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Exactly Solvable

Non-Equilibrium Steady States

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- Model I: with R. Frassek, J. Kurchan (ENS Paris)
- Model II: with C. Franceschini, R. Frassek (Modena)

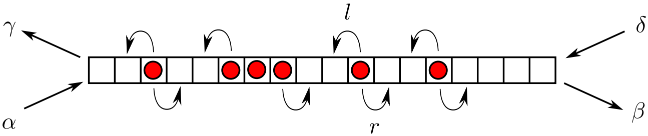
Outline

1. Introduction
2. Model I : ‘harmonic’ process
3. Results
4. Proof (ideas)
5. Model II: work in progress

1. Introduction

Non-equilibrium steady states (NESS)

open Symmetric Exclusion Process (SEP)



- Boundary densities: $\rho_L = \frac{\alpha}{\alpha+\gamma}, \quad \rho_R = \frac{\delta}{\delta+\beta}$

$$\rho_L = \rho_R \Longrightarrow \text{Equilibrium}$$

$$\rho_L \neq \rho_R \Longrightarrow \text{Non-Equilibrium}$$

- exactly solvable by matrix product ansatz:
  [Derrida-Evans-Hakim-Pasquier], [Derrida-Lebowitz-Speer], ...

Non-equilibrium steady states (NESS)

(Q1): How different are the stationary states of a system evolving in contact with equilibrium baths, from one evolving in contact with non-equilibrium baths?

> I shall argue that, as a consequence of Markov duality, there exists a non-local transformation relating non-equilibrium to equilibrium.

If the system is integrable, then the transformation can be explicitly written and the NESS can be exactly solved.

Emerging properties of NESS

- Long-range correlations

e.g. in the open SEP model: $\eta_x(t) = \#$ particles at site x at time t

$$0 < y_1 < \ldots < y_n < 1$$

Covariance $$\mathbb{C}ov(\eta_{Ny_1}, \eta_{Ny_2}) \sim -\frac{1}{N} y_1(1-y_2)(\rho_R - \rho_L)^2$$

Cumulants $$\kappa_n(\eta_{Ny_1}, \ldots, \eta_{Ny_n}) \sim \frac{1}{N^{n-1}} f_n(y_1, \ldots, y_n)(\rho_R - \rho_L)^n$$

- Non-local large deviation functions (Macroscopic Fluctuation Theory)

[Bertini-De Sole-Gabrielli-Landim-J.Lasinio]

e.g. in the open SEP model:

$$\mathbb{P}\mathrm{rob}\left[\rho(y)\right] \sim e^{-N\mathcal{I}(\rho(y))}$$

$$\mathcal{I}(\rho(y)) = \sup_{\substack{F \\ F(0)=\rho_L \\ F(1)=\rho_R}} \int_0^1 dy \left[\rho(y) \ln \frac{\rho(y)}{F(y)} + (1-\rho(y)) \ln \frac{1-\rho(y)}{1-F(y)} + \ln \frac{F'(y)}{\rho_R - \rho_L}\right]$$

Quantum spin chains

- Symmetric (partial) exclusion process:

$$\mathcal{H} = \sum_i \left[S_i^+ S_{i+1}^- + S_i^- S_{i+1}^+ + 2 S_i^o S_{i+1}^o - 2s^2 \right] \qquad [s = \frac{1}{2} \text{ integrable}]$$

$\mathfrak{su}(2)$ Lie algebra $\qquad [S^0, S^\pm] = \pm S^\pm, \qquad [S^+, S^-] = 2S^0$

- "KMP family" of processes: $\quad$ ($\longrightarrow$ Franceschini talk 26/04)

$$\mathcal{H} = \sum_i \left[S_i^+ S_{i+1}^- + S_i^- S_{i+1}^+ - 2 S_i^0 S_{i+1}^0 + 2s^2 \right] \qquad \text{[non-integrable ...]}$$

$\mathfrak{su}(1,1)$ Lie algebra $\qquad [S^0, S^\pm] = \pm S^\pm, \qquad [S^+, S^-] = -2S_0$

Quantum spin chains

(Q2): Is there a process with integrable Hamiltonian and $\mathfrak{su}(1,1)$ symmetry?

Can we add boundaries ?

- "Harmonic family" of processes (⟶ Frassek talk 28/04)

$$\mathcal{H} = \sum_i 2\Big(\psi(\mathbb{S}_{i,i+1}) - \psi(2s)\Big) \qquad [s > 0 \text{ integrable}]$$

$$\mathbb{S}_{i,i+1}(\mathbb{S}_{i,i+1} - 1) = S_i^0 S_{i+1}^0 - \frac{1}{2}(S_i^+ S_{i+1}^- + S_i^- S_{i+1}^+)$$

[Faddeev, Lipatov, Korchemsky, Derkachov, Beisert, ...]

I shall argue this is the "bosonic" exactly solvable counterpart of the open SEP model; the $\mathfrak{su}(1,1)$ family is richer.

Markov duality & integrability

- $\mathfrak{su}(2)$ Heisenberg spin chain $s = 1/2$: integrable, stochastic, dual process.

- Heisenberg chains with

  - $\mathfrak{su}(2)$ spins of higher values $s \neq 1/2$
  - $\mathfrak{su}(1,1)$ spins of all values $s > 0$

  they are stochastic, they all have a dual process, none of them is integrable.

- Integrable $\mathfrak{su}(2)$ chains of higher spin values, e.g. for spin $s = 1$

$$H = \sum_i \left((\vec{S}_i \cdot \vec{S}_{i+1})^2 - (\vec{S}_i \cdot \vec{S}_{i+1}) \right)$$

  Yet, they are not stochastic (as far as we could tell).

