(W-Z)]}$$



Formulas for KPZ models (half space)

- We use the correspondence between q-Whittaker measure and free fermions to get formulas



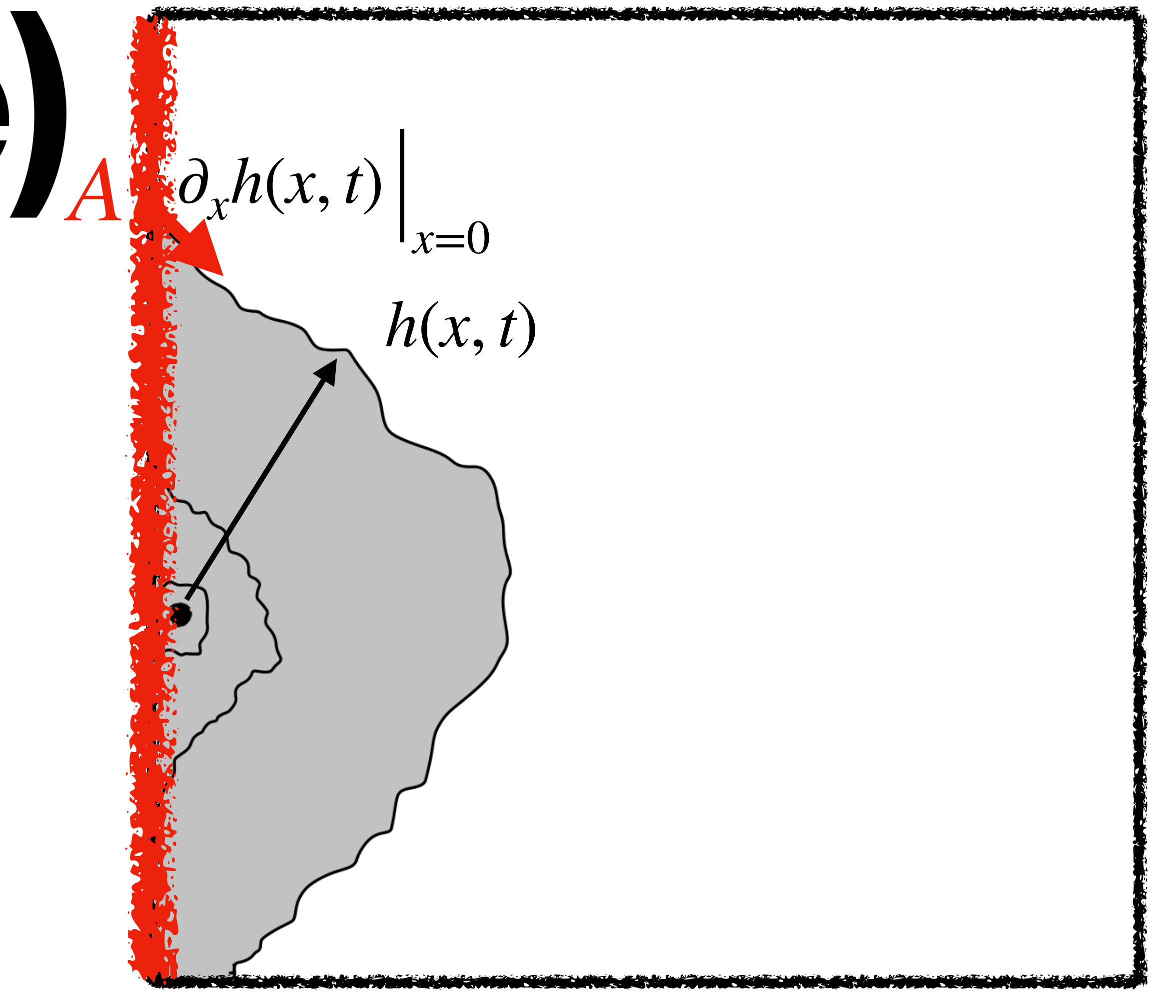
THEOREM (Imamura-M.-Sasamoto '22)

$$\mathbb{P}\left(\mu_1^{\mathrm{hs}} + \chi + 2S \leq r\right) = \mathrm{Pf}\,(J - L)_{\ell^2(r+1, r+2, \ldots)}$$

$$L(x,y) = \begin{pmatrix} k(x,y) & -\nabla_y k(x,y) \\ -\nabla_x k(x,y) & \nabla_x \nabla_y k(x,y) \end{pmatrix}$$

$$k(x,y) = \frac{1}{(2\pi \mathrm{i})^2} \oint_{|z|=r} \frac{dz}{z^{x+3/2}} \oint_{|w|=r} \frac{dw}{w^{y+5/2}} F(z) F(w) \kappa^{\mathrm{hs}}(z,w)$$

