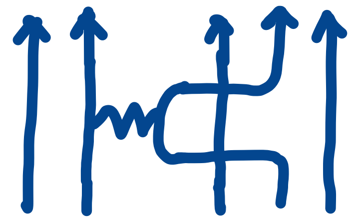
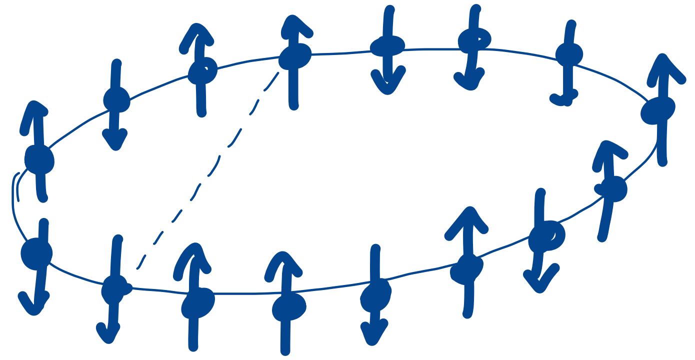
The Heisenberg spin chain and its long-range friends

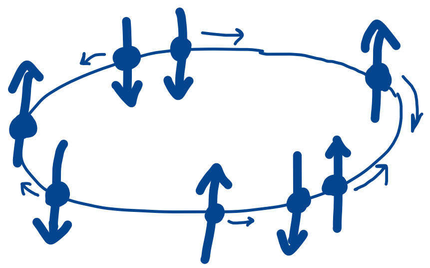




by

Jules Lamers

Institut de Physique Théorique



based on

JL
PRB 97 ('18) 214416
arXiv.1801 05728

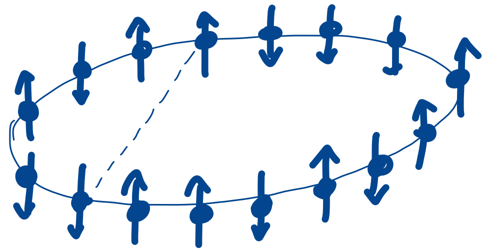
JL, V Pasquier, D Serban
to appear in CMP
arXiv 2004.13210

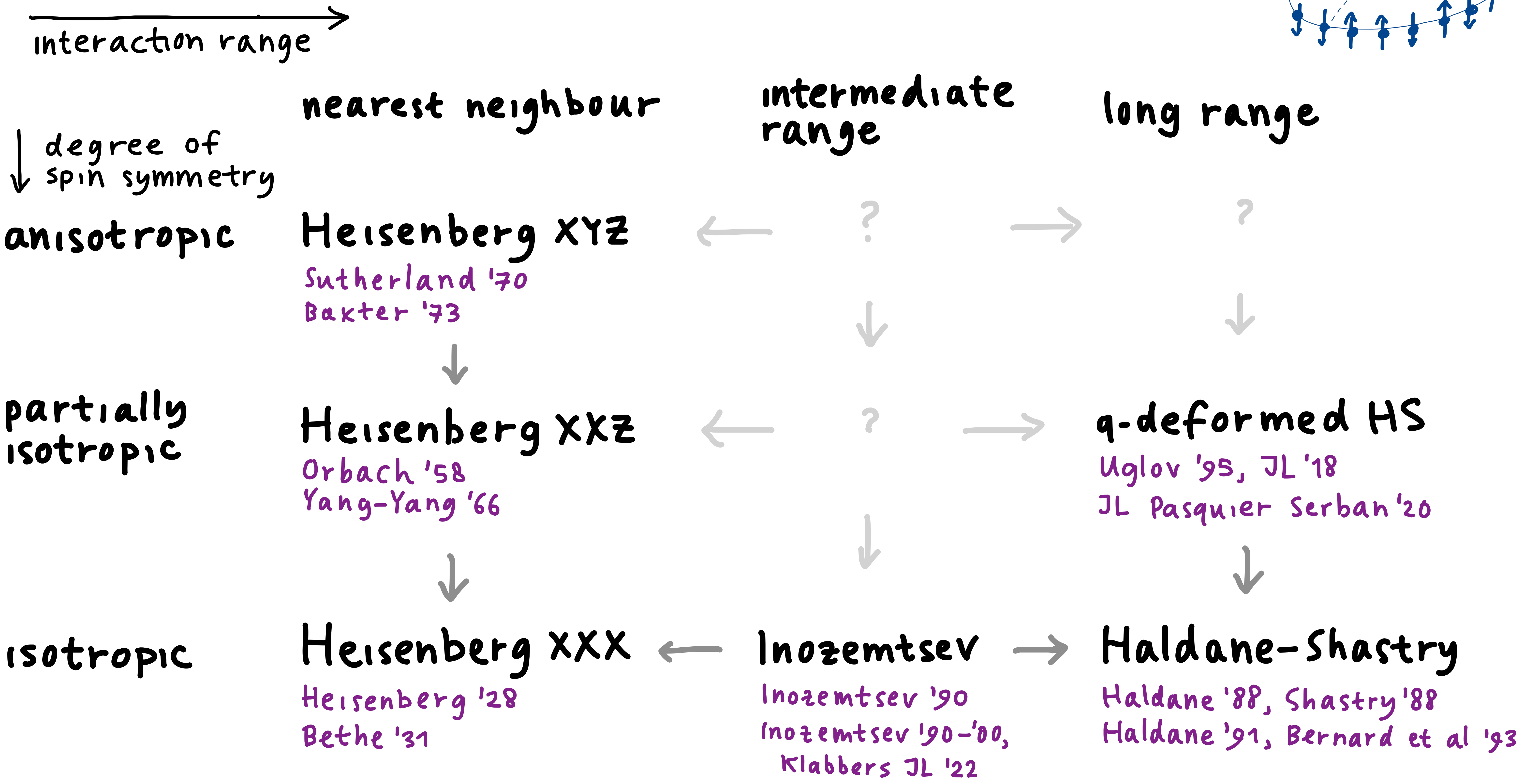
R Klabbers, JL
CMP 390 ('22) 827
arXiv 2009 14513

and ongoing

randomness, integrability, universality GGI '22·V·11

Overview landscape of long-range spin chains





interaction range →

↓ degree of spin symmetry	nearest neighbour	intermediate range	long range
anisotropic	Heisenberg XYZ Sutherland '70 Baxter '73	?	?
partially isotropic	Heisenberg XXZ Orbach '58 Yang–Yang '66	?	q-deformed HS Uglov '95, JL '18 JL Pasquier Serban '20
isotropic	Heisenberg XXX Heisenberg '28 Bethe '31	Inozemtsev Inozemtsev '90 Inozemtsev '90–'00, Klabbers JL '22	Haldane–Shastry Haldane '88, Shastry '88 Haldane '91, Bernard et al '93

Isotropic level Hamiltonians

typical anatomy of spin chain:
(spin $\frac{1}{2}$)

potential

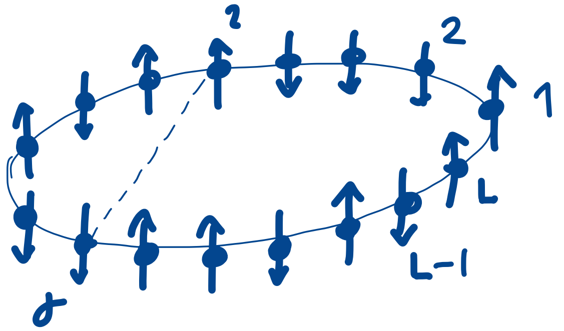
spin 'exchange' interaction

$$H = \sum_{i<j}^{L} V(d_L(i,j)) \, \overbrace{(1-P_{ij})}^{(1-\vec{\sigma}_i \cdot \vec{\sigma}_j)/2}$$

pairwise

homogeneous (transl invariant)

isotropic ($\mathfrak{sl}_2$-invariant)



e.g.

Heis XXX '28 ← **Inozemtsev** '90 → **Haldane** '88 **– Shastry** '88

nearest neighbour — intermediate range — long range

$$V_{\text{Heis}}(d) = \delta_{d,1} \;\xleftarrow{\kappa\to\infty}\; V_{\text{Ino}}(d) \sim \wp(d) \sim (\log \vartheta(d))' \;\xrightarrow{\kappa\to 0}\; V_{HS}(d) = \frac{(\pi/L)^2}{\sin(\frac{\pi}{L}d)^2} = \frac{1}{\text{crd}^2}$$

periods $(L, \frac{i\pi}{\kappa}) \in \mathbb{Z}_{\geq 2} \times i\mathbb{R}_{>0}$

Isotropic level Hamiltonians

typical anatomy of spin chain:
(spin $\frac{1}{2}$)

spin 'exchange' interaction

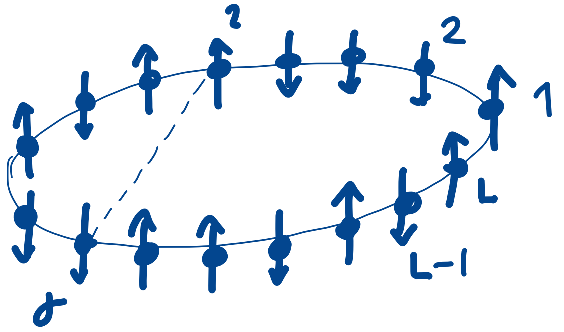
potential

$$H=\sum_{i<j}^{L} V(d_L(i,j))\,(1-\overbrace{P_{ij}}^{(1-\vec{\sigma}_i\cdot\vec{\sigma}_j)/2})$$

pairwise

homogeneous
(transl invariant)

isotropic
($\mathfrak{sl}_2$-invariant)



e.g.

Heis XXX '28
nearest neighbour
$V_{Heis}(d)=\delta_{d,1}$

←

Inozemtsev '90
intermediate range
$V_{Ino}(d)\sim\wp(d)$ ~ $(\log\vartheta(d))'$
periods
$(L,\frac{i\pi}{\kappa})\in\mathbb{Z}_{\geq 2}\times i\mathbb{R}_{>0}$

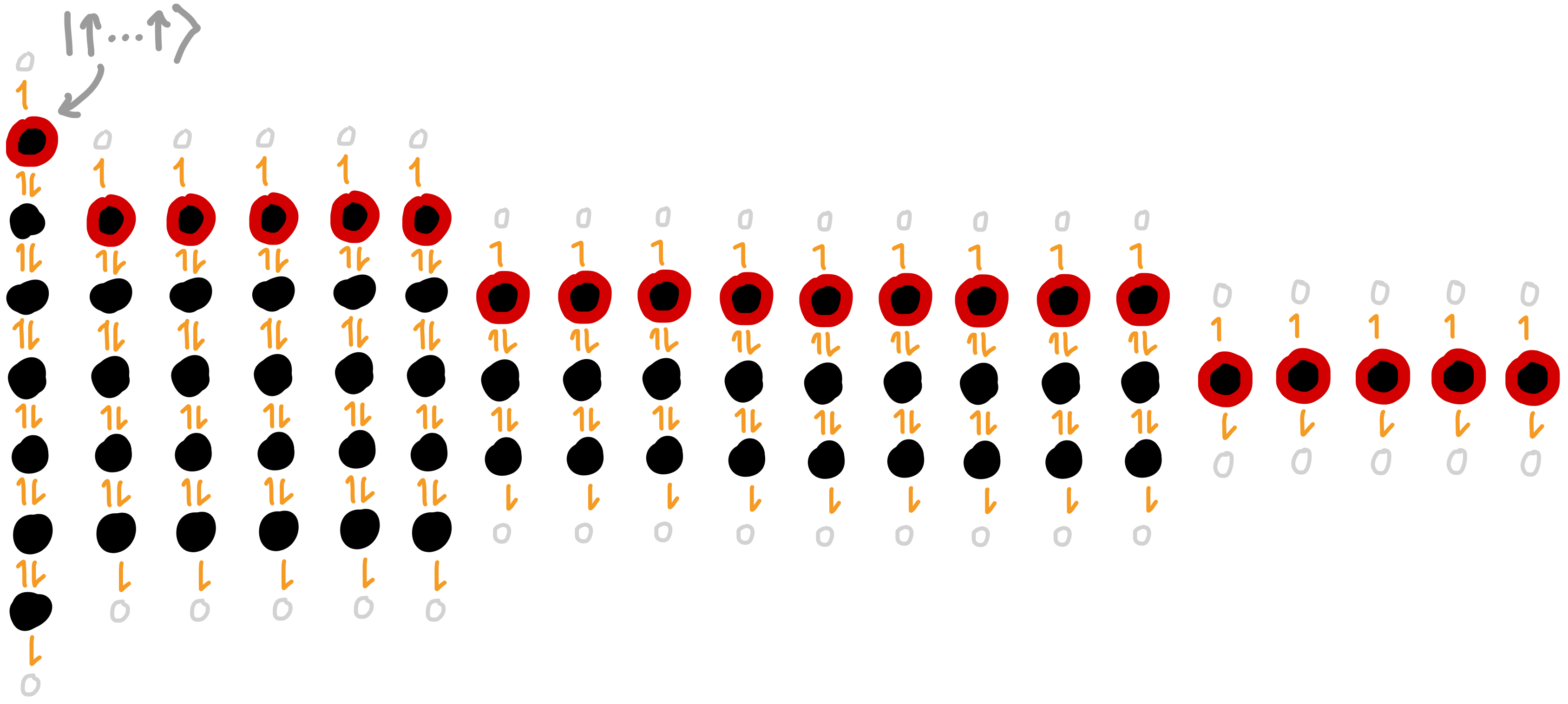
→

Haldane '88 – **Shastry** '88
long range
$$V_{HS}(d)=\frac{(\pi/L)^2}{\sin(\frac{\pi}{L}d)^2}=\frac{1}{\mathrm{crd}^2}$$