2. Model I

The open

symmetric ‘harmonic’ process

[FGK (2020a)] Frassek, G., Kurchan, *Non-compact quantum spin chains as integrable stochastic particle processes*, J. Stat. Phys. 180, 366–397 (2020)

The basic model

Markov process $\{\eta(t), t \geq 0\}$ taking values on $\Omega_N = \mathbb{N}_0^N$ with generator

$$\mathscr{L} = \mathscr{L}_1 + \sum_{i=1}^{N-1} \left(\mathscr{L}_{i,i+1}^{\rightarrow} + \mathscr{L}_{i,i+1}^{\leftarrow} \right) + \mathscr{L}_N$$

$$\mathscr{L}_{i,i+1}^{\rightarrow} f(\eta) = \sum_{k=1}^{\eta_i} \frac{1}{k} \left[f(\eta - k\delta_i + k\delta_{i+1}) - f(\eta) \right]$$

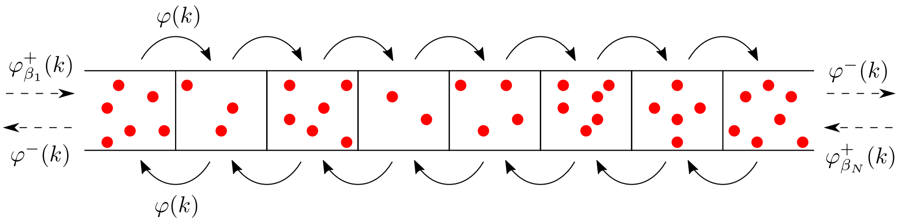
$$\mathscr{L}_{i,i+1}^{\leftarrow} f(\eta) = \sum_{k=1}^{\eta_{i+1}} \frac{1}{k} \left[f(\eta + k\delta_i - k\delta_{i+1}) - f(\eta) \right]$$

$$\mathscr{L}_1 f(\eta) = \sum_{k=1}^{\eta_i} \frac{1}{k} \left[f(\eta - k\delta_1) - f(\eta) \right] + \sum_{k=1}^{\infty} \frac{\beta_L^k}{k} \left[f(\eta + k\delta_1) - f(\eta) \right]$$

$$\mathscr{L}_N f(\eta) = \sum_{k=1}^{\eta_i} \frac{1}{k} \left[f(\eta - k\delta_N) - f(\eta) \right] + \sum_{k=1}^{\infty} \frac{\beta_R^k}{k} \left[f(\eta + k\delta_N) - f(\eta) \right]$$

Remark: $0 < \beta_L, \beta_R < 1$; the holding time in η is an Exp. R.V. with parameter $\sum_{i=1}^{N} 2h(\eta_i) - \log(1 - \beta_L) - \log(1 - \beta_R)$ $\quad h(n) = \sum_{k=1}^{n} \frac{1}{k}$ harmonic numbers

The basic model



A chain of length $N = 8$, the i^{th} site corresponds to the i^{th} box.

$$\varphi(k) = \varphi^-(k) = \frac{1}{k} \qquad \varphi^+_{\beta_i}(k) = \frac{\beta_i^k}{k}$$

The basic model

- If $\beta_L = \beta_R = \beta$: *equilibrium* set-up. The product geometric distribution

$$\mu^{eq}(\eta) = \prod_{i=1}^{N} \left[\beta^{\eta_i}(1-\beta)\right] \qquad 0 < \beta < 1$$

is reversible, and thus stationary, with density

$$\rho(\beta) = \frac{\beta}{1-\beta}$$

- If $\beta_L \neq \beta_R$: *boundary driven non-equilibrium*

$$\mu(\eta) \ = ?$$

Remark: non-product law (cf. standard zero-range [Levine, Mukamel,Schütz])

The general

open symmetric ‘harmonic’ process

The general model (spin s)

$$\mathscr{L} = \mathscr{L}_1 + \sum_{i=1}^{N-1} \mathscr{L}_{i,i+1} + \mathscr{L}_N$$

$$\begin{aligned}\mathscr{L}_{i,i+1} f(\eta) &= \sum_{k=1}^{\eta_i} \varphi_s(k,\eta_i)\Big[f(\eta - k\delta_i + k\delta_{i+1}) - f(\eta)\Big] \\ &+ \sum_{k=1}^{\eta_{i+1}} \varphi_s(k,\eta_{i+1})\Big[f(\eta + k\delta_i - k\delta_{i+1}) - f(\eta)\Big]\end{aligned}$$

$$\mathscr{L}_i f(\eta) = \sum_{k=1}^{\eta_i} \varphi_s(k,\eta_i)\Big[f(\eta - k\delta_i) - f(\eta)\Big] + \sum_{k=1}^{\infty} \frac{\beta_i^k}{k}\Big[f(\eta + k\delta_i) - f(\eta)\Big]$$

$$\varphi_s(k,n) = \frac{1}{k}\frac{\Gamma(n+1)\Gamma(n-k+2s)}{\Gamma(n-k+1)\Gamma(n+2s)}$$

$$\psi(z) = \frac{\partial}{\partial z}\log\Gamma(z) \qquad \psi(z+1) = \psi(z) + \frac{1}{z}$$

$$\sum_{k=1}^{n} \varphi_s(k,n) = \psi(n+2s) - \psi(2s) = \sum_{k=1}^{n} \frac{1}{k+2s-1} =: h_s(n)$$

The general model (spin s)

- If $s = \frac{1}{2}$ then we recover the basic model
- If $\beta_L = \beta_R = \beta$: *equilibrium* set-up. The product negative-binomial distribution

$$\mu^{eq}(\eta) = \prod_{i=1}^{N} \left[\frac{\beta^{\eta_i}}{\eta_i!} \frac{\Gamma(\eta_i + 2s)}{\Gamma(2s)} (1-\beta)^{2s} \right] \qquad 0 < \beta < 1$$

  is reversible.
- If $\beta_L \neq \beta_R$: *boundary driven* particle system

$$\mu(\eta) \;= ?$$

Relation to previous models

Relation to previous models

- The bulk part of the basic model is the $q \to 1$ limit of the MADM model

[Sasamoto-Wadati]

$$\mathscr{L}^{MADM} = \sum_{i \in \mathbb{Z}} \mathscr{L}_{i,i+1}$$

$$\begin{aligned} \mathscr{L}_{i,i+1} f(\eta) &= \sum_{k=1}^{\eta_i} \frac{1}{[k]_q} \Big[f(\eta - k\delta_i + k\delta_{i+1}) - f(\eta) \Big] \\ &+ \sum_{k=1}^{\eta_{i+1}} \frac{q^k}{[k]_q} \Big[f(\eta + k\delta_i - k\delta_{i+1}) - f(\eta) \Big] \end{aligned}$$

q-number $\quad [k]_q = \dfrac{1-q^k}{1-q} \to k \quad \text{as} \quad q \to 1$

Relation to previous models

- The bulk part of the spin s model is the $q \to 1$ limit of the q-Hahn model

[Barraquand-Corwin], [Povolotsky]