Formulas for KPZ models (half space)



$$\begin{cases} \partial_t h = \frac{1}{2}\partial_x^2 h + \frac{1}{2}(\partial_x h)^2 + \eta & x \in \mathbb{R}_+ \\ h(x,0) = \log(\delta_x), \qquad \partial_x h(x,t)\Big|_{x=0} = A \end{cases}$$

THEOREM (Imamura-M.-Sasamoto '22)

$$\mathbb{P}\left(h^{\mathrm{hs}}(0,t) + \frac{t}{24} + \mathcal{G} \le r\right) = \mathrm{Pf}\left(J - \mathcal{L}\right)_{\mathbb{L}^2(r,+\infty)} \qquad \mathcal{G} = \text{indep . Gumbel r . v .}$$

$$\mathcal{L}(X,Y) = \begin{pmatrix} \mathcal{K}^{\mathrm{hs}}(X,Y) & -\partial_y \mathcal{K}^{\mathrm{hs}}(X,Y) \\ -\partial_x \mathcal{K}^{\mathrm{hs}}(X,Y) & \partial_x\partial_y \mathcal{K}^{\mathrm{hs}}(X,Y) \end{pmatrix}$$

$$\mathcal{K}^{\mathrm{hs}}(X,Y) = \int_{\mathrm{i}\mathbb{R}+d} \frac{\mathrm{d}Z}{2\pi \mathrm{i}} \int_{\mathrm{i}\mathbb{R}+d} \frac{\mathrm{d}W}{2\pi \mathrm{i}} e^{-ZX-WY} g_{t,A}(Z) g_{t,A}(W) \frac{\sin[\pi(Z-W)]}{\sin[\pi(Z+W)]} \qquad g_A(Z) = \exp\left\{\frac{t}{2}\frac{Z^3}{3}\right\} \frac{\Gamma(\frac{1}{2}+A-Z)}{\Gamma(\frac{1}{2}+A+Z)} \Gamma(2Z)$$

Idea of proof

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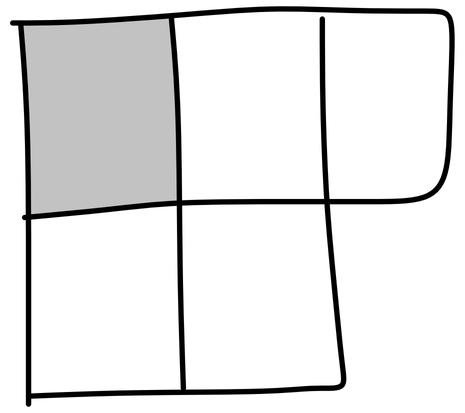
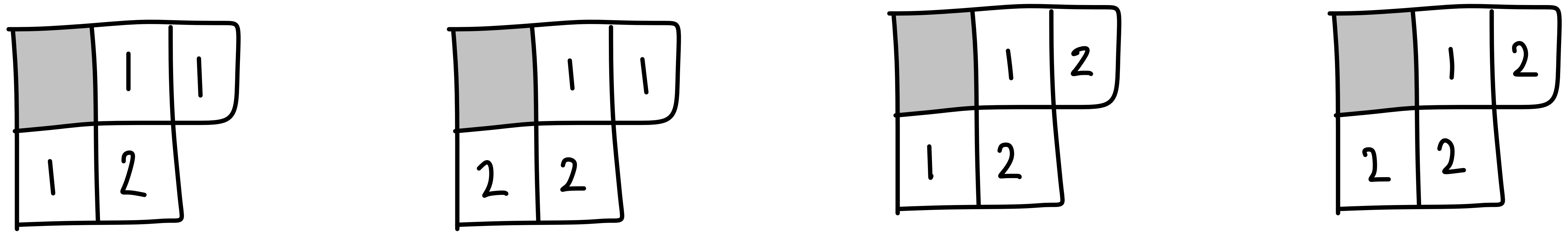
$\lambda/\rho =$ 



$$s_{\lambda/\rho}(x_1, x_2) = x_1^3 x_2 + x_1^2 x_2^2 + x_1^2 x_2^2 + x_1 x_2^3$$