$\kappa\to\infty$ $\kappa\to 0$

$\mathfrak{sl}_2$: $S^+ \rightleftharpoons S^-$

wt	$M=\#\downarrow$
L	0
$L-2$	1
$L-4$	2
⋮	
$2-L$	$L-1$
$-L$	L



so it remains to find
$\mathfrak{sl}_2$-highest weight vectors
at $M\geq 1$

Isotropic level Hamiltonians

typical anatomy of spin chain:
(spin $\frac{1}{2}$)

potential

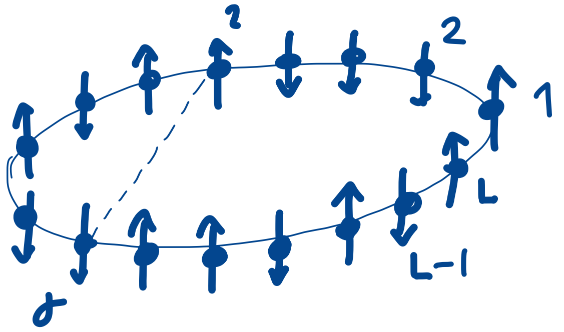
spin 'exchange' interaction

$$H = \sum_{i<j}^{L} V(d_L(i,j))\,\overbrace{(1-P_{ij})}^{(1-\vec{\sigma}_i\,\vec{\sigma}_j)/2}$$

pairwise

homogeneous
(transl invariant)

isotropic
($\mathfrak{sl}_2$-invariant)



e.g.

Heis XXX '28 ← **Inozemtsev '90** → **Haldane '88 – Shastry '88**

nearest neighbour

$V_{\mathrm{Heis}}(d) = \delta_{d,1}$

$\kappa \to \infty$

intermediate range

$V_{\mathrm{Ino}}(d) \sim \wp(d) \sim (\log\vartheta(d))'$

periods

$(L, \frac{i\pi}{\kappa}) \in \mathbb{Z}_{\geq 2} \times i\mathbb{R}_{>0}$

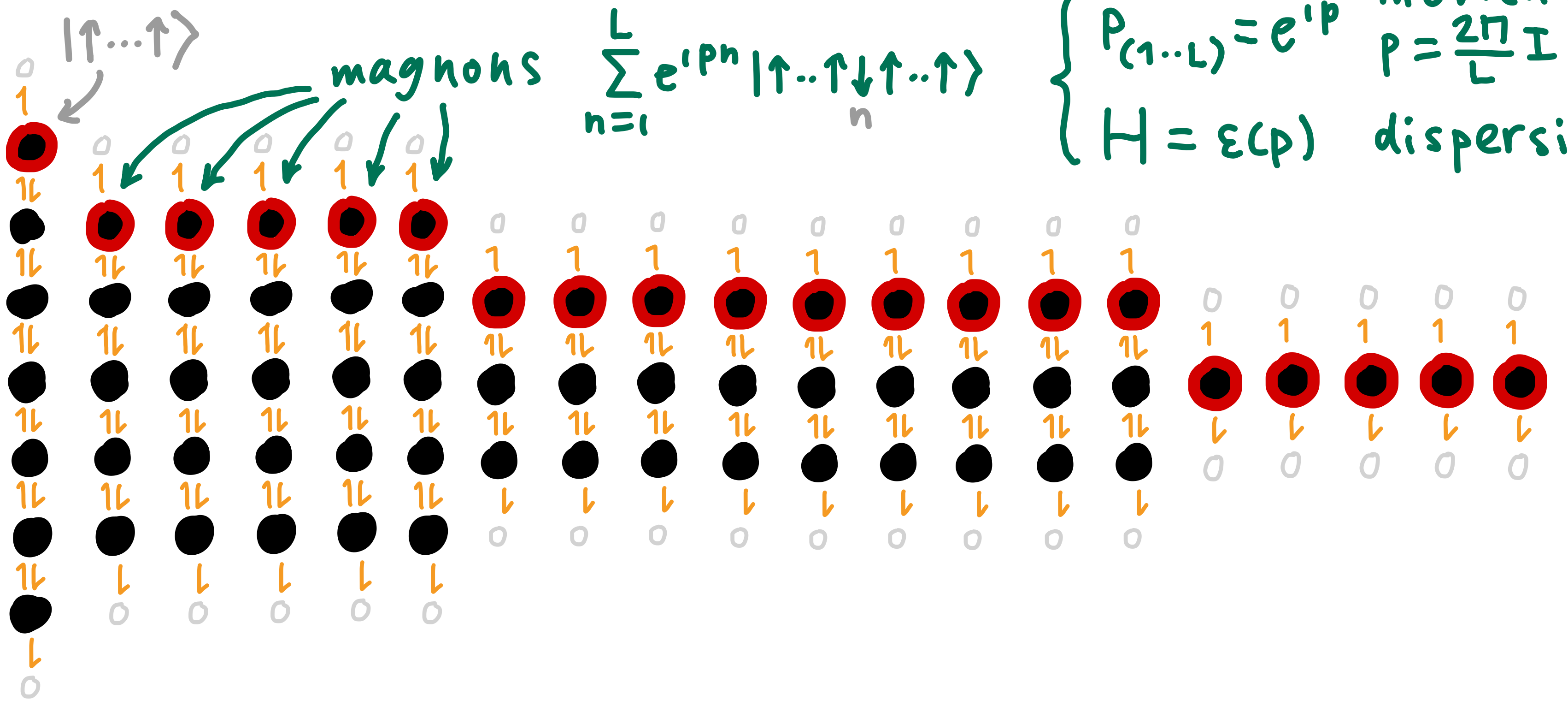
$\kappa \to 0$

long range

$$V_{HS}(d) = \frac{(\pi/L)^2}{\sin(\frac{\pi}{L}d)^2} = \frac{1}{\mathrm{crd}^2}$$

$\mathfrak{sl}_2$: $S^+ \rightleftarrows S^-$

wt	$M = \#\downarrow$
L	0
L-2	1
L-4	2
⋮	
2-L	L-1
-L	L



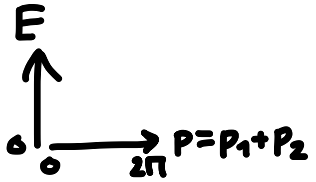
magnons $\sum_{n=1}^{L} e^{ipn}\,|\uparrow..\uparrow\underset{n}{\downarrow}\uparrow..\uparrow\rangle$

$$\begin{cases} P_{(1..L)} = e^{ip} & \text{momentum } p = \frac{2\pi}{L} I \quad 1 \leq I \leq L-1 \\ H = \varepsilon(p) & \text{dispersion} \end{cases}$$

so it remains to find $\mathfrak{sl}_2$-highest weight vectors at $M \geq \cancel{1}\ 2$

Isotropic level M=2

$$\begin{cases} p_{1,2} = \frac{2\pi}{L} I_{1,2} \pm \underset{\text{known}}{\theta(p_1,p_2)} \\ 0 \le I_1 \le I_2 \le L-1 \end{cases}$$

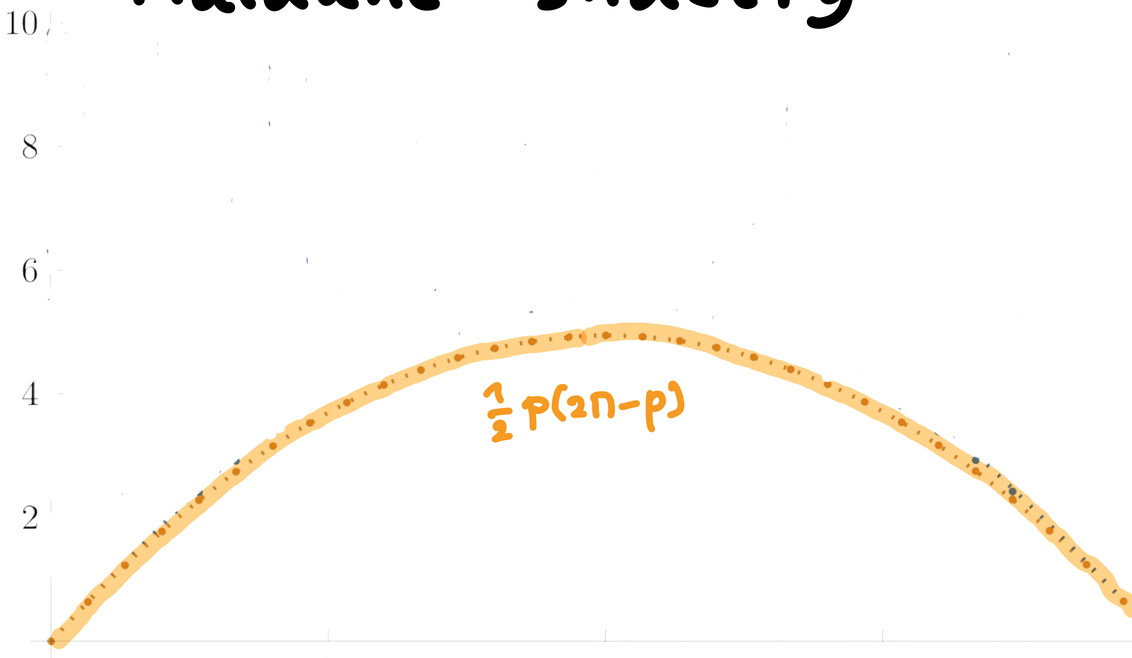


Haldane et al '91-2
Klabbers JL '22

Heis XXX

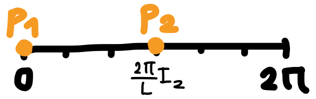
Inozemtsev

Haldane-Shastry



$I_1=0$ # = L

$\theta_{Heis}=0$



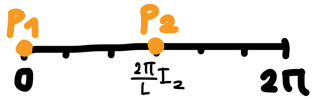
$E_{Heis}=\varepsilon_{Heis}(p_2)$ as at M=1

$\mathcal{Y}_2$-descendant

=

←

$\theta_{Ino}=0$



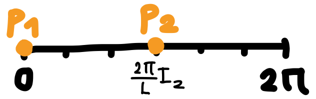
$E_{Ino}=\varepsilon_{Ino}(p_2)$ as at M=1

$\mathcal{Y}_2$-descendant

=

→

$\theta_{HS}=0$



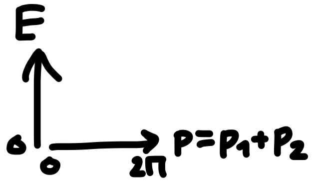
$E_{HS}=\varepsilon_{HS}(p_2)$ as at M=1