$$\mathscr{L}^{q-Hahn} = \sum_{i \in \mathbb{Z}} \mathscr{L}_{i,i+1}$$

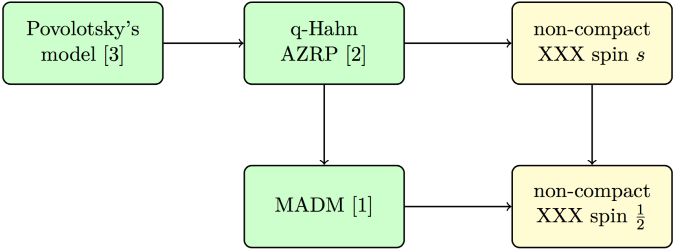
$$\begin{aligned} \mathscr{L}_{i,i+1} f(\eta) &= \sum_{k=1}^{\eta_i} \varphi^{r,q,\nu}(k,\eta_i)\Big[f(\eta - k\delta_i + k\delta_{i+1}) - f(\eta)\Big] \\ &+ \sum_{k=1}^{\eta_{i+1}} \varphi^{\ell,q,\nu}(k,\eta_{i+1})\Big[f(\eta + k\delta_i - k\delta_{i+1}) - f(\eta)\Big] \end{aligned}$$

$$\varphi^{r,q,\nu}(k,n) = \frac{\nu^k (\nu;q)_{n-k}(q;q)_n}{[k]_q(\nu;q)_n(q,q)_{n-k}} \qquad \varphi^{\ell,q,\nu}(k,n) = \frac{(\nu;q)_{n-k}(q;q)_n}{[k]_q(\nu;q)_n(q,q)_{n-k}}$$

$$(\nu;q)_n = \prod_{j=0}^{n-1}(1-\nu q^j) \qquad \lim_{q\to 1}\frac{(q^{2s};q)_n}{(1-q)^n} = \frac{\Gamma(2s+n)}{\Gamma(2s)}$$

$$\lim_{q\to 1} \varphi^{r/\ell,q,q^{2s}}(k,n) = \varphi_s(k,n)$$

Relation to previous models



[FGK (2020a)], see also [Frassek '19]

3. Results

Preliminaries:

i) duality

Duality

Definition [Liggett]

$(\eta_t)_{t\geq 0}$ Markov process on Ω with generator $\mathscr{L}$,

$(\xi_t)_{t\geq 0}$ Markov process on Ω_{dual} with generator $\mathscr{L}^{dual}$

ξ_t is dual to η_t with duality function $D : \Omega \times \Omega_{dual} \to \mathbb{R}$ if $\forall t \geq 0$

$$\mathbb{E}_\eta(D(\eta_t, \xi)) = \mathbb{E}_\xi(D(\eta, \xi_t)) \qquad \forall (\eta, \xi) \in \Omega \times \Omega_{dual}$$

η_t is self-dual if $\mathscr{L}^{dual} = \mathscr{L}$.

In terms of generators:

$$\mathscr{L} D(\cdot, \xi)(\eta) = \mathscr{L}^{dual} D(\eta, \cdot)(\xi)$$

Example [Lévy]

$(X_t)_{t\geq 0}$ Brownian motion on $[0,\infty)$ started at $x>0$, reflected at the origin

$(Y_t)_{t\geq 0}$ Brownian motion on $[0,\infty)$ started at $y>0$, absorbed at the origin

$D(x,y) = 1_{\{x\leq y\}}$ ('Sigmund' duality fct)

$$\mathbb{E}_x(D(X_t,y)) = \mathbb{E}_y(D(x,Y_t))$$

$$\mathbb{P}_x[X_t \leq y] = \mathbb{P}_y[Y_t \geq x]$$

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$$\mathbb{P}_x[X_t \leq y] = \mathbb{P}_y[Y_t \geq x]$$

$$\int_0^y \left(e^{-\frac{(z-x)^2}{2t}} + e^{-\frac{(z+x)^2}{2t}}\right)\frac{dz}{\sqrt{2\pi t}} = \int_x^\infty \left(e^{-\frac{(z-y)^2}{2t}} - e^{-\frac{(z+y)^2}{2t}}\right)\frac{dz}{\sqrt{2\pi t}}$$

Duality for Markov chains

Assume state spaces Ω, Ω_{dual} are countable sets, then the Markov generator L is a matrix $L(\eta, \eta')$ s.t.

$$L(\eta, \eta') \geq 0 \quad \text{if} \quad \eta \neq \eta', \qquad \qquad \sum_{\eta' \in \Omega} L(\eta, \eta') = 0$$

$$\mathscr{L} D(\cdot, \xi)(\eta) = \mathscr{L}^{dual} D(\eta, \cdot)(\xi)$$

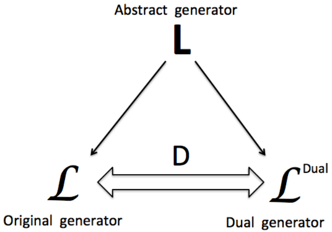
amounts to

$$LD = DL_{dual}^T$$

(Lie) Algebraic approach to duality

Overarching ideas:

- self-duality is derived from symmetries [Schütz]
- duality arises as a change of representation



- Monograph in preparation with G. Carinci and F. Redig: *'Duality of Markov processes: an algebraic approach'*.
- The approach is constructive: [Carinci, G., Redig, Sasamoto] quantum groups rank 1; [Kuan] higher rank quantum groups; [Franceschini, G., Groenevelt, Redig, Sau] orthogonal polynomials
- See also [Z. Chen, J. de Gier, M. Wheeler] for Macdonald (type) polynomials

Preliminaries:

ii) factorial moments

Factorial moments

The multivariate factorial moments of order $(\xi_1, \ldots, \xi_N) \in \mathbb{N}^N$ of an integer-valued random vector with distribution $\mu(\eta_1, \ldots, \eta_N)$ are defined as

$$F(\xi_1, \ldots, \xi_N) = \sum_{\eta \in \mathbb{N}^N} \left[\prod_{i=1}^{N} \eta_i(\eta_i - 1) \cdots (\eta_i - \xi_i + 1) \right] \mu(\eta_1, \ldots, \eta_N)$$