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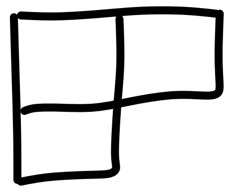
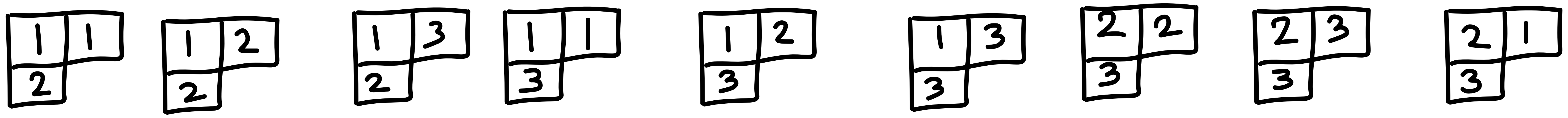
- **Combinatorial formulas**

  - $$s_{\lambda/\rho}(x) = \sum_{T \in SST(\lambda/\rho)} x^T$$ sum over semi-standard tableaux

  - $$\mathscr{P}_\mu(x;q) = \sum_{V \in VST(\mu)} q^{\mathscr{H}(V)} x^V$$ sum over "vertically strict tableaux"

    $\mathscr{H}$ = intrinsic energy

  Example : $\mu =$  $n = 3$

$V =$ 

$\mathscr{H} =$ 0 0 0 0 0 0 0 0 1

$$\mathscr{P}_\mu(x;q) = x_1^2x_2 + x_1x_2^2 + x_1x_2x_3 + x_1^2x_3 + x_1x_2x_3 + x_1x_3^2 + x_2^2x_3 + x_2x_3^2 + qx_1x_2x_3$$

Cauchy Identities

- $$\sum_{\lambda,\rho} q^{|\rho|} s_{\lambda/\rho}(x) s_{\lambda/\rho}(y) = \frac{1}{(q;q)_\infty} \prod_{i,j} \frac{1}{(x_i y_j; q)_\infty}$$
- $$\sum_{\mu} b_\mu \mathscr{P}_\mu(x) \mathscr{P}_\mu(y) = \prod_{i,j} \frac{1}{(x_i y_j; q)_\infty}$$

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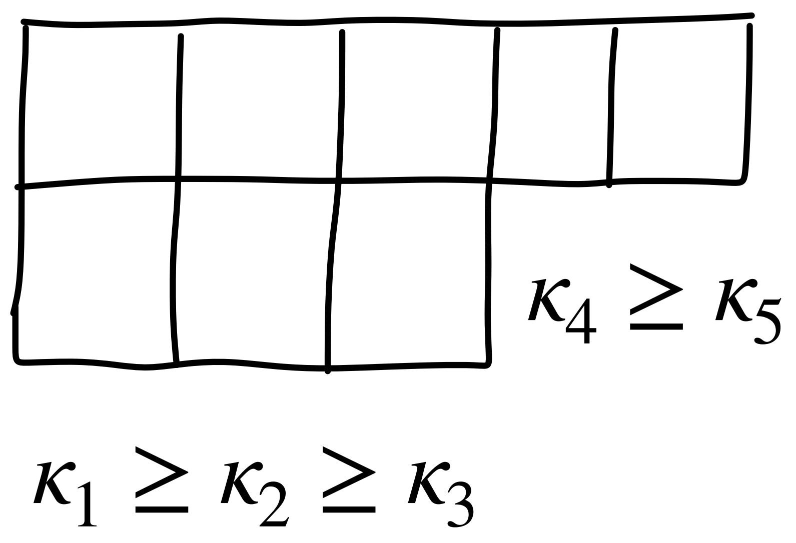
$$b_\mu = \prod_{i \geq 1} (q;q)^{-1}_{\mu_i - \mu_{i+1}} = \sum_{\kappa \in \mathscr{K}(\mu)} q^{\kappa_1 + \kappa_2 + \cdots + \kappa_{\mu_1}}$$

$\mu =$ 

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IDEA: $(P, Q) \overset{\Upsilon}{\longleftrightarrow} (V, W; \kappa, \nu)$

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Properties of Υ

- $(P, Q) \longleftrightarrow (V, W; \kappa, \nu)$ has the following properties
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  - $|\rho| = \mathscr{H}(V) + \mathscr{H}(W) + |\kappa| + |\nu|$ $\qquad (\mathscr{H} = \mathscr{H}_1 + \mathscr{H}_2 + \cdots)$
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Conclusion

- Correspondence between solvable non-free fermionic models and positive temperature free fermionic models

Some achievements

- Solutions of KPZ equation derived without Bethe Ansatz

- Proof of pfaffian formulas for the KPZ equation in half space

Iffy points

- Correspondence proved for discrete models. Can it be described nicely at the KPZ eq. Level?

- Refine the correspondence to describe multi point observables