$\mathcal{Y}_2$-descendant

$$\sum_{n=1}^{L} e^{ipn} |\uparrow\cdots\uparrow\underset{n}{\downarrow}\uparrow\cdots\uparrow\rangle \quad \begin{cases} P_{(1\cdots L)} = e^{ip} & \text{momentum } p = \frac{2\pi}{L} I \quad 1 \le I \le L-1 \\ H = \varepsilon(p) & \text{dispersion} \end{cases}$$

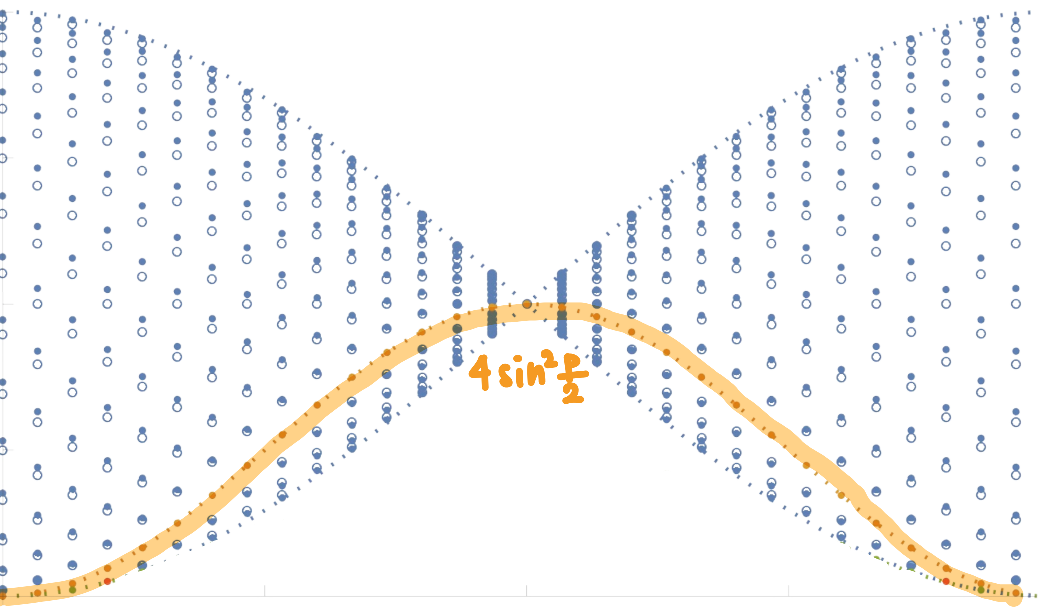
Isotropic level $M=2$

$$\begin{cases} p_{1,2} = \frac{2\pi}{L} I_{1,2} \pm \underset{\text{known}}{\theta(p_1,p_2)} \\ 0 \leq I_1 \leq I_2 \leq L-1 \end{cases}$$

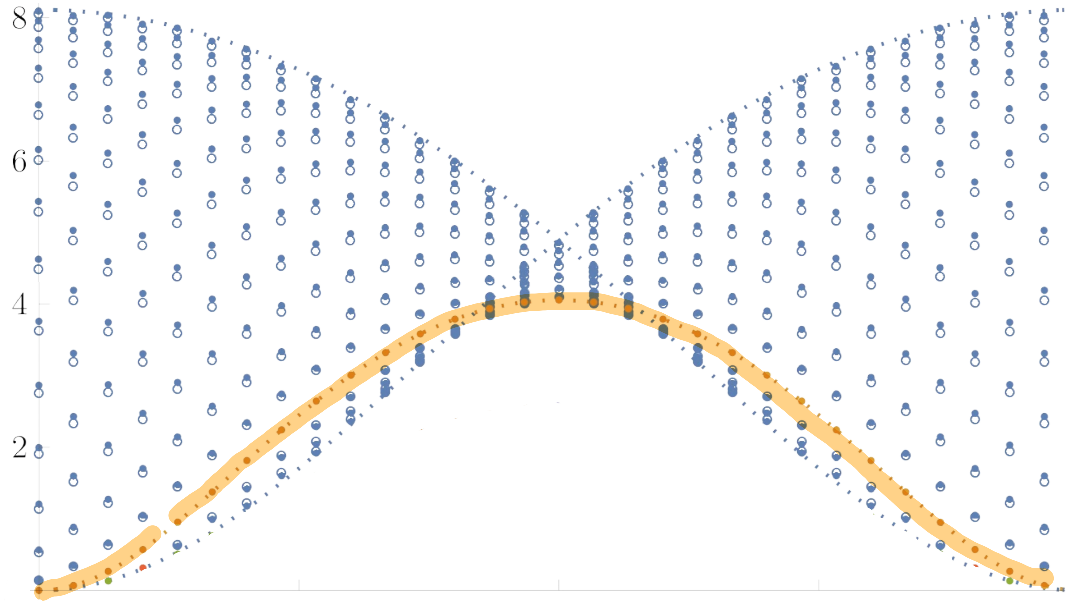


Haldane et al '91-2
Klabbers JL '22

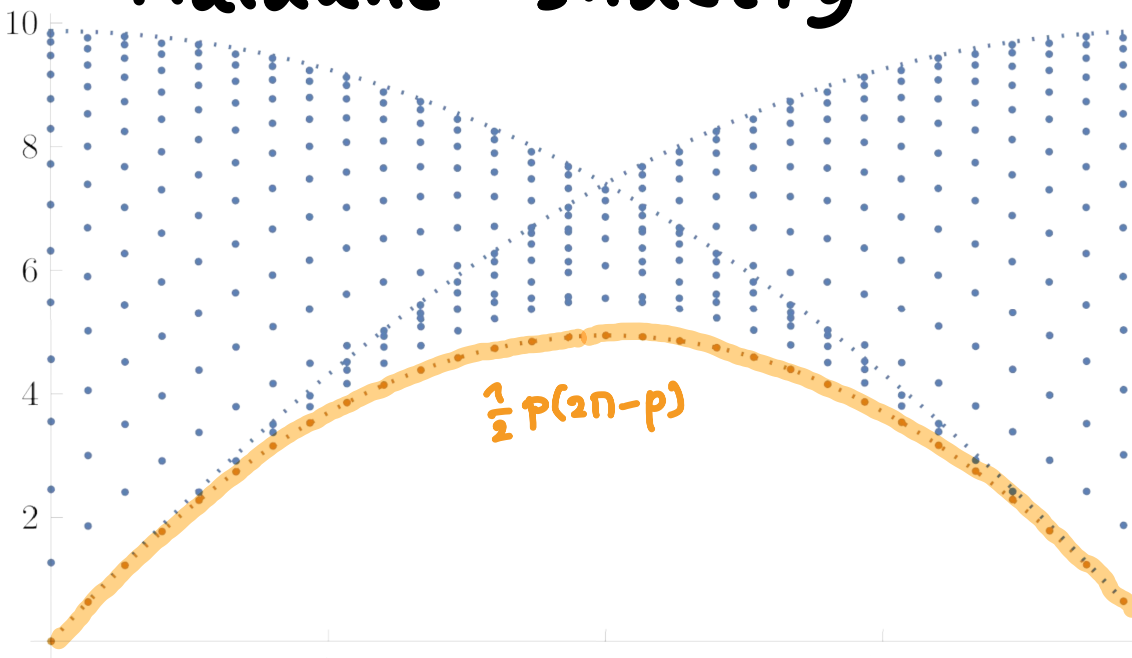
Heis XXX



Inozemtsev



Haldane–Shastry

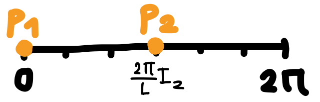


$I_1=0$ $\#=L$

$\theta_{Heis}=0$

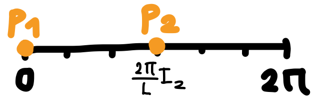
$=$

$\theta_{Ino}=0$



$=$

$\theta_{HS}=0$



$E_{Heis}=\varepsilon_{Heis}(p_2)$ as at $M=1$ ← $E_{Ino}=\varepsilon_{Ino}(p_2)$ as at $M=1$ → $E_{HS}=\varepsilon_{HS}(p_2)$ as at $M=1$

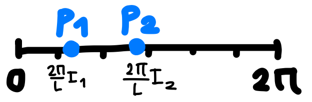
$\mathfrak{sl}_2$-descendant $\mathfrak{sl}_2$-descendant $\mathfrak{sl}_2$-descendant

$\mathfrak{sl}_2$-hw

$I_1>0$, $I_2>I_1+1$

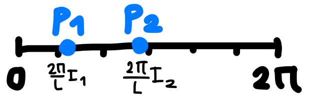
$\#=\binom{L-2}{2}$

$\exists!\,\theta_H \in (0,\frac{\pi}{L})$



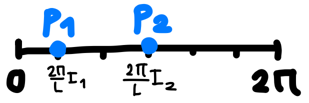
←

$\exists!\,\theta_{Ino} \in (0,\theta_{Heis})$



→

$\theta_{HS}=0$

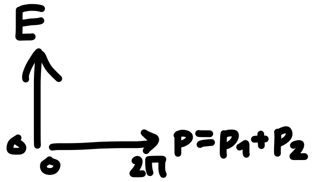


$E_{Heis} \approx \varepsilon_{Heis}(\frac{2\pi}{L}I_1)+\varepsilon_{Heis}(\frac{2\pi}{L}I_2)$ ← $E_{Ino} \approx \varepsilon_{Ino}(\frac{2\pi}{L}I_1)+\varepsilon_{Ino}(\frac{2\pi}{L}I_2)$ → $E_{HS} = \varepsilon_{HS}(\frac{2\pi}{L}I_1)+\varepsilon_{HS}(\frac{2\pi}{L}I_2)$

scattering **scattering**

Isotropic level $M=2$

$$\begin{cases} p_{1,2}=\frac{2\pi}{L}I_{1,2}\pm\underset{\text{known}}{\theta(p_1,p_2)} \\ 0\leq I_1\leq I_2\leq L-1 \end{cases}$$

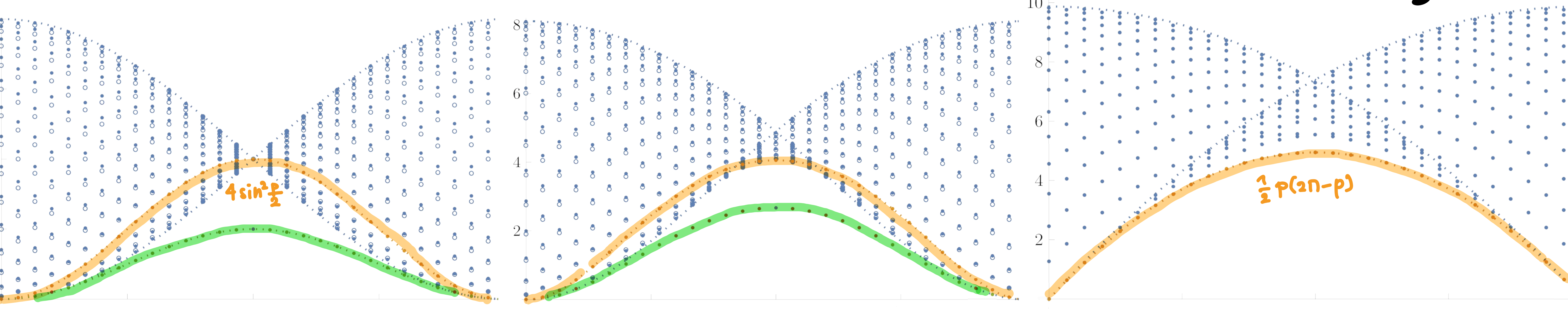


Haldane et al '91-2
Klabbers JL '22

Heis XXX

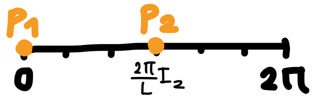
Inozemtsev

Haldane–Shastry



$I_1=0$ $\#=L$

$\theta_{Heis}=0$

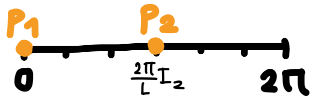


$E_{Heis}=\varepsilon_{Heis}(p_2)$ as at $M=1$

$\mathfrak{sl}_2$-descendant

=

$\theta_{Ino}=0$

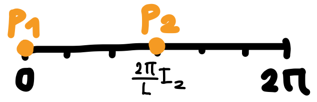


$E_{Ino}=\varepsilon_{Ino}(p_2)$ as at $M=1$

$\mathfrak{sl}_2$-descendant

=

$\theta_{HS}=0$



$E_{HS}=\varepsilon_{HS}(p_2)$ as at $M=1$

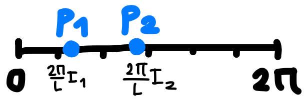
$\mathfrak{sl}_2$-descendant

$\mathfrak{sl}_2$-hw

$I_1>0,\ I_2>I_1+1$

$\#=\binom{L-2}{2}$

$\exists!\,\theta_H\in(0,\frac{\pi}{L})$

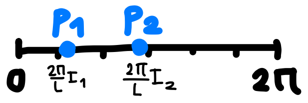


$E_{Heis}\approx\varepsilon_{Heis}(\frac{2\pi}{L}I_1)+\varepsilon_{Heis}(\frac{2\pi}{L}I_2)$