Inversion formula

$$\mu(\eta_1, \ldots, \eta_N) = \sum_{\xi \in \mathbb{N}^N} F(\xi_1, \ldots, \xi_N) \prod_{i=1}^{N} \binom{\xi_i}{\eta_i} \frac{(-1)^{\xi_i - \eta_i}}{\xi_i!}$$

Factorial moments

For us it will be convenient to consider

$$G(\xi_1, \ldots, \xi_N) = \sum_{\eta \in \mathbb{N}^N} \left[\prod_{i=1}^{N} \frac{\eta_i}{2s} \frac{(\eta_i - 1)}{2s+1} \cdots \frac{(\eta_i - \xi_i + 1)}{(2s + \xi_i - 1)} \right] \mu(\eta_1, \ldots, \eta_N)$$

At equilibrium

$$\mu^{eq}(\eta) = \prod_{i=1}^{N} \left[\frac{\beta^{\eta_i}}{\eta_i !} \frac{\Gamma(\eta_i + 2s)}{\Gamma(2s)} (1-\beta)^{2s} \right] \qquad 0 < \beta < 1$$

$$G^{eq}(\xi_1, \ldots, \xi_N) = \prod_{i=1}^{N} \rho^{\xi_i} \qquad \rho := \rho(\beta) = \frac{\beta}{1-\beta}$$

Preliminaries:

iii) dual absorbing process

The dual absorbing process [1]

Markov process $\{\xi(t), t \geq 0\}$ taking values on $\Omega_{N+2} = \mathbb{N}^{N+2}$

Configurations: $\xi = (\xi_0, \xi_1, \ldots, \xi_N, \xi_{N+1}) \in \Omega_{N+2}$

Generator: $$\mathscr{L}^{dual} = \mathscr{L}_{0,1}^{\leftarrow} + \sum_{i=1}^{N-1} \mathscr{L}_{i,i+1} + \mathscr{L}_{N,N+1}^{\rightarrow}$$

$$\mathscr{L}_{0,1}^{\leftarrow} f(\xi) = \sum_{k=1}^{\xi_1} \varphi_s(k, \xi_1) \Big[f(\xi + k\delta_0 - k\delta_1) - f(\xi) \Big]$$

$$\mathscr{L}_{N,N+1}^{\rightarrow} f(\xi) = \sum_{k=1}^{\xi_N} \varphi_s(k, \xi_N) \Big[f(\xi - k\delta_N + k\delta_{N+1}) - f(\xi) \Big]$$

[1] Spohn, Presutti, De Masi .... ; for a review see [Carinci, G., Giberti, Redig, JSP13]

Results

Duality of the open symmetric harmonic process

Theorem [FGK (2020a)]

1. The open symmetric harmonic process $\{\eta(t), t \geq 0\}$ with generator $\mathscr{L}$ and the absorbing process $\{\xi(t), t \geq 0\}$ with generator $\mathscr{L}^{dual}$ are dual

$$D(\eta, \xi) = \rho_L^{\xi_0} \left[\prod_{i=1}^{N} \frac{\eta_i!}{(\eta_i - \xi_i)!} \frac{\Gamma(2s)}{\Gamma(\xi_i + 2s)} \right] \rho_R^{\xi_{N+1}}$$

$$\rho_L = \frac{\beta_L}{1 - \beta_L} \qquad \rho_R = \frac{\beta_R}{1 - \beta_R}$$

Namely

$$\mathbb{E}_\eta \Big[D(\eta(t), \xi) \Big] = \mathbb{E}_\xi \Big[D(\eta, \xi(t)) \Big]$$

Factorial moments and absorption probabilities

Theorem [FGK (2020a)]

2. Let $G(\xi_1, \ldots, \xi_N)$ be scaled factorial moments of non-equilibrium state μ_N.

   Consider dual configuration $\check{\xi} = (0, \xi_1, \ldots, \xi_N, 0)$ with $|\xi| = \sum_{i=1}^N \xi_i$ particles.

   Then

$$G(\xi_1, \ldots, \xi_N) = \sum_{k=0}^{|\xi|} \rho_L^k \, \rho_R^{|\xi|-k} \; p_{\check{\xi}}(k)$$

   where

$$p_{\check{\xi}}(k) = \mathbb{P}\Big[\xi(\infty) = k\delta_0 + (|\xi| - k)\delta_{N+1} \,\Big|\, \xi(0) = \check{\xi}\Big]$$

absorption probabilities

Factorial moments: explicit expression I

Theorem [FG (arXiv:2107.01720)]

3. Let $G(\xi_1, \ldots, \xi_N)$ be scaled factorial moments of non-equilibrium state μ_N. Then [assuming $2s \in \mathbb{N}$]

$$G(\xi_1, \ldots, \xi_N) = \sum_{n=0}^{|\xi|} \rho_R^{|\xi|-n} (\rho_L - \rho_R)^n g_\xi(n)$$

with

$$g_\xi(n) = \sum_{\substack{n_1, \ldots, n_N \\ \sum_i n_i = n}} \prod_{i=1}^{N} \binom{\xi_i}{n_i} \prod_{j=1}^{2s} \frac{2s(N+2-i)-j}{2s(N+2-i)-j+\sum_{k=i}^{N} n_k}$$

In other words

$$p_{\xi}(k) = \sum_{n=k}^{|\xi|} (-1)^{n-k} \binom{n}{k} g_\xi(n)$$

Factorial moments: explicit expression II

Theorem [FG (arXiv:2107.01720)]

4. If we identify $\xi = (\xi_1, \dots, \xi_N)$ with the the ordered set $x = (x_1, x_2, \dots, x_{|\xi|})$ with $1 \leq x_1 \leq x_2 \leq \dots \leq x_{|\xi|} \leq N$, then

$$G(\xi_1, \dots, \xi_N) = \sum_{n=0}^{|\xi|} \rho_R^{|\xi|-n} (\rho_L - \rho_R)^n g_x(n)$$

with

$$g_x(n) = \sum_{1 \leq i_1 < \dots < i_n \leq |\xi|} \prod_{\alpha=1}^{n} \frac{n - \alpha + 2s(N + 1 - x_{i_\alpha})}{n - \alpha + 2s(N + 1)}$$

Mapping non-equilibrium to equilibrium

Theorem [FGK(2020b),FG (arXiv:2107.01720)]

5. As a consequence of duality, there exists a matrix P such that

$$H^{eq} = P^{-1} H P$$

where

$$H^{eq} = (L^{eq})^T \qquad H = L^T$$

- The mapping P was observed macroscopically by [Tailleur, Kurchan, Lecomte] in the context of the MFT

MACRO:

Local Equilibrium & Fick's law

Local equilibrium & Fick's law

Let μ_N be the invariant measure.

(i) For all f bounded cylindrical function on $\mathbb{N}_0^{\mathbb{N}}$

$$\lim_{N\to\infty} \mu_N(\tau_{[uN]} f) = \nu_{\rho(u)}(f) \qquad u \in (0,1)$$

where

- τ_i translation by i
- ν_ρ is the product measure on $\mathbb{N}_0^{\mathbb{N}}$ with marginals NegBin $(2s, \rho)$
- $\rho(u) = \rho_L + (\rho_R - \rho_L)u$

(ii) Fick's law

$$J = -K_s \frac{d\rho(u)}{du} \qquad u \in (0,1)$$

where

- $K_s = 2s$
- $J = \lim_{N\to\infty} \sum_{i=1}^{N-1} \left[\sum_{\eta\in\mathbb{N}_0^{\mathbb{N}}} \mu_N(\eta)[\eta_i - \eta_{i+1}] \right]$

Proof of local equilibrium

Given $(\xi_1, \ldots, \xi_N) \in \mathbb{N}^N$ made of $|\xi|$ dual particles located at $1 \le x_1 \le \ldots \le x_{|\xi|} \le N$

define $(\xi_1^u, \ldots, \xi_N^u) \in \mathbb{N}^N$ with dual particles located at $x_1 + [uN] \le \ldots \le x_{|\xi|} + [uN]$

Then we need to prove that

$$\lim_{N\to\infty} G(\xi_1^u, \ldots, \xi_N^u) = [\rho(u)]^{|\xi|}$$

The explicit formula gives

$$\lim_{N\to\infty} g_{x^u}(n) = \lim_{N\to\infty} \sum_{1\le i_1<\ldots<i_n\le|\xi|} \prod_{\alpha=1}^{n} \frac{n-\alpha+2s(N+1-x_{i_\alpha}-[uN])}{n-\alpha+2s(N+1)} = \binom{|\xi|}{n}(1-u)^n$$

and therefore

$$\lim_{N\to\infty} G(\xi_1^u, \ldots, \xi_N^u) = \sum_{n=0}^{|\xi|} \rho_R^{|\xi|-n}(\rho_L-\rho_R)^n \binom{|\xi|}{n}(1-u)^n = [\rho(u)]^{|\xi|}$$

MICRO:

Correlation functions

$(s = \frac{1}{2})$

1 dual particle : $\xi = \delta_x$

$$D(\eta, \delta_x) = \eta_x$$

$$\mathbb{E}[\eta_x] = \rho_L \, p_x(1) + \rho_R \, p_x(0)$$

$$p_x(1) = 1 - \frac{i}{N+1} \qquad\qquad p_x(0) = \frac{i}{N+1}$$

$$\rho_x := \langle \eta_x \rangle = \rho_L + \frac{\rho_R - \rho_L}{N+1} x$$

Linear profile

$$J := -\langle \eta_{x+1} - \eta_x \rangle = -\frac{\rho_R - \rho_L}{N+1}$$

Fick's law

2 dual particles : $\xi = \delta_{x_1} + \delta_{x_2}$

$$\mathbb{C}ov(\eta_{x_1}, \eta_{x_2}) = \mathbb{E}[\eta_{x_1}\eta_{x_2}] - \mathbb{E}[\eta_{x_1}]\mathbb{E}[\eta_{x_2}]$$

$$G_{x_1,x_2} = \mathbb{E}[D(\eta, \delta_{x_1} + \delta_{x_2})] \qquad G_{x_1} = \mathbb{E}[D(\eta, \delta_{x_1})]$$

$$\mathbb{C}ov(\eta_{x_1}, \eta_{x_2}) = \begin{cases} G_{x_1,x_2} - G_{x_1}G_{x_2} & \text{if} \quad x_1 \neq x_2 \\ 2G_{x_1,x_1} + G_{x_1}[1 - G_{x_1}] & \text{if} \quad x_1 = x_2 \end{cases}$$

$$= \frac{x_1(N+1-x_2)}{(N+1)^2(N+2)}(\rho_L - \rho_R)^2 \qquad \text{if} \quad x_1 \neq x_2$$

Remark: Long range correlations

$$\lim_{N\to\infty} N\,\mathbb{C}ov(\eta_{y_1N}, \eta_{y_2N}) = y_1(1-y_2)(\rho_L - \rho_R)^2 \qquad 0 \leq y_1 < y_2 \leq 1$$

3 dual particles : $\xi = \delta_{x_1} + \delta_{x_2} + \delta_{x_3}$

$$\begin{aligned}\kappa_3(\eta_{x_1}, \eta_{x_2}, \eta_{x_3}) &= \mathbb{E}[\eta_{x_1}\eta_{x_2}\eta_{x_3}] + \mathbb{E}[\eta_{x_1}]\mathbb{E}[\eta_{x_2}]\mathbb{E}[\eta_{x_3}] \\ &- \mathbb{E}[\eta_{x_1}\eta_{x_2}]\mathbb{E}[\eta_{x_3}] - \mathbb{E}[\eta_{x_1}\eta_{x_3}]\mathbb{E}[\eta_{x_2}] - \mathbb{E}[\eta_{x_2}\eta_{x_3}]\mathbb{E}[\eta_{x_1}]\end{aligned}$$

Using three dual particles started at $1 \leq x_1 < x_2 < x_3 \leq N$

$$\kappa_3(\eta_{x_1}, \eta_{x_2}, \eta_{x_3}) = \frac{2x_1(N+1-2x_2)(N+1-x_3)}{(N+1)^3(N+2)(N+3)}(\rho_R - \rho_L)^3$$

Remark: for $0 \leq y_1 < y_2 < y_3 \leq 1$

$$\lim_{N\to\infty} N^2\, \kappa_3(\eta_{y_1 N}, \eta_{y_2 N}, \eta_{y_3 N}) = 2y_1(1-2y_2)(1-y_3)(\rho_R - \rho_L)^3$$

$$N^{n-1}\kappa_n \sim f_n(y_1, \ldots, y_n)(\rho_R - \rho_L)^n$$

proved by orthogonal duality in [Floreani, Redig, Sau '20]