scattering

←

$\exists!\,\theta_{Ino}\in(0,\theta_{Heis})$

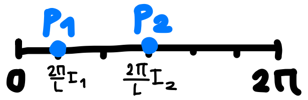


$E_{Ino}\approx\varepsilon_{Ino}(\frac{2\pi}{L}I_1)+\varepsilon_{Ino}(\frac{2\pi}{L}I_2)$

scattering

→

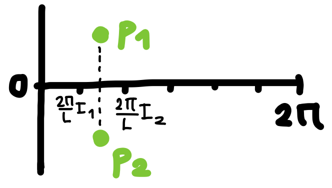
$\theta_{HS}=0$



$E_{HS}=\varepsilon_{HS}(\frac{2\pi}{L}I_1)+\varepsilon_{HS}(\frac{2\pi}{L}I_2)$

$I_1>0$, suitable
$I_2\in\{I_1,I_1+1\}$
$\#=L-3$

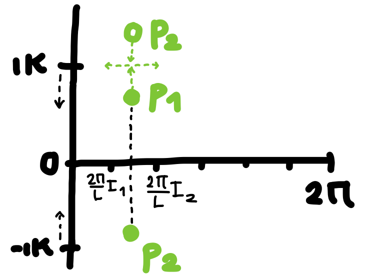
$\exists!\,\theta_{Heis}\in\frac{2\pi}{L}\frac{I_2-I_1}{2}+i\,\mathbb{R}_{>0}$ L suff low



$E_{Heis}\approx\frac{1}{2}\varepsilon_{Heis}(\frac{2\pi}{L}(I_1+I_2))$

bound state

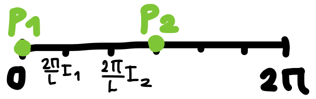
←



$E_{Ino}>E_{Heis}$

→

$\theta_{HS}=-\frac{2\pi}{L}I_1$



$E_{HS}=\varepsilon_{HS}(p_2)$ as at $M=1$

Isotropic level hidden symmetry of HS

$$H_{HS} = \sum_{i<j}^{L} \frac{(\pi/L)^2}{\sin(\frac{\pi}{L}(i-j))^2}(1-P_{ij})$$

Ha et al '92
Bernard et al '93

Yangian symmetry

$\mathfrak{sl}_2$ $\qquad S^{\pm} = \sum_{i=1}^{L} \sigma_i^{\pm} \qquad S^{z} = \sum_{i=1}^{L} \frac{\sigma_i^{z}}{2}$

$$\begin{cases} [S^z, S^{\pm}] = \pm S^{\pm} \\ [S^+, S^-] = 2 S^z \end{cases}$$

$\cap$

$Y\mathfrak{sl}_2$

(∞-dim)

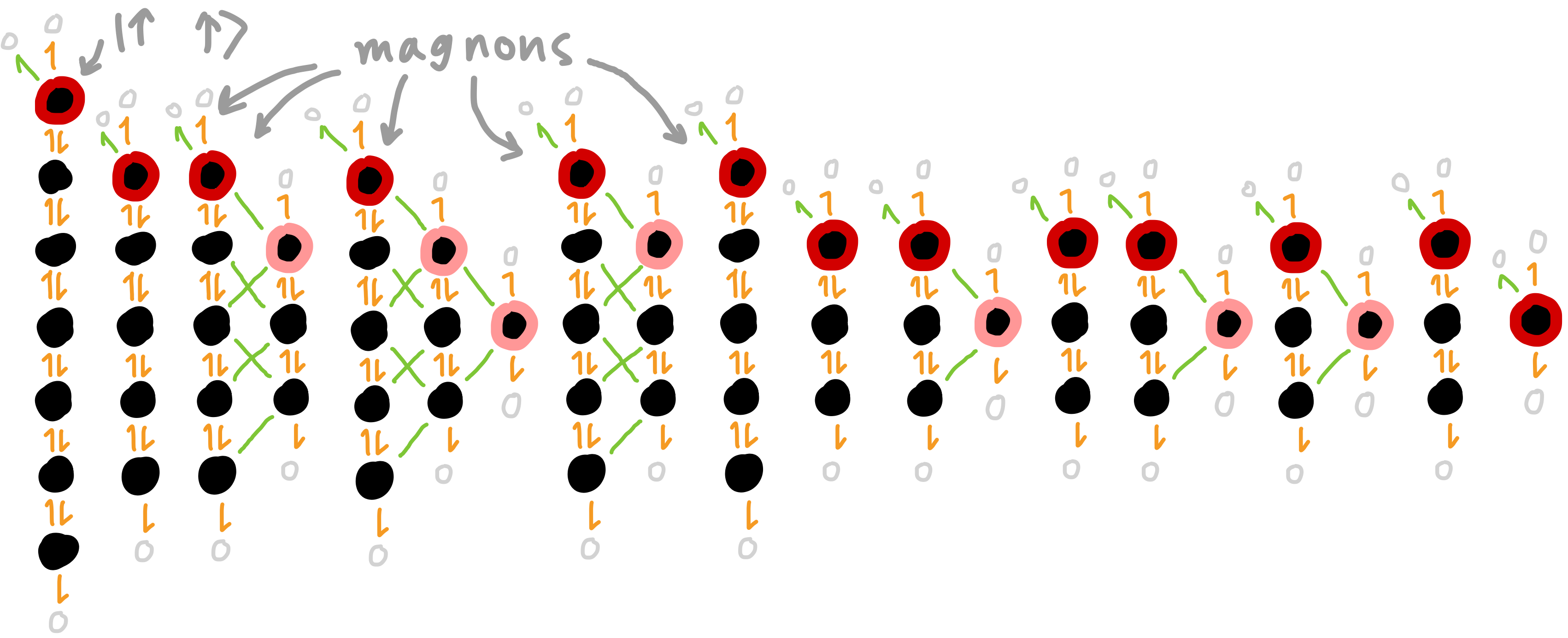
$$Q^{\pm} = \pm\frac{1}{2}\sum_{i<j}^{L} \cot\left(\tfrac{\pi}{L}(i-j)\right)\left(\sigma_i^z\sigma_j^{\pm} - \sigma_i^{\pm}\sigma_j^{z}\right)$$

$$Q^{z} = \frac{1}{2}\sum_{i<j}^{L} \cot\left(\tfrac{\pi}{L}(i-j)\right)\left(\sigma_i^+\sigma_j^{-} - \sigma_i^{-}\sigma_j^{+}\right)$$

$$\begin{cases} [S^z, Q^{\pm}] = \pm Q^{\pm} \\ [S^{\pm}, Q^{\mp}] = \pm 2 Q^z \\ [S^{\pm}, Q^{z}] = \mp Q^{\pm} \\ \text{\& Serre-like relation} \end{cases}$$

organises $(\mathbb{C}^2)^{\otimes L}$ into irreps

wt	$M = \#\downarrow$
L	0
$L-2$	1
$L-4$	2
$\vdots$	
$2-L$	$L-1$
$-L$	L

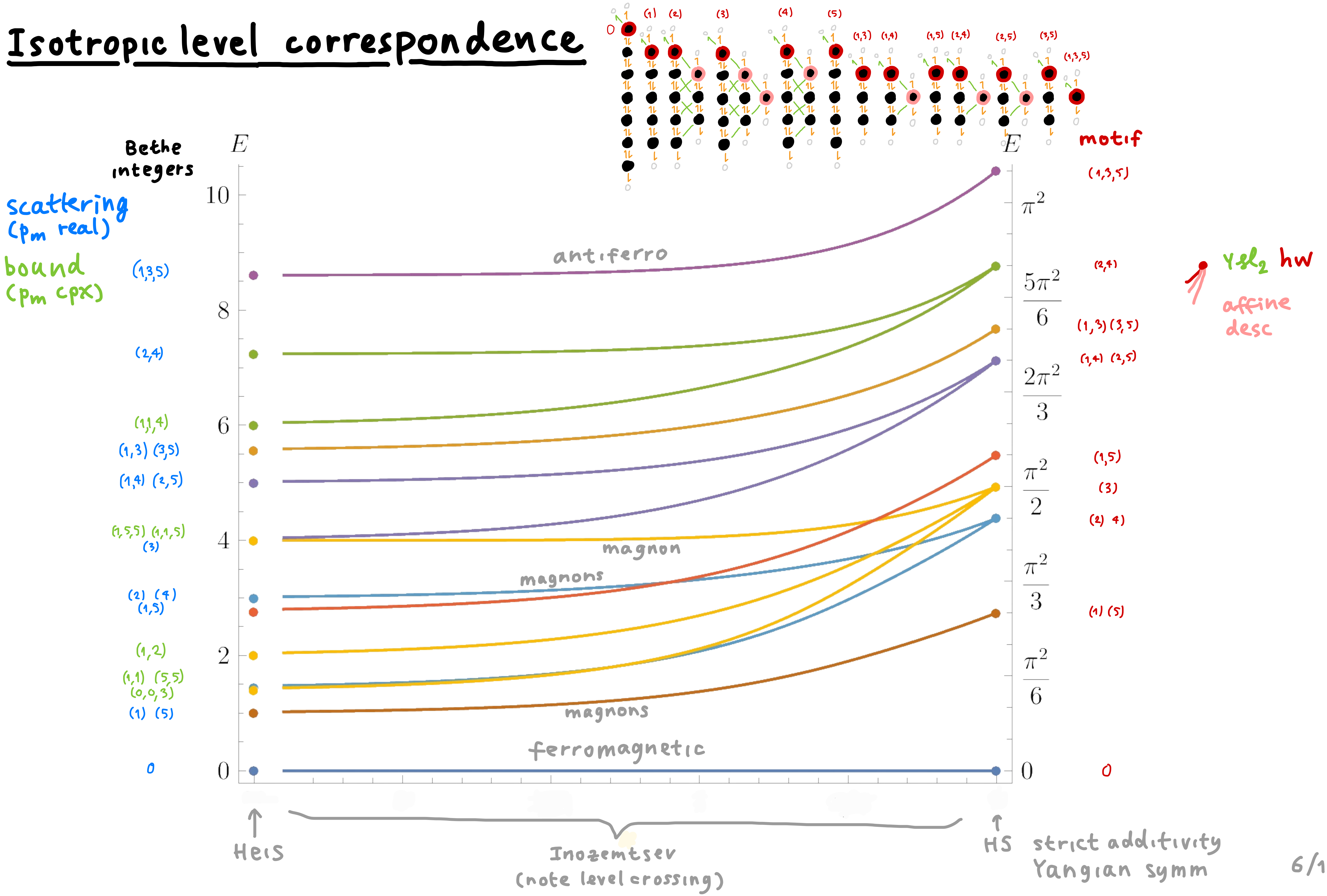


so it remains to find $Y\mathfrak{sl}_2$-highest weight vectors at $M \geq \cancel{1}\ 2$

labelled by motifs

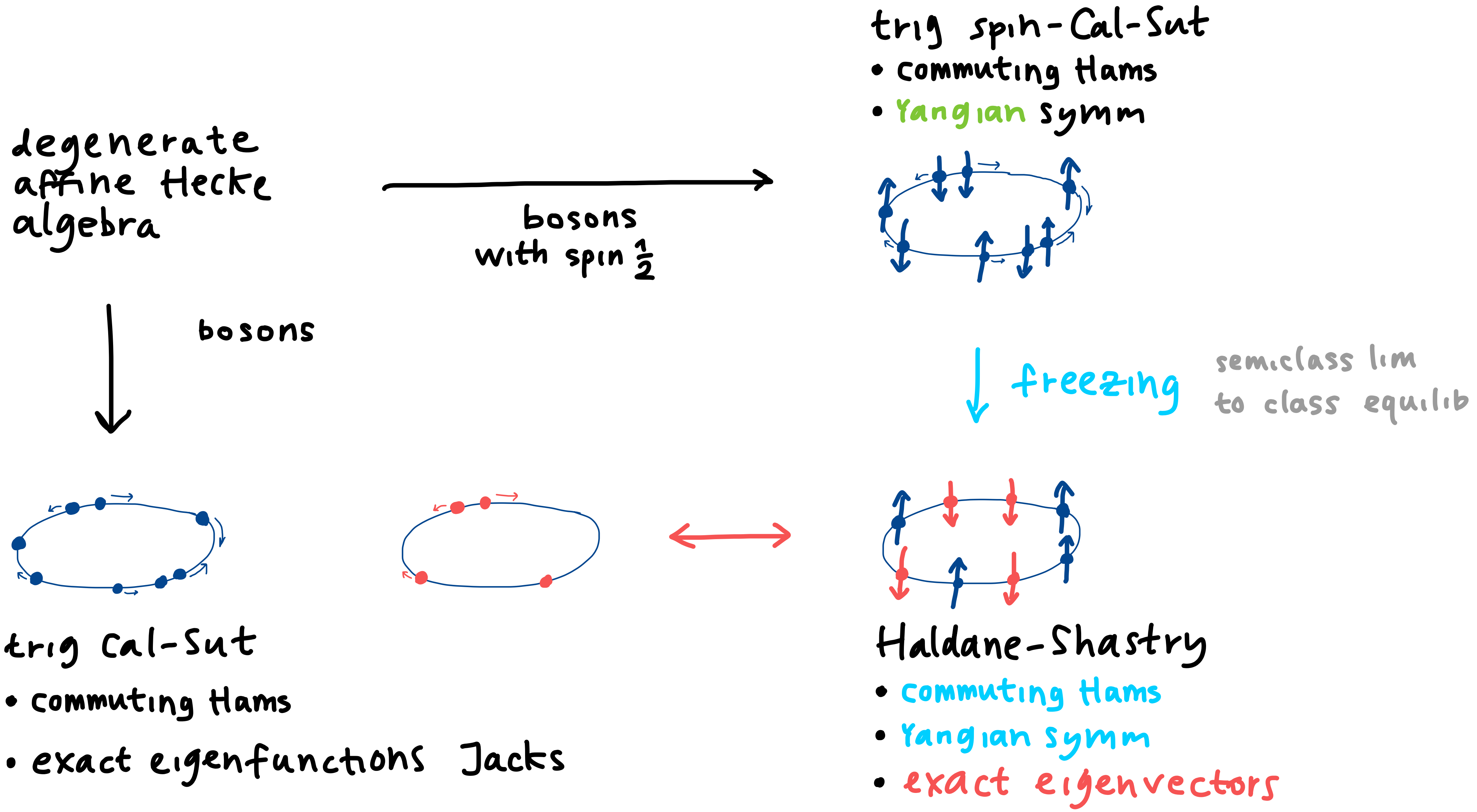
$(\mu_1, \dots, \mu_M): \mu_{m+1} > \mu_m + 1$

Isotropic level correspondence



Isotropic level algebraic structure of HS

Bernard et al '93
Talstra Haldane '95



degenerate affine Hecke algebra

bosons with spin $\frac{1}{2}$

bosons

trig spin-Cal-Sut
- commuting Hams
- Yangian symm

freezing

semiclass lim to class equilib

trig Cal-Sut
- commuting Hams
- exact eigenfunctions Jacks

Haldane-Shastry
- commuting Hams
- Yangian symm
- exact eigenvectors

Isotropic level algebraic structure of HS

Bernard et al '93
Talstra Haldane '95

permutations
Dunkl operators
$\Uparrow$
degenerate affine Hecke algebra

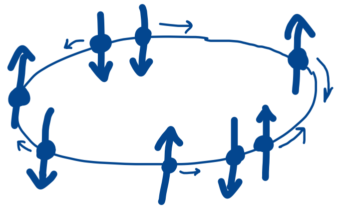
$\xrightarrow{k=\alpha^{-1}\in\mathbb{C}}$

N bosons with spin $\frac{1}{2}$

$\mathrm{Sym}^N\mathbb{C}^2[e^{ix}]$
$=(\mathbb{C}^2)^{\otimes N}\underset{S_N}{\otimes}\mathbb{C}[e^{ix_1},\ ,e^{ix_N}]$

trig spin-Cal-Sut

- commuting Hams
- Yangian symm



$$\tilde{H}_{tCS}=-\frac{1}{2}\sum_{j=1}^{N}\partial_{x_j}^2+\sum_{i<j}^{N}\frac{k(k-P_{ij})}{4\sin((x_i-x_j)/2)^2}$$

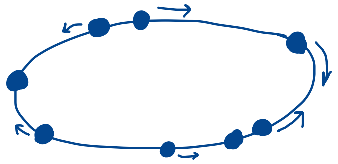
$\exists Q^\alpha$ Dunkls instead of inhomogenities

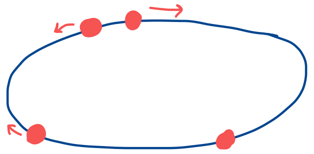
$k=\alpha^{-1}\in\mathbb{C}$
reduced coupling
$\downarrow$
N bosons
$\mathrm{Sym}^N\mathbb{C}[e^{ix}]$
$=\mathbb{C}[e^{ix_1},\ ,e^{ix_N}]^{S_N}$

$\downarrow$ freezing

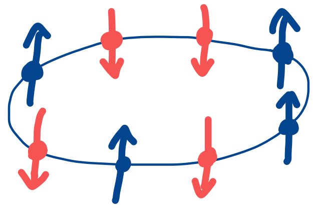
semiclass lim
to class equilib

$N^*=L$
$k\to\infty$
$x_j^*=\frac{2\pi}{L}j$





$\longleftrightarrow$
$N^*=M$
$k^*=2$
$x_m^*=\frac{\pi}{L}n_m$



trig Cal-Sut

- commuting Hams
- exact eigenfunctions Jacks

$$\tilde{P}_{CS}=-i\sum_{j=1}^{N}\partial_{x_j}$$

$$\tilde{H}_{tCS}=-\frac{1}{2}\sum_{j=1}^{N}\partial_{x_j}^2+\sum_{i<j}^{N}\frac{\overbrace{k(k-1)}^{\text{coupling}}}{4\sin((x_i-x_j)/2)^2}$$

Haldane-Shastry

- commuting Hams
- Yangian symm
- exact eigenvectors

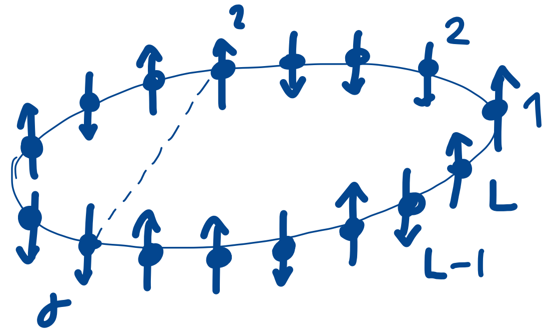
$$H_{HS}=\sum_{i<j}^{L}\frac{(\pi/L)^2(1-P_{ij})}{\sin(\frac{\pi}{L}(i-j))^2}$$

$$\Psi(n_1,\ ,n_M)=\tilde{\Psi}(\omega^{n_1},\ ,\omega^{n_M}),\quad \omega=e^{2\pi i/L}$$

$$\underset{\underset{\text{motif}}{\uparrow}}{\tilde{\Psi}_{\mu}}(z_1,\ ,z_M)=\underbrace{\prod_{m<m'}^{M}(z_m-z_{m'})^2}_{\substack{\text{Vandermonde}\\ \text{squared}}}\times\underbrace{P_{\lambda(\mu)}^{(\frac{1}{2})}(z_1,\ ,z_M)}_{\substack{\text{Jack polynomial}\\ \text{(zonal spherical)}}}$$

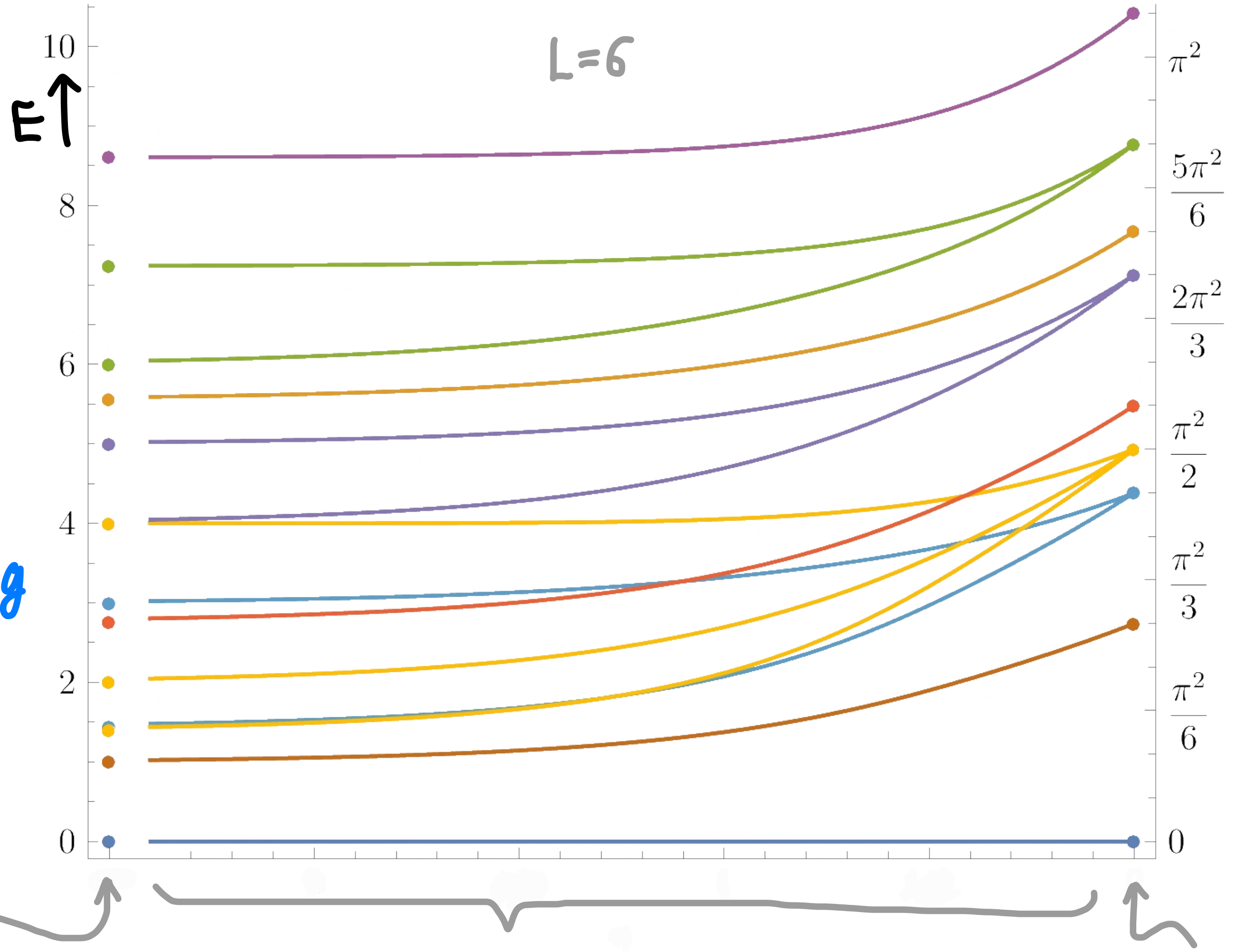
Isotropic level summary

$$H = \sum_{i<j}^{L} V(d_L(i,j))\,(1-P_{(i,j)})$$



scattering (pm real)

bound (pm cpx)