Comparison to

inhomogeneous product meausure

Fluctuation of total mass $|\eta| = \sum_{x=1}^{N} \eta_x$

$\mathbb{E}[\cdot]$ non-equilibrium steady state

$\mathbb{E}_{loc}[\cdot]$ inhomog. product state $\otimes_{x=1}^{N} NegBin(2s, \rho_x)$

- average

$$\mathbb{E}[|\eta|] = \mathbb{E}_{loc}[|\eta|] = \sum_{x=1}^{N} \rho_x = \sum_{x=1}^{N} 2s\Big(\rho_L + \frac{\rho_R - \rho_L}{N+1} x\Big) = 2sN\Big(\frac{\rho_L + \rho_R}{2}\Big)$$

- fluctuations

$$\mathbb{V}ar[|\eta|] = \sum_{x_1, x_2 = 1}^{N} \mathbb{C}ov[\eta_{x_1}, \eta_{x_2}]$$

$$\lim_{N\to\infty} \frac{1}{N} \mathbb{V}ar[|\eta|] = \lim_{N\to\infty} \frac{1}{N} \mathbb{V}ar_{loc}[|\eta|] + \frac{s}{6}(\rho_L - \rho_R)^2$$

For $s = -\frac{1}{2}$ one recovers open SEP [Derrida, Lebowitz, Speer, PRL01]

4. Proof

(Ideas)

Algebraic description

$\mathfrak{su}(1,1)$ Lie algebra

Non-compact Lie algebra with generators satisfying

$$[S^0, S^\pm] = \pm S^\pm \qquad [S^+, S^-] = -2S^0$$

Representation with ∞-dimensional matrices (spin $s > 0$)

$$S^+|n\rangle = (2s+n)|n+1\rangle \qquad S^-|n\rangle = n|n-1\rangle \qquad S^0|n\rangle = (n+s)|n\rangle$$

$$|n\rangle = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \end{pmatrix} \quad S^+ = \begin{pmatrix} 0 & & & \\ 2s & \ddots & & \\ & 2s+1 & \ddots & \\ & & \ddots & \ddots \end{pmatrix} \quad S^- = \begin{pmatrix} 0 & 1 & & \\ & \ddots & 2 & \\ & & \ddots & \ddots \\ & & & \ddots \end{pmatrix} \quad S^0 = \begin{pmatrix} s & 0 & & \\ 0 & s+1 & \ddots & \\ & \ddots & s+2 & \ddots \\ & & \ddots & \ddots \end{pmatrix}$$

Open XXX chain with $\mathfrak{su}(1,1)$ spins

$$H = H_1 + \sum_{i=1}^{N-1} H_{i,i+1} + H_N$$

- Bulk: [Beisert, Faddeev, Korchemsky, Derkachov ...]

$$H_{i,i+1} = 2\Big(\psi(\mathbb{S}_{i,i+1}) - \psi(2s)\Big)$$

$$\mathbb{S}_{i,i+1}(\mathbb{S}_{i,i+1} - 1) = S_i^0 S_{i+1}^0 - \frac{1}{2}(S_i^+ S_{i+1}^- + S_i^- S_{i+1}^+)$$

- Boundary [FGK(2020a)]

$$H_1 = e^{-S_1^-}\, e^{\rho_L S_1^+} \Big(\psi(S_1^0 + s) - \psi(2s)\Big)\, e^{-\rho_L S_1^+}\, e^{S_1^-}$$

$$H_N = e^{-S_N^-}\, e^{\rho_R S_N^+} \Big(\psi(S_N^0 + s) - \psi(2s)\Big)\, e^{-\rho_R S_N^+}\, e^{S_N^-}$$

In the discrete representation

$$L = H^T$$

Open XXX chain with $\mathfrak{su}(1,1)$ spins: rewriting of the bulk

$$H = H_1 + \sum_{i=1}^{N-1} H_{i,i+1} + H_N$$

- Bulk: [FG (arXiv:2107.01720)]

$$H_{i,i+1} = H^{\rightarrow}_{i,i+1} + H^{\leftarrow}_{i,i+1}$$

$$H^{\rightarrow}_{x,y} = e^{-S_y^+ (S_y^0 + s)^{-1} S_x^-} \Big(\psi(S_x^0 + s) - \psi(2s) \Big) e^{S_y^+ (S_y^0 + s)^{-1} S_x^-}$$

$$H^{\leftarrow}_{x,y} = e^{-S_x^+ (S_x^0 + s)^{-1} S_y^-} \Big(\psi(S_y^0 + s) - \psi(2s) \Big) e^{S_x^+ (S_x^0 + s)^{-1} S_y^-}$$

Duality explained

$$LD = DL_{dual}^{T}$$

$$\Updownarrow$$

$$H^{T}D = DH_{dual}$$

$$\Updownarrow$$

$$\left(H_1 + \mathbf{H} + H_N\right)^{T} D = D\left(H_{0,1}^{\leftarrow} + \mathbf{H} + H_{N,N+1}^{\rightarrow}\right)$$

Bulk (self-)duality: $\mathbf{H}^T D = D\,\mathbf{H}$

- Bulk Hamiltonian $\mathfrak{su}(1,1)$ symmetry:

$$\mathbf{H} := \sum_{i=1}^{N-1} H_{i,i+1} \qquad \mathbf{S}^+ := \sum_{i=1}^{N} S_i^+$$

$$[\mathbf{H}, \mathbf{S}^0] = [\mathbf{H}, \mathbf{S}^+] = [\mathbf{H}, \mathbf{S}^-] = 0$$

- Reversibility

$$\mathbf{H}^T \mathbf{d} = \mathbf{d}\mathbf{H}$$

$$\mathbf{d}(\eta, \xi) = \prod_{i=1}^{N} \eta_i! \frac{\Gamma(2s)}{\Gamma(\eta_i + 2s)} \delta_{\eta,\xi}$$