Y𝔰𝔩₂ hw

affine desc

exact solvability (spectrum)

↑

(Yang-Baxter) **quantum integrab**

↓

higher Ham's

Heis XXX '28

nearest neighbour

Bethe ansatz; up to solving BAE

Bethe '31

↑ alg Bethe ansatz

Yangian structure

Faddeev et al, late '70s

↓ transfer matrix

known

Inozemtsev '90

intermediate range

mix of both methods; up to solving BAE

Inozemtsev '90, '95, '00

Klabbers JL '22

unknown

conjectured, partial proof

Dittrich Inozemtsev '08

Haldane '88 – Shastry '88

long range

connection to Cal-Sut; in closed form

Haldane '91

↑

degenerate affine Hecke alg

Yangian symmetry

Bernard et al '93

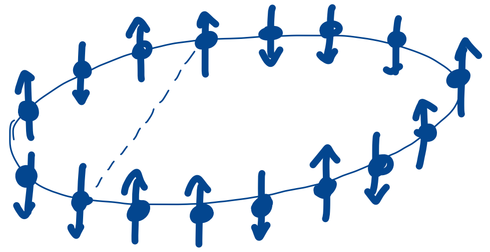
↓

known

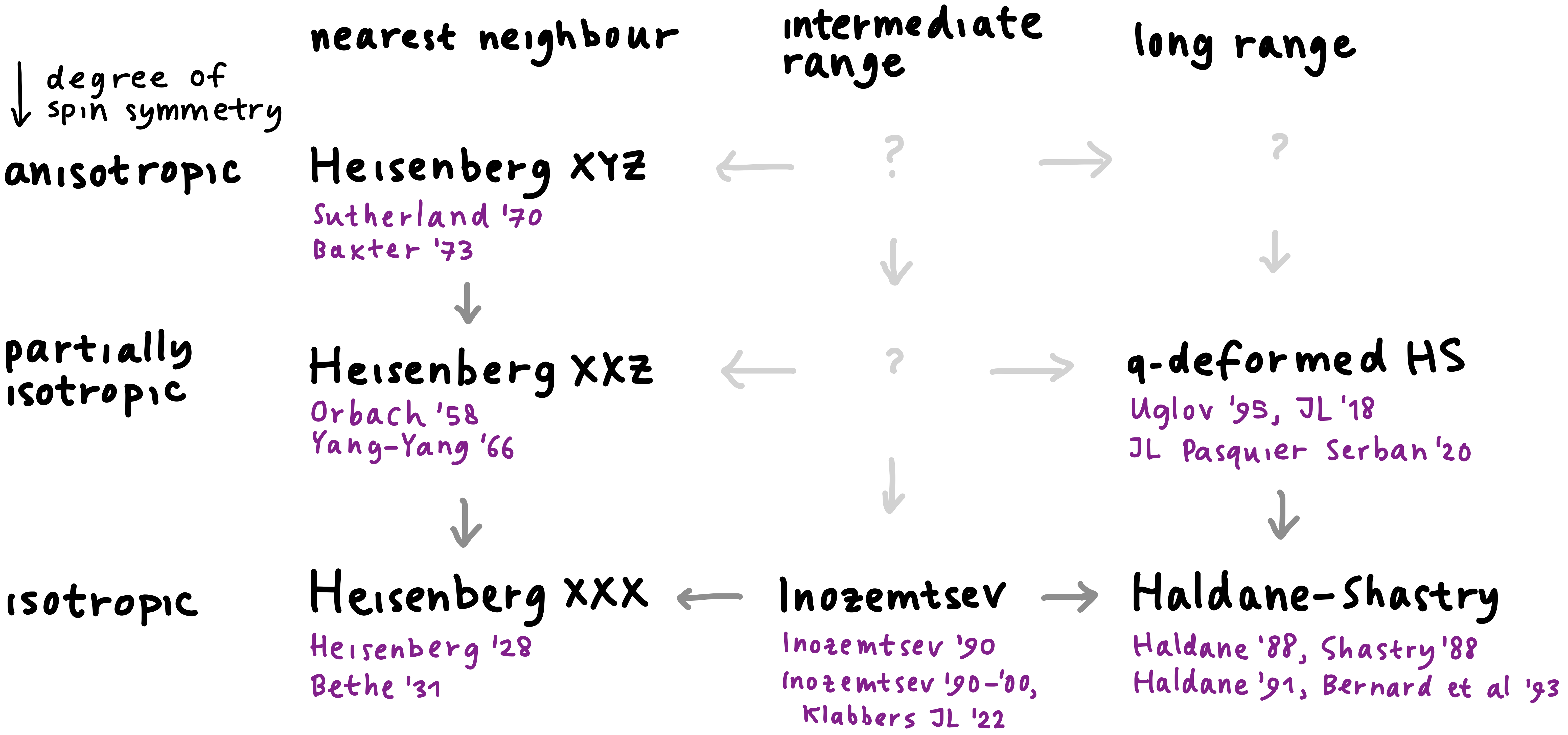
Bernard et al '93

Talstra Haldane '95

Overview landscape of long-range spin chains

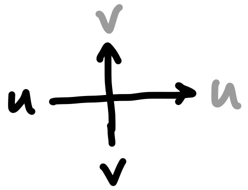


interaction range →

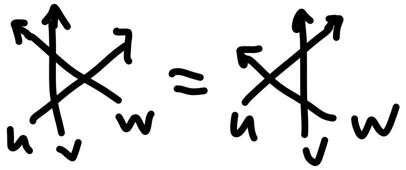


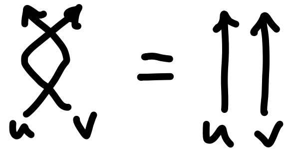
↓ degree of spin symmetry	nearest neighbour		intermediate range		long range
anisotropic	**Heisenberg XYZ** Sutherland '70 Baxter '73	←	?	→	?
	↓		↓		↓
partially isotropic	**Heisenberg XXZ** Orbach '58 Yang–Yang '66	←	?	→	**q-deformed HS** Uglov '95, JL '18 JL Pasquier Serban '20
	↓		↓		↓
isotropic	**Heisenberg XXX** Heisenberg '28 Bethe '31	←	**Inozemtsev** Inozemtsev '90 Inozemtsev '90–'00, Klabbers JL '22	→	**Haldane–Shastry** Haldane '88, Shastry '88 Haldane '91, Bernard et al '93

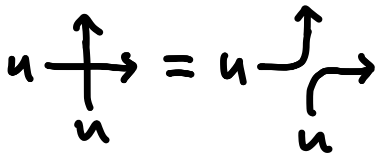
Partially isotropic level inhomogeneous Heis XXZ

$R(u/v) =$


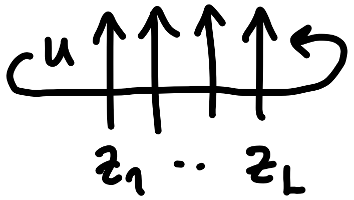
$= P\left(1-\frac{u-v}{qu-q^{-1}v}\,e^{TL}\right)$ six-vertex

Yang-Baxter


unitarity


'initial'


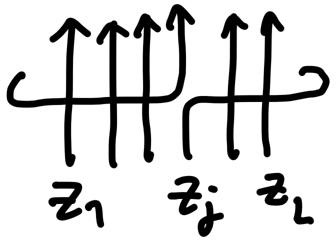
$= P$

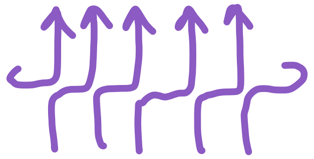
$t(u;\underline{z}) =$


$= tr_0\left[R_{0L}(u/z_L)\cdot\cdot R_{01}(u/z_1)\right]$

$[t(u;\underline{z}), t(v;\underline{z})] = 0$

conserved charges:

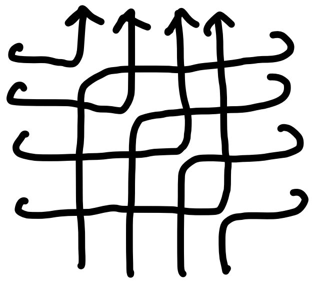
$G_j(\underline{z}) := t(z_j;\underline{z}) =$


$\longrightarrow$


$= R_{jj-1}(z_j/z_{j-1})\cdot\cdot R_{j1}(z_j/z_1)$
$\times R_{jL}(z_j/z_L)\cdot\cdot R_{jj+1}(z_j/z_{j+1})$

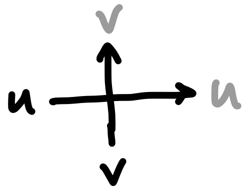
$z_j \mapsto 1$

$[G_i, G_j] = 0$

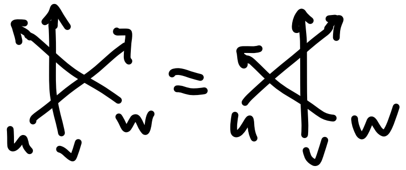
$G_1 \cdots G_L =$

$= 1$

$\downarrow$
G^L

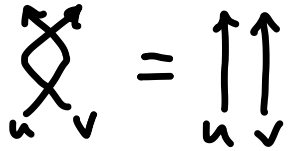
Partially isotropic level inhomogeneous Heis XXZ

$R(u/v) =$  $= P\left(1 - \frac{u-v}{qu - q^{-1}v} e^{TL}\right)$ six-vertex

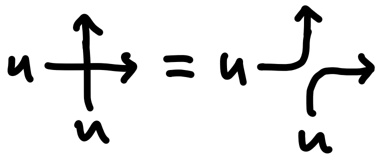
Yang-Baxter

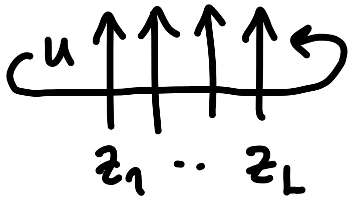


unitarity



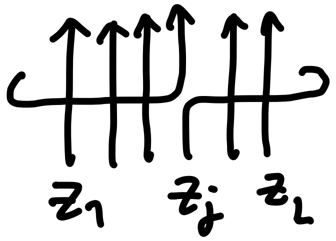
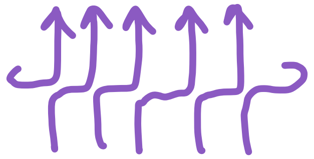
'initial'

 $= P$

$t(u;\underline{z}) =$  $= tr_0\left[R_{0L}(u/z_L)\cdots R_{01}(u/z_1)\right]$

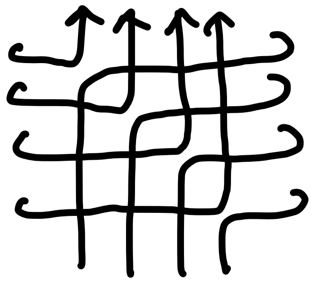
$[t(u;\underline{z}), t(v;\underline{z})] = 0$

conserved charges:

$G_j(\underline{z}) := t(z_j;\underline{z}) =$  →  $z_j \mapsto 1$

$= R_{jj-1}(z_j/z_{j-1})\cdots R_{j1}(z_j/z_1)$
$\times R_{jL}(z_j/z_L)\cdots R_{jj+1}(z_j/z_{j+1})$

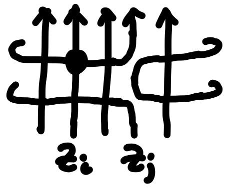
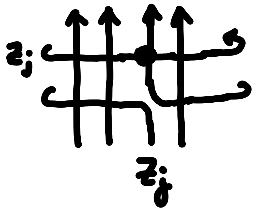
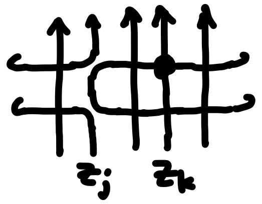
$[G_i, G_j] = 0$

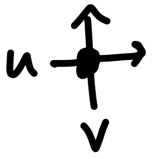
$G_1 \cdots G_L =$  $= 1$

$\downarrow$

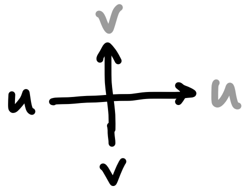
G^L

$H_j(\underline{z}) := \partial_u|_{u=z_j} \log t(u;\underline{z}) = t'(z_j;\underline{z})\, G_i(\underline{z})^{-1}$

$= \sum_{i(<j)}$  $+$  $+ \sum_{k(>j)}$ 

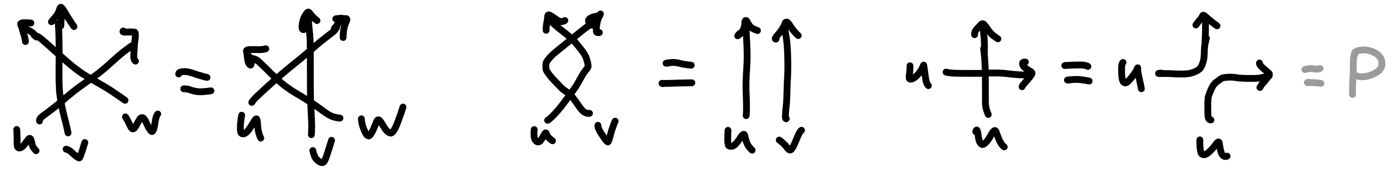
 $= R'(u/v)$

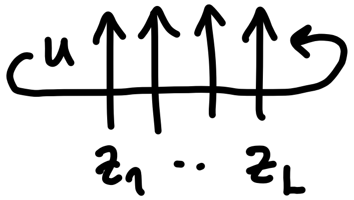
Partially isotropic level inhomogeneous Heis XXZ

$R(u/v) =$ 

$= P\left(1 - \frac{u-v}{qu-q^{-1}v}e^{TL}\right)$ six-vertex

Yang-Baxter unitarity 'initial'

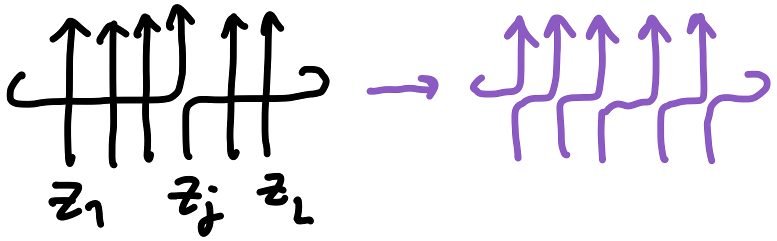


$t(u;\underline{z}) =$ 

$= tr_0\left[R_{0L}(u/z_L)\cdots R_{01}(u/z_1)\right]$

$[t(u;\underline{z}), t(v;\underline{z})] = 0$

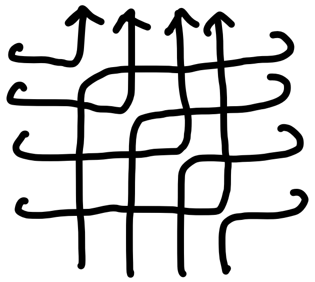
conserved charges:

$G_j(\underline{z}) := t(z_j;\underline{z}) =$ 

$z_j \mapsto 1$

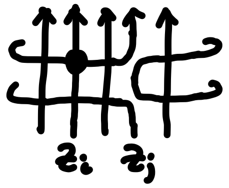
$= R_{jj-1}(z_j/z_{j-1})\cdots R_{j1}(z_j/z_1)$
$\times R_{jL}(z_j/z_L)\cdots R_{jj+1}(z_j/z_{j+1})$

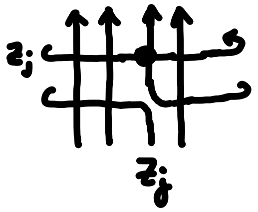
$[G_i, G_j] = 0$

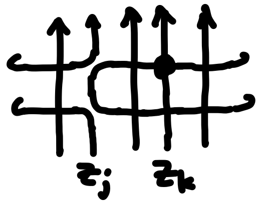
$G_1\cdots G_L =$  $= 1$

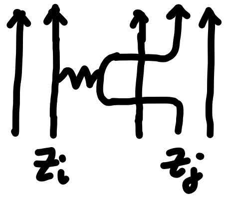
$\downarrow$
G^L

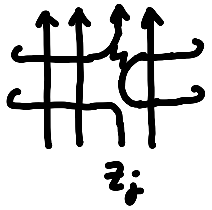
$H_j(\underline{z}) := \partial_u|_{u=z_j} \log t(u;\underline{z}) = t'(z_j;\underline{z})G_i(\underline{z})^{-1}$

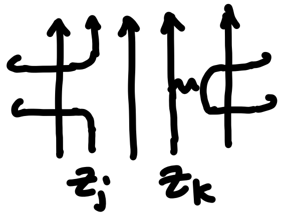
$= \sum_{i(<j)}$ 

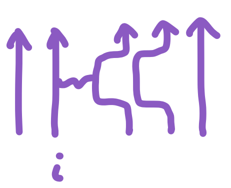
$+$ 

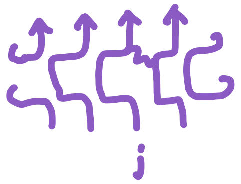
$+ \sum_{k(>j)}$ 

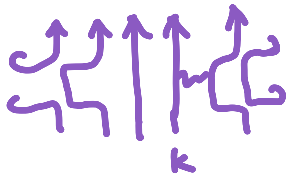
$\propto \sum_{i(<j)} \frac{V(z_i,z_j)}{V(1,1)} \times$ 

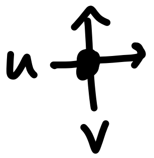
$+$ 

$+ \sum_{k(>j)} \frac{V(z_j,z_k)}{V(1,1)} \times$ 

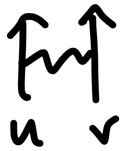








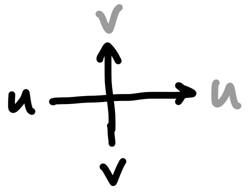
$= R'(u/v)$



$= -(q-q^{-1})PR'(1) = e^{TL}$

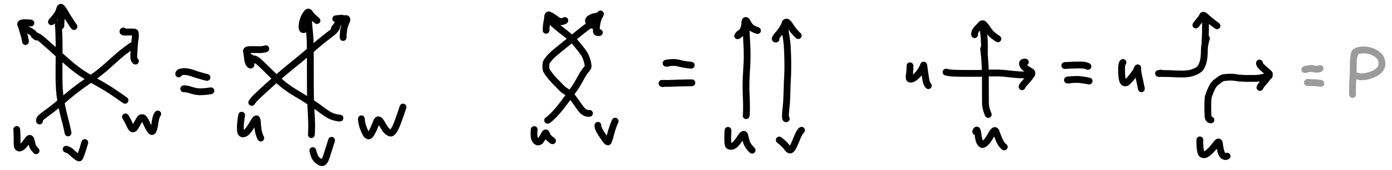
$$V(z,z') = \frac{-zz'}{(qz-q^{-1}z')(q^{-1}z-qz')}$$

Partially isotropic level q-deformed HS

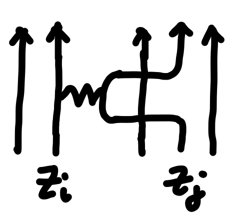
$R(u/v) =$ 

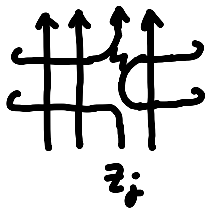
$= P\left(1 - \frac{u-v}{qu-q^{-1}v}\, e^{TL}\right)$ six-vertex

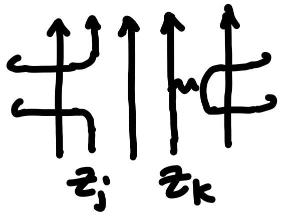
Yang-Baxter unitarity 'initial'

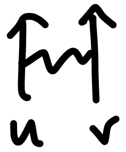


inhomog Heis XXZ:

$H_j(z) \propto \sum_{i(<j)} \frac{V(z_i,z_j)}{V(1,1)} \times$ 

$+$ 

$+ \sum_{k(>j)}^{L} \frac{V(z_j,z_k)}{V(1,1)} \times$ 

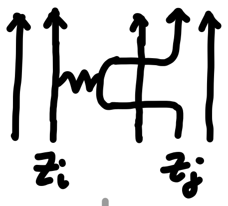


$= -(q-q^{-1})\, P\, R'(1) = e^{TL}$

$$V(z,z') = \frac{-z\,z'}{(qz-q^{-1}z')(q^{-1}z-qz')}$$

exact spectrum from alg Bethe ansatz
up to solving BAE

q-deformed Haldane-Shastry:

$H^{qHS} \propto \sum_{i<j}^{L} \Big[V(z_i,z_j) \times$ 

$\Big]_{z_j \mapsto (e^{2\pi i/L})^j}$

$\downarrow$ H^{HS}

$\downarrow$ $\frac{1}{4\sin(\frac{\pi}{L}(i-j))^2}$

$q \to 1$

$\downarrow$ $1-P_{ij}$

exact spectrum via connection to Ruijsenaars-Macdonald
in closed form

Partially isotropic level q-deformed HS

Bernard et al '93
Uglov '95, JL '18
JL Pasquier Serban '20

$q \in \mathbb{C}$ q-permutations T_i
q-Dunkl operators Y_i
$\cap$

affine Hecke algebra

$p = q^{2/k} \in \mathbb{C}$
$q_{\text{Macd}} = t_{\text{Macd}}^{1/k}$

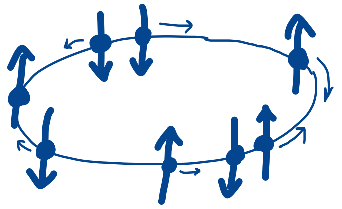
$\longrightarrow$

N q-bosons with spin $\frac{1}{2}$

$\mathrm{Sym}_q^N \mathbb{C}^2[e^{ix}]$
$= (\mathbb{C}^2)^{\otimes N} \otimes_{H_N} \mathbb{C}[e^{ix_1}, \ , e^{ix_N}]$

trig spin-Ruijsenaars-Macdonald

- commuting Hams explicit
- quantum-loop symm q-Dunkls instead of inhomogenities



$p = q^{2/k} \in \mathbb{C}$
$q_{\text{Macd}} = t_{\text{Macd}}^{1/k}$

$\downarrow$

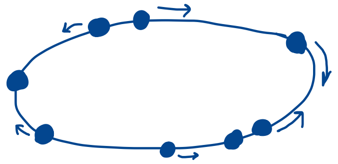
relativistic
N (q-)bosons

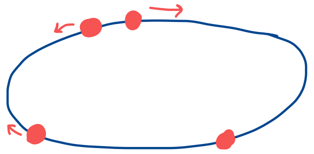
$\mathrm{Sym}_q^N \mathbb{C}[e^{ix}]$
$= \mathbb{C}[e^{ix_1}, \ , e^{ix_N}]^{H_N}$
$= \mathbb{C}[e^{ix_1}, \ , e^{ix_N}]^{S_N}$

$\downarrow$ freezing

semiclass lim
to class equilib

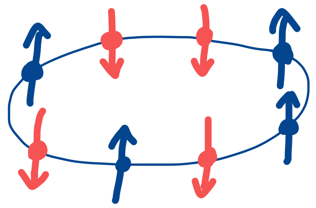
$N^* = L$
$k \to \infty$
$x_j^* = \frac{2\pi}{L} j$