- Bulk (self-)duality

$$\mathbf{H}^T \mathbf{d}e^{\mathbf{S}^+} = \mathbf{d}\mathbf{H}e^{\mathbf{S}^+} = \mathbf{d}e^{\mathbf{S}^+}\mathbf{H}$$

$$\mathbf{d}e^{\mathbf{S}^+}(\eta, \xi) = \prod_{i=1}^{N} \frac{\eta_i!}{(\eta_i - \xi_i)!} \frac{\Gamma(2s)}{\Gamma(\xi_i + 2s)}$$

Boundary duality: $H_1^T D = D\, H_{0,1}^{\leftarrow}$

$$(d_1 e^{S_1^+})^{-1} H_1^T (d_1 e^{S_1^+}) = e^{-\rho_L S_1^-} \left(\psi(S_1^0 + s) - \psi(2s)\right) e^{\rho_L S_1^-}$$

Boundary duality: $H_1^T D = D\, H_{0,1}^{\leftarrow}$

$$(d_1 e^{S_1^+})^{-1} H_1^T (d_1 e^{S_1^+}) = e^{-\rho_L S_1^-} \Big(\psi(S_1^0 + s) - \psi(2s)\Big)\, e^{\rho_L S_1^-}$$

Two representations of $\mathfrak{su}(1,1)$ with intertwiner $I_0 = \sum_{\xi_0} \rho_L^{\xi_0} \langle \xi_0 |$

$$\begin{cases} \mathscr{S}_0^+ = \rho_L \Big(\rho_L \frac{\partial}{\partial \rho_L} + 2s\Big) \\ \mathscr{S}_0^- = \frac{\partial}{\partial \rho_L} \\ \mathscr{S}_0^0 = \Big(\rho_L \frac{\partial}{\partial \rho_L} + s\Big) \end{cases} \qquad \begin{cases} S_0^+ |\xi_0\rangle = (\xi_0 + 2s)|\xi_0 + 1\rangle \\ S_0^- |\xi_0\rangle = \xi_0 |\xi_0 - 1\rangle \\ S_0^0 |\xi_0\rangle = (\xi_0 + s)|\xi_0\rangle \end{cases}$$

$$e^{-\mathscr{S}_0^+ (\mathscr{S}_0^0 + s)^{-1} S_1^-} \Big(\psi(S_1^0 + s) - \psi(2s)\Big)\, e^{\mathscr{S}_0^+ (\mathscr{S}_0^0 + s)^{-1} S_1^-}$$

is intertwined by I_0 to

$$e^{-S_0^+ (S_0^0 + s)^{-1} S_1^-} \Big(\psi(S_1^0 + s) - \psi(2s)\Big)\, e^{S_0^+ (S_0^0 + s)^{-1} S_1^-} = H_{0,1}^{\leftarrow}$$

Duality to absorbing process $\Longrightarrow$ Isospectrality

- $H = H_1 + \mathbf{H} + H_N$

$$H_i = e^{-S_i^-} \, e^{\rho_i S_i^+} \left(\psi(S_i^0 + s) - \psi(2s)\right) e^{-\rho_i S_i^+} \, e^{S_i^-}$$

- $H' = e^{\mathbf{S}^-} \, H \, e^{-\mathbf{S}^-} = H_1' + \mathbf{H} + H_N'$

$$H_i' = e^{\rho_i S_i^+} \left(\psi(S_i^0 + s) - \psi(2s)\right) e^{-\rho_i S_i^+}$$

upper triangular

- $H^\circ = H_1^\circ + \mathbf{H} + H_N^\circ$

$$H_i^\circ = \psi(S_i^0 + s) - \psi(2s)$$

block diagonal

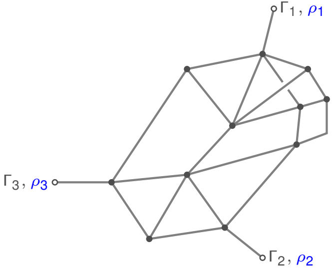
They all have the same spectrum (which is independent of ρ_L, ρ_R)!

As a consequence there exists P such that

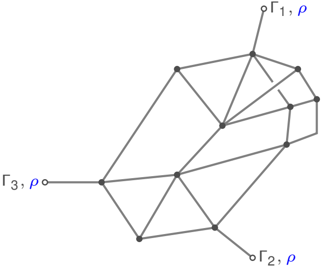
$$H^{eq} = P^{-1} H P$$

Mapping non-equilibrium to equilibrium

The mapping holds on any graph, with multiple reservoirs



$\Longrightarrow$



(ρ_1, ρ_2, ρ_3) (ρ, ρ, ρ)

$$H^{eq} = P^{-1} H P$$

Sequence of similarity transformations

- $H' = e^{\mathbf{S}^-} H e^{-\mathbf{S}^-}$

$$H'_1 = e^{\rho_L S_1^+} \Big(\psi(S_1^0+s)-\psi(2s)\Big)\, e^{-\rho_L S_1^+} \qquad H'_N = e^{\rho_R S_N^+} \Big(\psi(S_N^0+s)-\psi(2s)\Big)\, e^{-\rho_R S_N^+}$$

- $H'' = e^{-\rho_R \mathbf{S}^+} H' e^{\rho_R \mathbf{S}^+}$

$$H''_1 = e^{(\rho_L-\rho_R) S_1^+} \Big(\psi(S_1^0+s)-\psi(2s)\Big)\, e^{-(\rho_L-\rho_R) S_1^+} \qquad H''_N = \psi(S_N^0+s)-\psi(2s)$$

- $H^\circ = W^{-1} H'' W$

$$H^\circ_1 = \psi(S_1^0+s)-\psi(2s) \qquad H^\circ_N = \psi(S_N^0+s)-\psi(2s)$$

- Mapping non-equilibrium to equilibrium

$$P = e^{-\mathbf{S}_-} e^{\rho_R \mathbf{S}_+} W e^{-\rho \mathbf{S}_+} e^{\mathbf{S}_-}$$

Non-trivial symmetries (Quantum Inverse Scattering method)

- First non-local charge:

$$[H'', Q''] = 0 \qquad Q'' = Q^\circ - (\rho_L - \rho_R) Q^+$$

$$Q^\circ = \mathbf{S}^0 \left(\mathbf{S}^0 + 2s - 1\right)$$

$$Q^+ = s\mathbf{S}^+ + \sum_{i=1}^{N} S_i^+ \left(S_i^0 + 2 \sum_{j=i+1}^{N} S_j^0\right)$$

- To find W such that $H^\circ = W^{-1} H'' W$ we rather solve

$$Q^\circ = W^{-1} Q'' W$$

- Ansatz:

$$W = 1 + \sum_{k=1}^{\infty} (\rho_L - \rho_R)^k W_k \qquad \Longrightarrow \qquad [Q^\circ, W_k] = Q^+ W_{k-1}$$

- This difference equation can be solved

$$W = \sum_{k=0}^{\infty} (\rho_L - \rho_R)^k \frac{(Q^+)^k}{k!} \frac{\Gamma(2(\mathbf{S}^0 + s))}{\Gamma(k + 2(\mathbf{S}^0 + s))}$$

non-local!