$\longleftrightarrow$

$N^* = M$
$k^* = 2$
$x_m^* = \frac{\pi}{N} n_m$



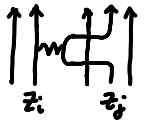
trig Ruijsenaars-Macdonald

- commuting Hams explicit
- exact eigenfunctions Macdonald

q-deformed Haldane-Shastry

- commuting Hams
- quantum-loop symm
- exact eigenvectors

$$\sum_{i<j}^{L} \Big[V(z_i, z_j) \times$$



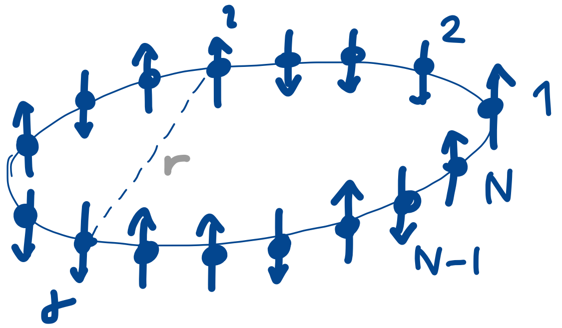
$$\Big]_{z_j \mapsto (e^{2\pi i/L})^j}$$

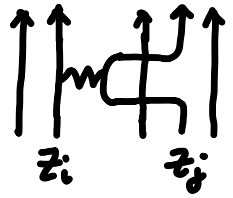
$$\tilde{\Psi}_{\mu}(z_1, \ , z_M) = \Big[\underbrace{\prod_{m<m'}^{M} (q z_m - q^{-1} z_{m'}) \times (q^{-1} z_m - q z_{m'})}_{\text{`symmetric square' of q-Vandermonde}} \times \underbrace{P_{\lambda(\mu)}(z_1, \ , z_M, q^{2/2}, q^2)}_{\text{Macdonald polynomial (quantum zonal spherical)}} \Big]_{z_j \mapsto (e^{2\pi i/L})^j}$$

$\uparrow$ motif

rmk rep-th framework:
affine Schur-Weyl duality

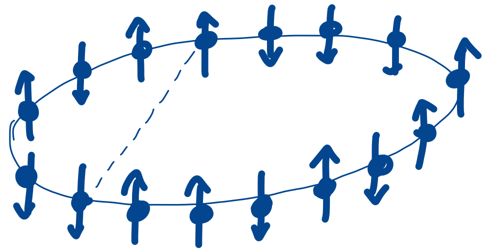
Partially isotropic level summary

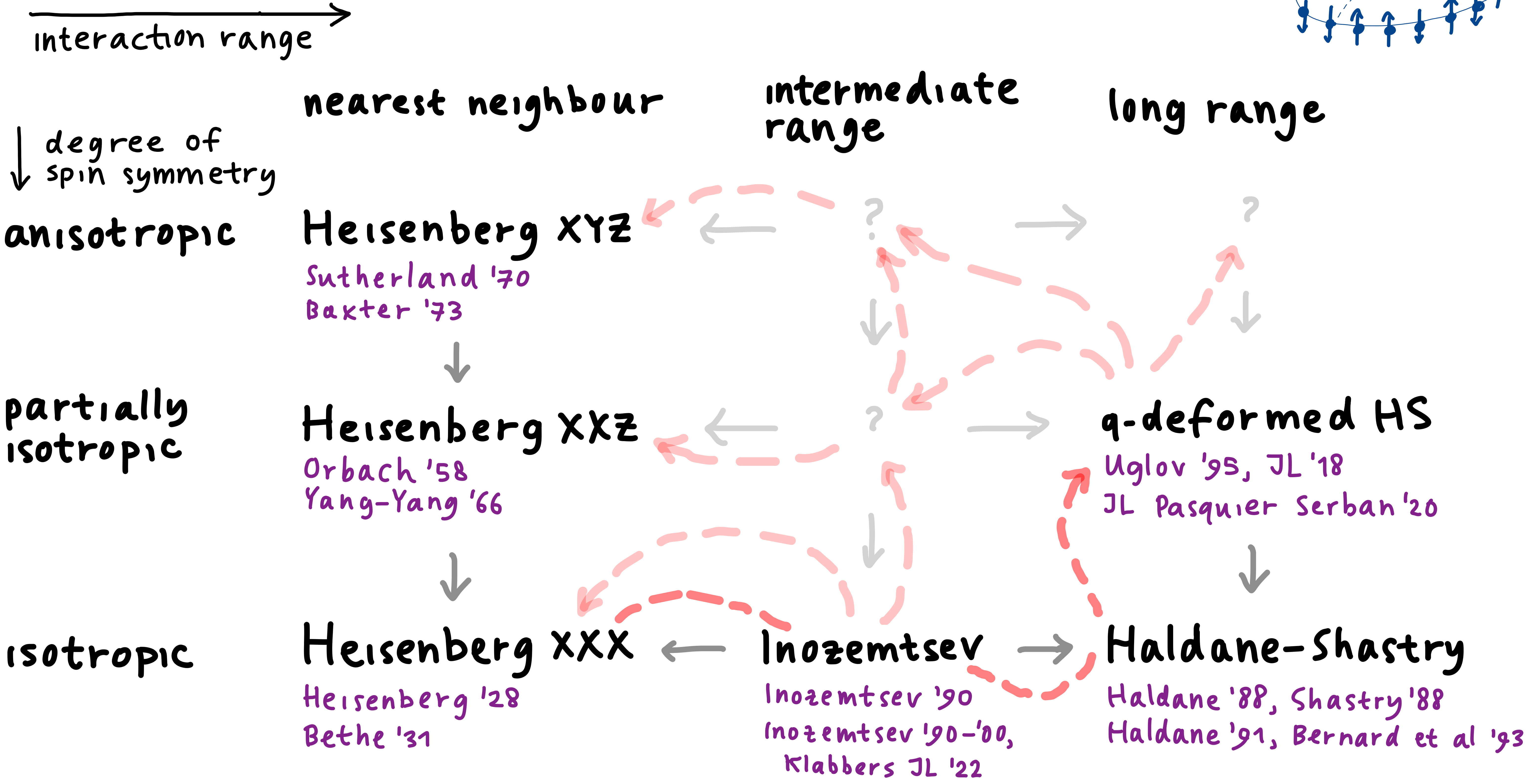


$H^{qHS} \propto \sum_{i<j}^{L} \Big[V(z_i, z_j) \times$  $\Big]_{z_j \mapsto (e^{2\pi i/L})^j}$

	homog: $z_j \mapsto 1$ **Heis XXZ** Orbach '58 nearest neighbour	unknown intermediate	**Uglov** '95 – **JL** '18 long range
potential	$V_{\text{Heis}}(d) = \delta_{d,1}$	← conjecture Klabbers JL '20 →	$V_{\text{UL}}(d) = \frac{1}{r_+(d)r_-(d)}$
exact solvability (spectrum)	Bethe ansatz; up to solving BAE Yang-Yang '66	unknown	via connection to Ruijsenaars-Macdonald in closed form JL et al '20
↑	↑ alg Bethe ansatz		↑
(Yang-Baxter) **quantum integrab**	quantum-loop structure Faddeev et al, late '70s	unknown	affine Hecke alg quantum-loop symmetry Bernard et al '93
↓	↓ transfer matrix		↓
higher Ham's	known	unknown	known JL et al '20

Outlook landscape of long-range spin chains





interaction range

nearest neighbour

intermediate range

long range

degree of spin symmetry

anisotropic

partially isotropic

isotropic

Heisenberg XYZ
Sutherland '70
Baxter '73

Heisenberg XXZ
Orbach '58
Yang–Yang '66

Heisenberg XXX
Heisenberg '28
Bethe '31

?

?

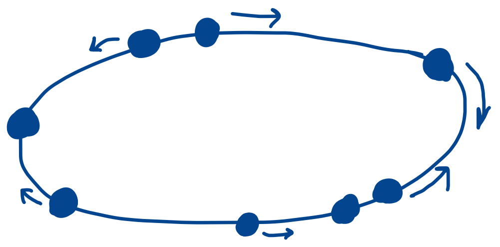
Inozemtsev
Inozemtsev '90
Inozemtsev '90–'00,
Klabbers JL '22

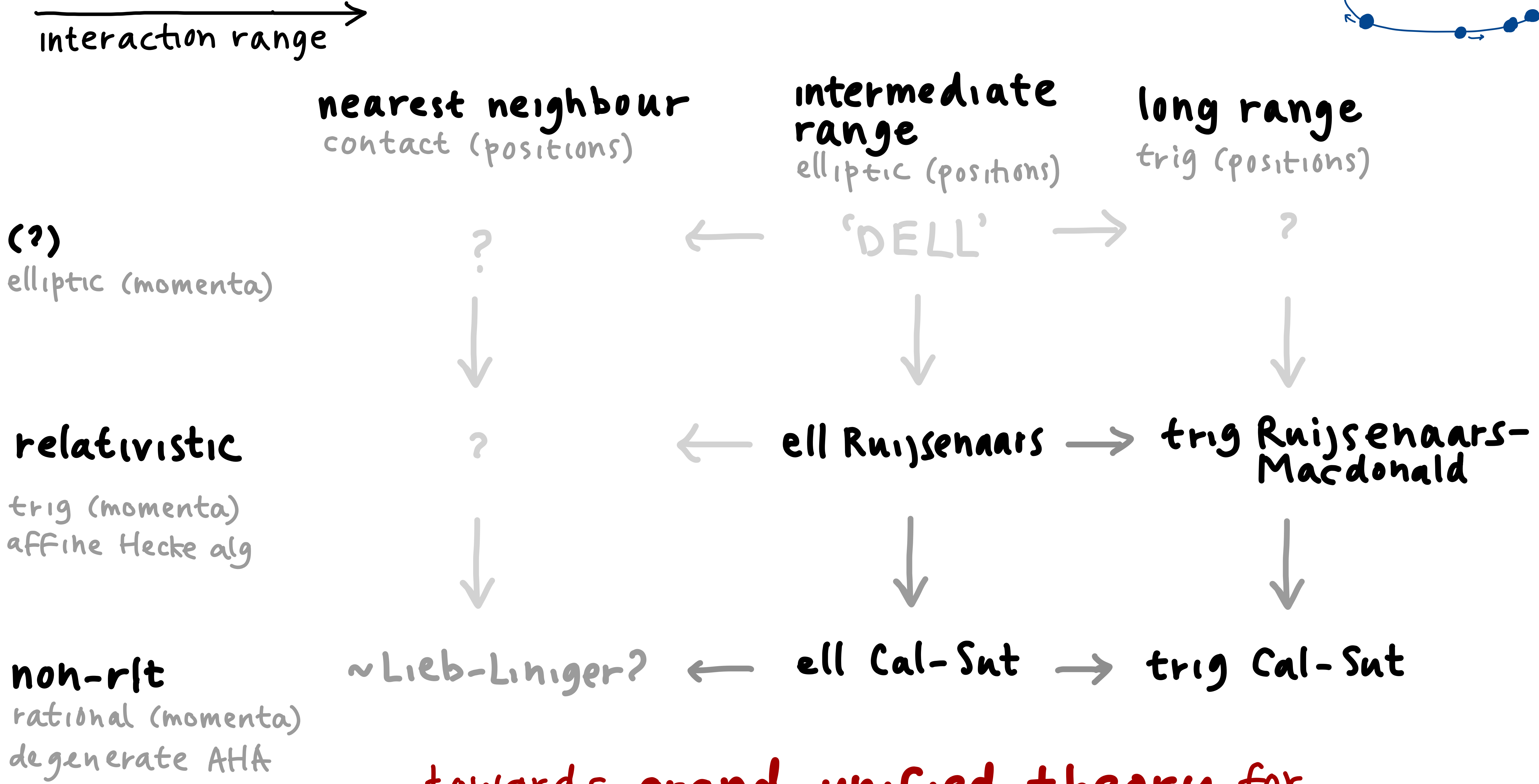
?

q-deformed HS
Uglov '95, JL '18
JL Pasquier Serban '20

Haldane–Shastry
Haldane '88, Shastry '88
Haldane '91, Bernard et al '93

Outlook landscape of quantum many-body systems





interaction range →

nearest neighbour
contact (positions)

intermediate range
elliptic (positions)

long range
trig (positions)

(?)
elliptic (momenta)

? ← 'DELL' → ?

relativistic
trig (momenta)
affine Hecke alg

? ← **ell Ruijsenaars** → **trig Ruijsenaars-Macdonald**

non-rlt
rational (momenta)
degenerate AHA

~Lieb-Liniger? ← **ell Cal-Sut** → **trig Cal-Sut**

towards **grand unified theory** for
quantum-integrable long-range spin chains?