Back tracking

- $H^\circ|\Omega\rangle = 0$ with $|\Omega\rangle = |0\rangle \otimes |0\rangle \otimes \ldots \otimes |0\rangle$
- $H''|g''\rangle = 0$ with $|g''\rangle = W|\Omega\rangle$
- $H'|g'\rangle = 0$ with $|g'\rangle = e^{\rho_R \mathbf{S}^+}|g''\rangle$
- $H|g\rangle = 0$ with $|g\rangle = e^{-\mathbf{S}^-}|g'\rangle$

- Mapping non-equilibrium to equilibrium

$$P = e^{-\mathbf{S}_-} e^{\rho_R \mathbf{S}_+} W e^{-\rho \mathbf{S}_+} e^{\mathbf{S}_-}$$

5. Work in progress

Energy redistribution using a Levy Beta process

Process $\{z(t), t \geq 0\}$ taking values on $\Omega_N = \mathbb{R}_+^N$

$z_i(t) \equiv$ energy at site $i \in \{1, 2, \ldots N\}$ at time $t \geq 0$

$$\mathscr{L} = \mathscr{L}_1 + \sum_{i=1}^{N-1} \mathscr{L}_{i,i+1} + \mathscr{L}_N$$

$$\begin{aligned}
\mathscr{L}_{i,i+1} f(z) &= \int_0^{z_i} \frac{d\alpha}{\alpha} \left(1 - \frac{\alpha}{z_i}\right)^{2s-1} \left[f(z - \alpha\delta_i + \alpha\delta_{i+1}) - f(z)\right] \\
&+ \int_0^{z_{i+1}} \frac{d\alpha}{\alpha} \left(1 - \frac{\alpha}{z_i}\right)^{2s-1} \left[f(z + \alpha\delta_i - \alpha\delta_{i+1}) - f(z)\right]
\end{aligned}$$

$$\begin{aligned}
\mathscr{L}_i f(z) &= \int_0^{z_i} \frac{d\alpha}{\alpha} \left(1 - \frac{\alpha}{z_i}\right)^{2s-1} \left[f(z - \alpha\delta_i) - f(z)\right] \\
&+ \int_0^{\infty} d\alpha \frac{e^{-\lambda_i \alpha}}{\alpha} \left[f(z + \alpha\delta_i) - f(z)\right]
\end{aligned}$$

Energy redistribution using a Levy Beta process

- Energy redistribution rule associated to $\mathscr{L}_{x,y} = \mathscr{L}^{\rightarrow}_{x,y} + \mathscr{L}^{\leftarrow}_{x,y}$

$$\mathscr{L}^{\rightarrow}_{x,y} : \qquad (Z_x, Z_y) \Longrightarrow (Z_x - U_1 Z_x, Z_y + U_1 Z_x)$$

$$\mathscr{L}^{\leftarrow}_{x,y} : \qquad (Z_x, Z_y) \Longrightarrow (Z_x + U_2 Z_y, Z_y - U_2 Z_y)$$

$U_1, U_2 \sim$ independent 'improper' Beta$(0, 2s)$ distributions

- Energy redistribution rule associated to KMP model $\mathscr{L}^{KMP}_{x,y}$

$$\mathscr{L}^{KMP}_{x,y} : \qquad (Z_x, Z_y) \Longrightarrow (V(Z_x + Z_y), (1 - V)(Z_x + Z_y))$$

$V \sim$ Beta$(2s, 2s)$ distribution

- The boundary driven process $\{z(t), t \geq 0\}$ arise as limit of the open harmonic process $\{\eta(t), t \geq 0\}$:

$$z(t) = \lim_{\epsilon \to 0} \epsilon \eta(t) \qquad \beta_i = 1 - \epsilon \lambda_i$$

Energy redistribution using a Levy Beta process

- The boundary driven process $\{z(t), t \geq 0\}$ has the same algebraic structure of the open symmetric 'harmonic process' $\{\eta(t), t \geq 0\}$, now in the representation of the $\mathfrak{su}(1,1)$ Lie algebra

$$\mathcal{S}_i^+ = z_i \qquad \mathcal{S}_i^- = z_i \frac{\partial^2}{\partial z_i^2} + 2s \frac{\partial}{\partial z_i} \qquad \mathcal{S}_i^0 = z_i \frac{\partial}{\partial z_i} + s$$

- If $\lambda_L = \lambda_R = \lambda$: *equilibrium* set-up, product of Gamma distribution is reversible

$$p^{eq}(z) = \prod_{i=1}^{N} \left[\frac{\lambda^{2s}}{\Gamma(2s)} z_i^{2s-1} e^{-\lambda z_i} \right] \qquad \lambda > 0$$

- If $\lambda_L \neq \lambda_R$: Duality with absorbing harmonic process with duality function

$$D(z, \xi) = \rho_L^{\xi_0} \left[\prod_{i=1}^{N} z_i^{\xi_i} \frac{\Gamma(2s)}{\Gamma(\xi_i + 2s)} \right] \rho_R^{\xi_{N+1}}$$

CONCLUSIONS

- Reasoning with Lie algebras is useful for duality and its consequences.

- Integrability (on top of duality) opens up the possibility of explicit formulae for boundary-driven models, connecting non-equilibrium to equilibrium.

- Some open problems:
  - matrix product ansatz
  - large deviations
  - open chain with $q \neq 1$
  - higher rank non-compact algebras
  - connection to population dynamics

- The algebraic approach to duality can be extended to quantum systems (e.g. FGK(2021), arXiv:2008.03476, quantum symmetric exclusion process).

Thank you for your attention